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Abstract

We introduce polyhedral cones associated with m-hemimetrics on n points, and, in particular, with m-hemimetrics coming from partitions of an n-set into m + 1blocks. We compute generators and facets of the cones for small values of m, n and study their skeleton graphs.

Key words: polyhedral cones; m-partitions; metrics, m-hemimetrics

1 Introduction

The notions of *m*-hemimetrics and *m*-partition hemimetrics are generalizations of the notions of metrics and cuts, which are well-known and central objects in Graph Theory, Combinatorial Optimization and, more generally, Discrete Mathematics.

Obviously the *m*-hemimetrics on an *n*-element set *E* form a cone. What are its facets and extreme rays? The calculations are rather complex due to the high dimension, $\binom{n}{m+1}$, of the cone even for small *n*; moreover, matrices can no longer be used.

Recall that a *metric* or a *metric space* is a pair (E, d) where E is a nonvoid set and $d: E^2 \longrightarrow \mathbb{R}_+$ (the set of nonnegative reals) satisfies for all $x, y, z \in E$: (d1) $d(x, y) = 0 \iff x = y$, (d2) d(x, y) = d(y, x) (symmetry), (d3) $d(x, y) \le d(x, z) + d(z, y)$ (the triangle inequality).

A basic example is (\mathbb{R}^2, d) , where d is the Euclidean distance of x and y; i.e., the length of the segment joining x and y. An immediate extension is (\mathbb{R}^3, d) , where d(x, y, z) is

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the area of the triangle with vertices x, y and z. This leads to the following definition (see [Me28,Bl53,Fr58,Gä63]). A 2-metric is a pair (E, d), where E is a nonempty set and $d: E^3 \longrightarrow \mathbb{R}_+$ satisfies for all $x, y, z, t \in E$ (d1') d(x, x, y) = 0, (d1'') $x \neq y \Longrightarrow d(x, y, u) > 0$ for some $u \in E$, (d2) d(x, y, z) is totally symmetric, (d3) $d(x, y, z) \leq d(t, y, z) + d(x, t, z) + d(x, y, t)$ (the tetrahedron inequality).

The axiom (d2) means that the value of d(x, y, z) is independent of the order of x, y and z. The axiom (d3) captures that fact that in \mathbb{R}^3 the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces. A 2-metric allows the introduction of several geometrical and topological concepts – e.g. the betweenness, convexity, line and neighborhood – which lead to interesting results.

For finite 2-metrics (the object of this study), for their polyhedral aspects and for applications, the axiom (d1') and (d1") seem to be too restrictive and so we drop them. The definition given below is formulated for an arbitrary positive integer m. A map $d: E^{m+1} \longrightarrow \mathbb{R}$ is totally symmetric if for all $x_1, ..., x_{m+1} \in E$ and every permutation π of $\{1, ..., m+1\}$

$$d(x_{\pi(1)}, ..., x_{\pi(m+1)}) = d(x_1, ..., x_{m+1}).$$

Definition. Let m > 0. An *m*-hemimetric is a pair (E, d), where $d : E^{m+1} \longrightarrow \mathbb{R}$ is totally symmetric and satisfies the simplex inequality: for all $x_1, ..., x_{m+2} \in E$

$$d(x_1, ..., x_{m+1}) \le \sum_{i=1}^{m+1} d(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{m+2}).$$
(1)

Call a *m*-hemimetric *d* nonnegative if *d* takes only nonnegative values. The notion of a *m*-hemimetric is new, but it is closely related to the notion of a *m*-metric, considered in about 200 references collected in [Gä90]. The study of *m*-hemimetrics is motivated (apart, of course, that it represents an extension of metrics) by applications in Statistics and Data Analysis (see [DeRo99] for some relevant references). Notice the following immediate:

Fact 1. If (E, d) and (E, d') are m-hemimetrics and $a, b \in \mathbb{R}_+$ then (E, ad + bd') is an *m*-hemimetric.

Here, as usual, for all $x_1, ..., x_{m+1} \in E$

$$(ad + bd')(x_1, ..., x_{m+1}) := ad(x_1, ..., x_{m+1}) + bd'(x_1, ..., x_{m+1}).$$

Given an (m+1)-partition S_1, \ldots, S_{m+1} of $V_n := \{1, 2, \ldots, n\}$, a partition *m*-hemimetric $\alpha(S_1, \ldots, S_{m+1})$ is defined by setting $\alpha(S_1, \ldots, S_{m+1})(i_1, \ldots, i_{m+1})$ is equal to 1 if for no $1 \le j < l \le m+1$ both i_j and i_l belong to the same S_k , and 0 otherwise. It is easy to

see that $\alpha(S_1, \ldots, S_{m+1})$ is a nonnegative *m*-hemimetric and for m = 1 it is the usual *cut* semimetric (see, for example, [DeLa97]).

For small values of n and m we consider the cone of all m-hemimetrics, the cone of all nonnegative m-hemimetrics and the cone, generated by all partition m-hemimetrics on V_n . Using computer search we list facets and generators for these cones and tables of their adjacencies and incidences. The different orbits were determined manually, using symmetries. We study two graphs, the 1-skeleton and the ridge graph, of these polyhedra: the number of their nodes and edges, their diameters, conditions of adjacency, inclusions among the graphs and their restrictions on some orbits of nodes. In fact, we would like to describe two graphs $G(C), G(C^*)$ for our three cones as fully as possible, but in the cases, when it is too difficult, we will give some partial information on adjacencies in those graphs. Especially we are interested in the diameters of the graphs, in a good criterion of adjacency, in their local graphs (i.e. in the subgraphs induced by all neighbors of a given vertex) and in their restrictions on some orbits. Finally, we compare obtained results with similar results for metric case (see [DeDe95,DDFu96,DeLa97]) and quasi-metric case (see [DePa99]). All computation was done using the programs cdd of [Fu95].

The following notation will be used below:

• the (m+1)-simplex inequality (1) and, in particular, for m = 2, the tetrahedron inequality

 $T_{ijk,l}: x_{ijl} + x_{ikl} + x_{jkl} - x_{ijk} \ge 0;$

- the nonnegativity inequality $N_{i_1,\dots,i_{m+1}}: x_{i_1,\dots,i_{m+1}} \ge 0;$
- the cone P_n^m of partition m-hemimetrics, generated by all (m+1)-partitions of V_n ;
- the cone NHM_n^m of nonnegative *m*-hemimetrics, defined by all $(n-m-1)\binom{n}{m+1}$ (m+1)-simplex inequalities and all $\binom{n}{m+1}$ nonnegativity inequalities on V_n ;
- the cone of all m-hemimetrics, HM_n^m , defined by all (m+1)-simplex inequalities on V_n .

Clearly, $P_n^m \subseteq NHM_n^m \subseteq HM_n^m$ and these 3 cones are of full dimension $\binom{n}{m+1}$ each. For m = 1 the last two cones coincide and the first two cones are the *cut cone* CUT_n and the *semimetric cone* MET_n , considered in detail in [DeLa97] and in the references listed there. The cone $P_n^1 = CUT_n$ has $\lfloor \frac{n}{2} \rfloor$ orbits of extreme rays (represented by the cuts $\alpha(1 \dots i, (i+1) \dots n)$).

To simplify the notation we keep n fixed and denote by E_k the family of all k-element subsets of V_n (k = 1, ..., n). Let d be a semimetric on the set V_n . Because of symmetry (d2) and since d(i, i) = 0 for all $i \in V_n$, we can view the semimetric d as a vector $(d_{i_1,i_2}) \in \mathbb{R}^{E_2}$. In the same way, we can view the m-hemimetric d on the set V_n as a vector $(d_{i_1,...,i_{m+1}}) \in \mathbb{R}^{E_{m+1}}$. In particular, each extreme ray of the cones, considered below, will be represented by an integer vector on the ray with relatively prime coordinates. We also will represent the facets of cones by such vectors. Any such vector $v = (v_{i_1,\ldots,i_{m+1}}) \in \mathbb{R}^{E_{m+1}}$ can be represented by the following vertexlabeled induced subgraph R(v) of the Johnson graph J(n, m + 1). The vertices of R(v)are all unordered (m+1)-tuples (i_1,\ldots,i_{m+1}) such that $(v_{i_1,\ldots,i_{m+1}})$ is not zero. This value will be the label of the vertex; we will omit the label when it is 1. Two vertices of the Johnson graph (and also of its induced subgraph R(v)) are adjacent if the corresponding (m+1)-tuples have m common elements.

For example, $R(\alpha(S_1, \ldots, S_{m+1}))$ is the complement to the Hamming graph $H(|S_1|, \ldots, |S_{m+1}|)$, i.e. the direct (Cartesian) product of the cliques $K_{|S_i|}$, $1 \le i \le m+1$. Another example: the graph R of the vector defining a nonnegativity facet is a vertex and, for a (m+1)simplex facet, it is the complete graph K_{m+2} with one vertex labeled -1.

2 Partition *m*-hemimetrics and related polyhedra

Recall that for a partition S_1, S_2 of V_n the cut semimetric $\alpha(S_1, S_2)$ satisfies $\alpha(S_1, S_2)_{ij} = 1$ if $\{i, j\} \cap S_1$ is a singleton and $\alpha(S_1, S_2)_{ij} = 0$ otherwise. We extend it as follows. Let $q \geq 2$ be an integer and let S_1, \ldots, S_q be pairwise disjoint nonvoid subsets of V_n , forming a partition of V_n . The multicut semimetric $\delta(S_1, \ldots, S_q)$ is the vector in \mathbb{R}^{E_2} , defined by $\delta(S_1, \ldots, S_q)_{ij} = 0$, if $i, j \in S_h$ for some $h, 1 \leq h \leq q$, and $\delta(S_1, \ldots, S_q)_{ij} = 1$, otherwise.

The connection between $\delta(S_1, \ldots, S_q)$ and $\alpha(S_1, \ldots, S_q)$ from Section 1 is given by

$$\alpha(S_1,\ldots,S_q)(i_1,\ldots,i_q) = \prod_{1 \le s < t \le q} \delta(S_1,\ldots,S_q)(i_s,i_t) =$$

$$\lfloor \frac{\sum_{1 \le s \le t \le q} \delta(S_1, \dots, S_q)(i_s, i_t)}{\binom{q}{2}} \rfloor; \text{ compare it with the half-perimeter } m \text{-semimetric from [DeRo99]}.$$

The cone generated by all multicut semimetrics $\delta(S_1, \ldots, S_q)$ $(q \ge 2)$ on V_n , is called *the* multicut cone and denoted by $MCUT_n$; it coincides with CUT_n (see [DeLa97], Proposition 4.2.9). The convex hull of the cut semimetrics (multicut semimetrics) on V_n , is called the *cut polytope* (*multicut polytope*) and is denoted by CUT_n^{\Box} ($MCUT_n^{\Box}$); the two polytopes not coincide.

3 Facets, extreme rays and their orbits in polyhedra

We recall some terminology. Let C be a polyhedral cone in \mathbb{R}^n . Given $v \in \mathbb{R}^n$, the inequality $v^T x \leq 0$ is said to be valid for C, if it holds for all $x \in C$. Then the set $\{x \in C | v^T x = 0\}$ is called the *face of* C, *induced by the valid inequality* $v^T x \leq 0$. A face of dimension dim(C) - 1 is called a *facet* of C; a face of dimension 1 is called an *extreme* ray of C. A face of dimension dim(C) - 2 is called a *ridge*. Two vertices x, y of C are said to be *adjacent*, if they generate a face of dimension 2 of C. Two facets of C are said to be *adjacent*, if their intersection has dimension $\dim(C) - 2$. The 1-skeleton graph of C is the graph G(C) whose nodes are the extreme rays of C and whose edges are the pairs of adjacent nodes. Denote by C^* the dual cone of C. The *ridge* graph of C is the graph whose nodes are the facets of C and with an edge between two facets if they are adjacent on C. So, the ridge graph of a cone C is the 1-skeleton $G(C^*)$ of its dual cone.

A mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a *symmetry* of a cone C (or a polytope P), if it is an isometry, satisfying f(C) = C (or f(P) = P). (An *isometry* of \mathbb{R}^n is a linear mapping preserving the Euclidean distance.) Given a face F, the *orbit* $\Omega(F)$ of F consists of all faces, that can be obtained from F by the group of all symmetries of C.

Clearly, all the faces of CUT_n and CUT_n^{\Box} are preserved by any permutation of V_n .

For m > 1 all orbits of faces of *m*-hemimetric cones P_n^m , NHM_n^m , HM_n^m on V_n are also preserved under any permutation of the set $V_n = \{1, \ldots, n\}$. We conjecture that the symmetry group consists only of permutations of V_n , i.e. it is the group Sym(n) of all permutations on V_n (see Theorem 3.3 in [DGLu91] stating that the symmetry group of a truncated multicut polytope is Sym(n)).

4 The case of n = m + 2

The minimal n for which the three cones are nontrivial is m + 2; the dimension of the cones is also m + 2 for n = m + 2.

First, we present a complete linear description for case (m, n) = (2, 4).

It turns out that $P_4^2 = NHM_4^2$. This cone has 6 extreme rays (all in the same orbit under Sym(4)): $\alpha(S_1, S_2, S_3)$ for the 3-partitions

(1, 2, 34), (1, 3, 24), (1, 4, 23), (2, 3, 14), (2, 4, 13), (3, 4, 12).

There are 8 facets, which form 2 orbits: the orbit F_1 of all 4 tetrahedron facets and the orbit F_2 of all 4 nonnegativity facets.

The edge graph G(C) is $K_6 - 3K_2$ (the octahedron); the 3 pairs of nonadjacent rays are of the form $\alpha(a, b, cd), \alpha(c, d, ab)$. Each extreme ray (say, $\alpha(1, 2, 34)$) is incident to 2 tetrahedron and to 2 nonnegativity facets (namely, to $T_{123,4}, T_{124,3}$ and N_{134}, N_{234}).

The ridge graph $G(C^*)$ is the cube. Adjacencies of facets of NHM_4^2 are shown in Table 1. For each orbit a representative and the number of adjacent facets from other orbits are given, as well as the total number of adjacent ones, the number of incident extreme rays

Table 1 The adjacencies of facets in the cone NHM_4^2

Orbit	Representative	F_1	F_2	Adj.	Inc.	$ F_i $
F_1	$T_{123,4}$	0	3	3	3	4
F_2	N_{123}	3	0	3	3	4

and the cardinality of orbits.

More precisely, for the ridge graph of NHM_4^2 it holds:

- (i) The tetrahedron facet $T_{ijk,l}$ is adjacent only to the facets $N_{ijl}, N_{ikl}, N_{jkl}$;
- (ii) The nonnegativity facet N_{ijk} is adjacent only to the facets $T_{ijl,k}, T_{ikl,j}, T_{jkl,i}$.

The cone HM_{m+2}^m is a simplex (m+2)-dimensional cone; so $G(C) = G(C^*) = K_{m+2}$. Its facets are all (1, -1)-valued (m+2)-vectors with only one -1, its generators are all (1-m, 1)-valued (m+2)-vectors with only one 1-m. Notice that $P_3^1 = HM_3^1 = CUT_3 = MET_3$.

In general, $P_{m+2}^m = NHM_{m+2}^m$ for any $m \ge 2$. This cone has $\binom{m+2}{2}$ extreme rays, all in the same orbit, represented by $\alpha(12, 3, \ldots, m+2)$, i.e. by any vector of length m+2, consisting of two ones and m zeros. The skeleton of P_{m+2}^m is the Johnson graph J(m+2, 2), called also the *triangular* graph T(m+2), which is is the line graph $L(K_{m+2})$. It is also the skeleton of the (m+1)-polytope (called $ambo - \alpha_{m+1}$), obtained from the (m+1)-simplex as the convex hull of the mid-points of all its edges; e.g. T(4) is the skeleton of the octahedron, T(5) is the complement of the Petersen graph.

The cone P_{m+2}^m has two orbits, F_1 and F_2 , of facets, containing m + 2 facets each and represented by the (m + 1)-simplex facet $T_{1...(m+1),(m+2)}$ and by the nonnegativity facet $N_{1...(m+1)}$. The orbit F_1 consists of simplex cones, i.e. facets from this orbit are incident to m + 1 linearly independent extreme rays. Any nonnegativity inequality N defines the cone $P_{m+1}^{m-1} = NHM_{m+1}^{m-1}$, i.e. it becomes equality on this smaller cone. So, N is non-facet only for m = 1 and it is a simplex cone only for m = 2; in general, N is incident to $\binom{m+1}{2}$ extreme rays. The ridge graph is $\overline{K_{m+2}}$ on F_1 ; on F_2 it is $\overline{K_4}$ for m = 2 and K_{m+2} for $m \geq 3$. Finally, the m + 2 pairs $(T_{i,i}, N_i)$ (of (m + 1)-simplex and nonnegativity facets) are the only non-edges for pairs of facets from different orbits.

5 Small 2-hemimetrics

5.1 The case of 5 points

We present here the complete linear description of P_5^2 , NHM_5^2 and HM_5^2 . The cone P_5^2 has 25 extreme rays, which form 2 orbits with representatives $\alpha(1, 2, 345)$ (orbit O_1) and $\alpha(1, 23, 45)$ (orbit O_2). The skeleton and the ridge graph of P_5^2 has 270 and 1185 edges, respectively. The cone P_5^2 has 120 facets divided into 4 orbits, induced by the 20 tetrahedron inequalities (orbit F_1), the 10 nonnegativity inequalities (orbit F_2), the 60 inequalities (orbit F_3), represented by

$$A: 2x_{123} - (x_{124} + x_{135}) + (x_{134} + x_{125} + x_{245} + x_{345}) \ge 0$$

and the 30 inequalities (orbit F_4), represented by

$$B: 2(x_{123} + x_{145} + x_{245} - x_{345}) + (x_{134} + x_{135} + x_{234} + x_{235} - x_{124} - x_{125}) \ge 0$$

The above two inequalities are the 2-hemimetric analogs of the following 5-gonal inequality (the simplest inequality, different from the triangle inequality), appearing in the cone CUT_n for $n \ge 5$:

$$(x_{13} + x_{14} + x_{15} + x_{23} + x_{24} + x_{25}) - (x_{12} + x_{34} + x_{35} + x_{45}) \ge 0$$

This facet and *B* have both the Petersen graph as their \overline{R} (i.e. the complement of their graph *R*). Clearly, the graphs *R* for partition 2-hemimetrics $\alpha(1, 2, 34)$, $\alpha(1, 2, 345)$, $\alpha(1, 23, 45)$ are the cycles C_2, C_3, C_4 .

The graphs $\overline{R(A)}$, $\overline{R(B)}$ are given on the Figure 1.

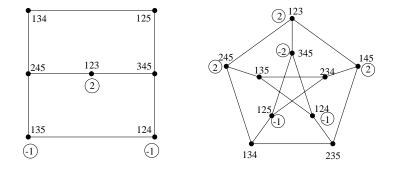


Figure 1: $\overline{R(A)}$, $\overline{R(B)}$ in the cone P_5^2

Table 2 The adjacencies of extreme rays in the cone P_5^2

Orbit	Representative	O_1	O_2	Adj.	Inc.	$ O_i $
O_1	$\alpha(1, 2, 345)$	9	12	21	54	10
O_2	$\alpha(1,23,45)$	8	14	22	54	15

Table 3

The adjacencies of facets in the cone P_5^2

Orbit	Representative	F_1	F_2	F_3	F_4	Adj.	Inc.	$ F_i $
F_1	$T_{123,4}$	16	9	18	6	49	16	20
F_2	N_{123}	18	3	18	3	42	16	10
F_3	A	6	3	4	2	15	10	60
F_4	В	4	1	4	0	9	9	30

Table 4

The adjacencies of extreme rays in the cone NHM_5^2

Orbit	Representative	O_1	O_2	O_3	Adj.	Inc.	$ O_i $
O_1	$\alpha(1,2,345)$	9	12	6	27	21	10
O_2	$\alpha(1,23,45)$	8	6	8	22	18	15
O_3	v_3	5	10	5	20	15	12

Table 5

The adjacencies of facets in the cone NHM_5^2

Orbit	Representative	F_1	F_2	Adj.	Inc.	$ F_i $
F_1	$T_{123,4}$	16	9	25	22	20
F_2	N_{123}	18	3	21	22	10

The facets from the orbit F_4 are simplex-cones, i.e. the extreme rays on them are linearly independent. Among the 9 neighbors of B, 4 are from the orbit F_1 , 4 are from the orbit F_3 and exactly one (actually, N_{123}) from the orbit F_2 . The local graph of a facet from F_4 (i.e. the subgraph of the ridge graph of P_5^2 , induced by all neighbors of B) is $K_9 - C_4$. In fact, all nonadjacencies in this local graph are the four edges of the 4-cycle of the 4 neighbors of B from the orbit F_3 .

Facets from orbits F_1, F_2, F_3, F_4 are incident, respectively, to 7,9; 7,9; 4,6; 3,6 extreme rays from orbits O_1, O_2 of P_5^2 .

The skeleton and the ridge graph of NHM_5^2 have 420 and 355 edges, respectively. The adjacencies of 37 extreme rays and of 30 facets of this cone are given in Tables 4, 5. The extreme rays are divided into 3 orbits O_1, O_2, O_3 , represented by (0,1)-valued vectors v_1, v_2, v_3 below; their *R*-graphs are C_3, C_4, C_5 , respectively.

The adjacencies of extreme rays in the cone HM_5^2										
Orbit	Representative	O_1	O_2	O_3	O_4	O_5	O_6	Adj.	Inc.	$ O_i $
O_1	$v_1 = \alpha(1, 2, 345)$	6	12	6	6	9	9	48	14	10
O_2	$v_2 = \alpha(1, 23, 45)$	8	2	4	2	4	8	28	12	15
O_3	v_3	5	5	0	5	5	5	25	10	12
O_4	v_4	6	3	6	0	3	6	24	12	10
O_5	v_5	6	4	4	2	0	4	20	12	15
O_6	v_6	3	4	2	2	2	0	13	10	30

Table 6 The adjacencies of extreme rays in the cone HM_5^2

Each facet (from both orbits) of NHM_5^2 is incident to 7,9,6 extreme rays from orbits O_1, O_2, O_3 , respectively. Each (tetrahedron) facet of HM_5^2 is incident to 7,9,6,6,9,15 extreme rays from orbits O_1, \ldots, O_6 .

For any cone, let I_{O_i,F_j} and I_{F_j,O_i} denote the number of facets from the orbit F_j , incident to an extreme ray of the orbit O_i , and, respectively, the number of extreme rays from O_i , incident to a facet from F_j . Clearly, $|O_i|I_{O_i,F_j} = |F_j|I_{F_j,O_i}$.

The cone HM_5^2 has 92 extreme rays divided into 6 orbits. Below we give some representatives v_1, \ldots, v_6 of those orbits O_1, \ldots, O_6 . The first two represent both orbits of P_5^2 , the first 3 represent the 3 orbits of NHM_5^2 .

 $x = (x_{123}, x_{124}, x_{125}, x_{134}, x_{135}, x_{145}, x_{234}, x_{235}, x_{245}, x_{345}):$

 $\begin{aligned} v_1 &= (1, 1, 1, 0, 0, 0, 0, 0, 0, 0); \\ v_2 &= (0, 1, 1, 1, 1, 0, 0, 0, 0, 0); \\ v_3 &= (1, 0, 1, 0, 0, 1, 1, 0, 0, 1); \\ v_4 &= (1, 1, 1, -1, 0, 0, 1, 0, 1, 1); \\ v_5 &= (1, 1, 1, -1, -1, 1, 1, 1, 1, 1); \\ v_6 &= (1, 0, 1, 0, 1, -1, 1, 1, 2, 1). \end{aligned}$

Proposition 1 The diameters of the skeleton graphs of P_5^2 and of NHM_5^2 are 2.

In fact, each of the orbits O_1, O_2 of P_5^2 is a dominating clique. There is only one type of a non-edge, represented by $\alpha(1, 23, 45), \alpha(2, 3, 145))$, but $\alpha(1, 3, 245)$ is one of common neighbors. The complement of the skeleton of P_5^2 turns out to be the Petersen graph with a new vertex (corresponding to a member of the orbit O_2) on each of 15 edges. The result

Table 7 The adjacencies of facets in the cone NHM_6^2

Orbit	Representative	F_1	F_2	Adj.	Inc.	$ F_i $
F_1	$T_{123,4}$	56	19	75	4001	60
F_2	N_{123}	57	10	67	3939	20

for NHM_5^2 comes also by finding out a common neighbor to each possible non-edge.

Proposition 2 For the ridge graphs of NHM_5^2 and HM_5^2 it holds:

- (i) The diameter of the ridge graph of NHM_5^2 is 2;
- (ii) Its restriction on the orbits F_1 and F_2 is $K_{4,4,4,4}$ and the Petersen graph, respectively;
- (iii) The ridge graph of HM_5^2 is $K_{4,4,4,4,4}$ (of diameter 2).

5.2 The case of 6 points

 NHM_6^2 has exactly 12492 extreme rays, with (*adjacency, incidence*) pairs being, respectively, (2278,64), (1321,56), (1030,40), (818,48), (731,48), (358,40), (270,36), (93,28), (66,28), (51,28), (47,28), (46,39), (37,31), (32,28), (30,27), (29,26), (27,23), (26,24), (26,23), (25,25), (23,22), (22,21), (21,21).

Three of the above pairs (1st, 2nd and 4th) are realized by orbits (say, 0_1 , O_2 and O_4), which are represented by 3-partition 2-hemimetrics $\alpha(1, 2, 3456)$, $\alpha(1, 23, 456)$, $\alpha(12, 34, 56)$ and have size 15, 60, 15, respectively. The graphs R of members of orbits O_1 , O_2 and O_4 are K_4 , $K_6 - C_6 = K_3 \times K_2$ (the skeletons of the tetrahedron and 3-prism) and the skeleton of the cube. Three other orbits consist also of (0,1)-valued extreme rays: O_3 (with R being the Petersen graph), O_5 (with R-graph being the skeleton of the simple polyhedron with p-vector $p = (p_3 = 2, p_4 = 2, p_5 = 2)$) and O_7 with graph R (non-planar, non-regular), given on Figure 2, together with one for O_5 . The extreme rays of the remaining orbits are (0,1,2)-valued and (0,1,2,3)-valued vectors.

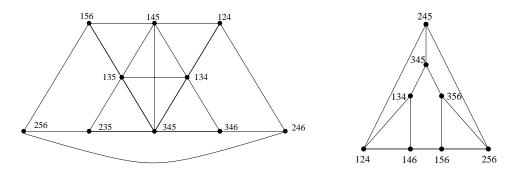


Figure 2 : The graphs R of extreme rays from orbits O_7, O_5 of the cone NHM_6^2

The cone P_6^2 has more than 950.000 facets (computer stopped, by lack of memory, after 72, out of 90, iterations). Here are two examples of a (0, 1, -1)-valued facets of P_6^2 ; see also Figure 3 (for the facet W).

 $W: (-x_{145} + x_{146} + x_{136} + x_{123} + x_{125}) + (x_{245} + x_{234} + x_{346} + x_{356} + x_{256}) - (x_{235} + x_{236}) \ge 0.$

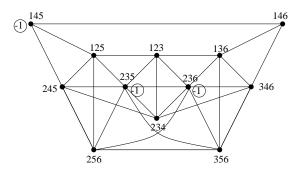


Figure 3 : R(W) in the cone P_6^2

$$Z: \sum x_{ijk} - (x_{124} + x_{125} + x_{145}) - (x_{234} + x_{235} + x_{345}) - 2(x_{146} + x_{156} + x_{456}) - 2x_{236} \ge 0.$$

Remark that the triples with coefficients zero, in W and Z, form the skeleton of 1- and 2-truncated tetrahedron, respectively; the triples with coefficient -1 form $K_3 + K_1$ and $K_2 + K_1$, respectively.

6 Small 3-hemimetrics

The cone NHM_6^3 has 287 extreme rays divided into 5 orbits. Below we give representatives u_1, \ldots, u_5 of the orbits O_1, \ldots, O_5 . These vectors are indexed by 4-subsets of the set $\{1, \ldots, 6\}$; the 4-subsets are given as the complements of 2-subsets. The first four are (0, 1)-valued; their *R*-graphs (in the Johnson graph J(6, 4) of all 4-tuples) are the cycles C_3, C_4, C_5, C_6 , respectively. The first two are partition 3-hemimetrics; they represent both orbits of P_6^3 . The graphs $R(u_4)$ and $R(u_5)$ are on Figure 4.

$$x = (x_{\overline{12}}, x_{\overline{13}}, x_{\overline{14}}, x_{\overline{15}}, x_{\overline{16}}, x_{\overline{23}}, x_{\overline{24}}, x_{\overline{25}}, x_{\overline{26}}, x_{\overline{34}}, x_{\overline{35}}, x_{\overline{36}}, x_{\overline{45}}, x_{\overline{46}}, x_{\overline{56}}):$$

 $u_1 = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0);$

 $u_2 = (0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0);$

 $u_3 = (0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1);$

 $u_4 = (0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0);$

 $u_5 = (0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 2, 0);$

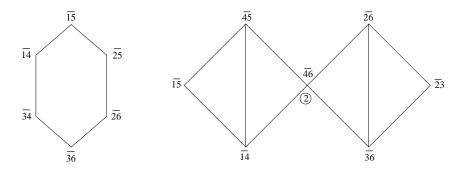


Figure 4 : $R(u_4), R(u_5)$ in the cone NHM_6^3

The cone P_6^3 has 4065 facets divided into at least 11 orbits. Below we give representatives f_1, \ldots, f_{11} of the orbits F_1, \ldots, F_{11} . Their (*adjacency, incidence*) pairs are, respectively, (1526,49), (703,41), (100,23), (37,19), (31,18), (30,18), (23,17), (23,15), (22,18), (18,16), (14,14). The facets f_1, f_2 are nonnegativity and 4-simplex facets; f_{11} is a simplex cone. The *R*-graphs of the facets f_3 and f_4 are on Figure 5.

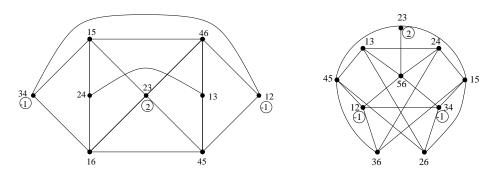


Figure 5 : $\overline{R(f_3)}$, $\overline{R(f_4)}$ in the cone P_6^3

$$\begin{split} f_1 &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0); \\ f_2 &= (0, 1, 0, 0, 0, 1, 0, 0, 0, -1, 1, 1, 0, 0, 0); \\ f_3 &= (-1, 1, 0, 1, 1, 2, 1, 0, 0, -1, 0, 0, 1, 1, 0); \\ f_4 &= (-1, 1, 0, 1, 0, 2, 1, 0, 1, -1, 0, 1, 1, 0, 1); \\ f_5 &= (-1, 1, 1, 2, 1, 2, 2, -1, 0, -2, 1, 0, 1, 2, 1); \\ f_6 &= (-1, 1, 0, 2, 2, 2, 1, -1, 1, -1, 1, -1, 2, 2, 0); \\ f_7 &= (-1, 1, 1, 3, 2, 2, 2, -2, 1, -2, 2, -1, 2, 3, 1); \\ f_8 &= (1, -1, 3, 1, 4, 2, 2, -2, 1, -2, 2, 3, 2, -1, 1); \end{split}$$

The adj	The adjacencies of extreme rays in the cone NHM_6^3								
Orbit	Representative	O_1	O_2	O_3	O_4	O_5	Adj.	Inc.	$ O_i $
O_1	$u_1 = \alpha(1, 2, 3, 456)$	19	36	36	18	27	136	21, 12	20
O_2	$u_2 = \alpha(1, 2, 34, 56)$	16	18	24	20	16	94	18, 11	45
O_3	u_3	10	15	20	15	10	70	15, 10	72
O_4	u_4	6	15	18	9	6	54	12, 9	60
O_5	u_5	6	8	8	4	0	26	10, 8	90

Table 8 The adjacencies of extreme rays in the cone NHM_{ϵ}^3

 $f_9 = (-1, 1, 1, 2, 2, 2, 2, -1, -1, -2, 1, 1, 1, 1, 2);$

 $f_{10} = (-1, 1, 1, 1, 2, 1, 1, 1, -1, -1, -1, 1, 2, 1, 1);$

 $f_{11} = (-1, 1, 1, 2, 0, 2, 2, -1, 1, -2, 1, 1, 1, 1, 2).$

Proposition 3 The diameter of the skeleton graph of P_6^3 is 2. Moreover:

(i) $G(O_1) = K_{20}, G(O_2) = K_{45} - 15K_3;$

(ii) all non-edges are represented by $\alpha(12, 34, 5, 6)$ that are nonadjacent to

 $\alpha(12, 3, 4, 56), \alpha(1, 2, 34, 56)$ (from the same orbit O_2) and to

 $\alpha(125, 3, 4, 6), \alpha(126, 3, 4, 5), \alpha(345, 1, 2, 6), \alpha(346, 1, 2, 5).$

In fact, both non-neighbors of $\alpha(12, 34, 5, 6)$ are in O_2 . For both types of non-edges - $\alpha(12, 34, 5, 6)$ with $\alpha(12, 3, 4, 56)$ and $\alpha(125, 3, 4, 6)$ - the ray $\alpha(13, 24, 5, 6)$ is a common neighbor. Also, all 9 non-neighbors of a ray from O_1 , form K_9 in the skeleton graph.

Notice that the skeleton of P_6^3 is not an induced subgraph of the skeleton of NHM_6^3 ; the only difference is in their restriction $G(O_2)$ to the orbit of rays, represented by u_2 . One can check that all neighbors of a partition hemimetric $\alpha(a_1, a_2, b_1b_2, c_1c_2)$ from the same orbit O_2 of NHM_6^3 are the 10 rays obtained by a transposition (xy) and the 8 rays obtained by a product $(a_1b_i)(a_2c_j)$ or $(a_1c_i)(a_2b_j)$ of two transpositions. But in the skeleton of P_6^3 , the ray $\alpha(a_1, a_2, b_1b_2, c_1c_2)$ is adjacent to all other members of O_2 , except for the two rays, obtained from it by $(a_1b_1)(a_2b_2)$ or $(a_1c_1)(a_2c_2)$. The complement of the graph, induced by all 18 neighbors of the ray $\alpha(a_1, a_2, b_1b_2, c_1c_2)$ from the same orbit O_2 of NHM_6^3 , is $C_4 + C_4$ on 8 rays, obtained by a product $(a_1b_i)(a_2c_j)$ or (a_ic_j) , and it is $\overline{K_2}$ on two rays obtained by (b_ic_j) .

The skeletons of P_6^3 and NHM_6^3 both contain a dominating clique O_1 ; so their diameters are 2 or 3. In order to see closer the skeleton of NHM_6^3 , we now describe the local graph, denoted by H, of the ray u_5 . All 26 neighbors are in orbits O_1, O_2, O_3, O_4 only. It will be easier to describe \overline{H} . The restrictions of \overline{H} on them are $\overline{K_6}$, C_8 , the skeleton of the cube

Table 9 The adjacencies of extreme rays in the cone P_6^3

Orbit	Representative	O_1	O_2	Adj.	Inc.	$ O_i $
O_1	$\alpha(1,2,3,456)$	19	36	55	1113	20
O_2	$\alpha(1,2,34,56)$	16	42	58	993	45

Table 10

The adjacencies of facets in the cone NHM_6^3

Orbit	Representative	F_1	F_2	Adj.	Inc.	$ F_i $
F_1	$T_{1234,5}$	25	14	39	131	30
F_2	N_{1234}	28	14	42	181	15

and $2K_2$, respectively. Two vertices from O_1 (say 15 and 16) are isolated; so the diameter of H is 2. Here we denote by ij the j-th member of the orbit O_i in H. All edges of \overline{H} (without isolated vertices 15 and 16) are presented on Figure 6. On the right picture the members of O_1 are excluded while on the left one the members of O_2 are excluded. \overline{H} does not contain cross-edges among orbits O_1 and O_2 .

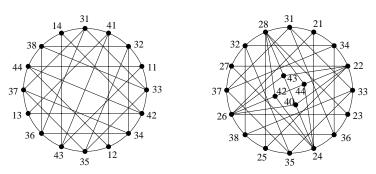


Figure 6 : A presentation of the local graph of a ray of the orbit O_5 of the cone NHM_6^3

Proposition 4 The ridge graph of NHM_6^3 has diameter 2. Moreover:

(i) any 4-simplex facet $T_{ijkl,m}$ is adjacent to all but 5 facets: N_{ijkl} and all 4 other 4-simplex facets with the same support;

(ii) the restrictions of the ridge graph to the orbits F_1 and F_2 are

 $K_{5,5,5,5,5,5}$ and K_{15} , respectively.

7 Small 4-hemimetrics

The cone NHM_7^4 has 3692 extreme rays divided into 8 orbits. We give below representatives w_1, \ldots, w_8 of their orbits O_1, \ldots, O_8 . These vectors are indexed by 5-subsets of the set

Table 11 The adjacencies of facets in the cone NHM_7^4

Orbit	Representative	F_1	F_2	Adj.	Inc.	$ F_i $
F_1	$T_{12345,6}$	36	20	56	1302	42
F_2	N_{12345}	40	20	60	2437	21

 $\{1, \ldots, 7\}$; the 5-subsets are given as the complements of 2-subsets. The (*adjacency, incidence*) pairs of those rays are, respectively, (985,48), (535,43), (315,38), (192,33), (126,28), (67,30), (43,25), (42,25). The first five vectors are (0, 1)-valued; their graphs R are C_3, C_4, C_5, C_6, C_7 , respectively. The first two are partition 4-hemimetrics; they represent both orbits of P_7^4 . The vectors $w_i, 1 \le i \le 4$, and w_6 have same R-graphs as the members of orbits $O_i, 1 \le i \le 5$, of NHM_6^3 , respectively; so, the graphs of Figure 5 represent also w_4 and w_6 . The graphs $R(w_7)$ and $R(w_8)$ are on Figure 7.

 $(\overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}, \overline{23}, \overline{24}, \overline{25}, \overline{26}, \overline{27}, \overline{34}, \overline{35}, \overline{36}, \overline{37}, \overline{45}, \overline{46}, \overline{47}, \overline{56}, \overline{57}, \overline{67}):$

 $w_2 = (0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0);$

 $w_3 = (0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0);$

 $w_4 = (0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0);$

 $w_5 = (0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0);$

 $w_6 = (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 2, 1, 0, 0, 1, 1, 0, 0);$

 $w_7 = (0, 0, 0, 2, 2, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0, 1);$

 $w_8 = (0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 2, 1, 1, 0, 0, 0, 1, 0, 0, 0).$

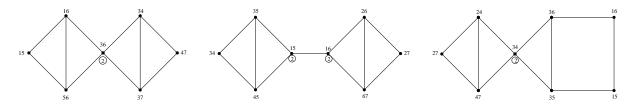


Figure 7 : $R(W_6), R(W_7), R(W_8)$ in the cone NHM_7^4

It is easy to check that the ridge graph of NHM_7^4 is $K_{6,6,6,6,6,6,6}$ on F_1 and K_{21} on F_2 . All non-edges among F_1, F_2 are of the form $T_{i_1...i_5,i_6}$ and $N_{i_1...i_5}$.

8 Comparison of the small cones

Now we compare some semimetric and m-hemimetric cones on n points for small n. The triangle inequalities suffice to describe the cut cones for $n \leq 4$, but $CUT_n \subset MET_n$ (strictly) for $n \geq 5$. The complete description of all the facets of the cut cone CUT_n is known for $n \leq 8$, the complete description of the semimetric cone MET_n is known for $n \leq 7$ (see, for example, the linear description of MET_7 in [Gr92]). Here the "combinatorial explosion" starts from n = 8. The number of orbits of facets and of extreme rays of those and other cones, when it is known, is given in Table 12.

In fact, $P_n^2 = NHM_n^2$ holds only for the smallest value n = 4. For n = 4, 5 we computed all facets, extreme rays and their adjacencies and incidences for three cones P_n^2 , NHM_n^2 , HM_n^2 . For 2-hemimetrics the "combinatorial explosion" (in terms of the amount of computation and memory) starts already for the cone P_6^2 .

In the Table 12 we compare the small 2-hemimetric cones P_n^2 , NHM_n^2 with the 1-hemimetric cones CUT_n , MET_n and their generalization in another direction: the cones $OMCUT_n$, $QMET_n$. Last two cones consist of all quasi-semimetrics on V_n and of those obtained from oriented multicuts; see [DePa99] for the notions and results for them given in the Table 12. The cones NHM_n^2 and $QMET_n$ have, besides of generalizations of the usual triangle inequality, only nonnegativity facets. In the Table 12, columns 3 and 4 give the number of extreme rays and facets, respectively; in parenthesis are given the numbers of their orbits. In column 5 are given the diameters of the skeleton and the ridge graphs of the cone specified in the row. In the Table 12, the number of orbits of extreme rays and the diameter for cones $QMET_5$, NHM_6^2 , P_6^2 , P_7^4 and dual P_6^3 , $OMCUT_5$ are taken from recent work [DuDe01], as well as the exact value of the diameter for NHM_6^3 , NHM_7^4 and for the duals of P_5^2 , Cut_7 .

Incidences (to the extreme rays) of facets $T_{ijk,l}$ and N_{ijk} on the cones $P_4^2 = NHM_4^2$, P_5^2 , NHM_5^2 amount to 3, 14 and 22, respectively, but they are different (4001 and 3939) on HM_6^2 . Incidences of similar facets $T_{ij,k}$ (oriented triangular inequality, i.e. $d(x, y) \leq d(x, z) + d(z, y)$ for a quasimetric d), N_{ij} (nonnegativity inequality) are equal (to 7, 43) on cones $OMCUT_3 = QMET_3$, $OMCUT_4$, but they are different (78 and 80) on $QMET_4$.

For n = 4, 5 we observe that the ridge graphs of HM_n^2 and NHM_n^2 are induced subgraphs of the ridge graphs of NHM_n^2 and P_n^2 , respectively. The similar property does *not* hold for the 1-skeletons of those cones. For example, any extreme ray of the orbit O_2 is adjacent to 14,6,2 members of the same orbit in the cones P_n^2, NHM_n^2, HM_n^2 , respectively. Also, the ridge graph of $QMET_4$ is an induced subgraph of the ridge graph of $OMCUT_4$, but the skeleton of $OMCUT_4$ is not an induced subgraph of the skeleton of $QMET_4$ (see [DePa99]). On the other hand, the ridge graph of MET_n and the skeleton of CUT_n (for any n) have diameters 2 and 1, respectively, and those graphs are induced subgraphs of the ridge graph of CUT_n and of the skeleton of MET_n , respectively (see Lemma 2.1 and Theorem 3.5 in [DeDe94])

Some parameters of	t cones for	small n		
cone	dimension	ext. rays (orbits)	facets (orbits)	diameters
$P_{m+2}^m = NHM_{m+2}^m$	m+2	$\binom{m+2}{2}$ (1)	2m + 4 (2)	2; 2
$m \ge 3$				
$CUT_3 = MET_3$	3	3(1)	3(1)	1; 1
$P_4^2 = NHM_4^2$	4	6(1)	8(2)	2; 3
$CUT_4 = MET_4$	6	7(2)	12(1)	1; 2
$OMCUT_3 = QMET_3$	6	12(2)	12(2)	2; 2
CUT_5	10	15(2)	40(2)	1; 2
MET_5	10	25(3)	30(1)	2; 2
P_{5}^{2}	10	25(2)	120(4)	2; 3
NHM_5^2	10	37(3)	30(2)	2; 2
$OMCUT_4$	12	74(5)	72(4)	2; 2
$QMET_4$	12	164(10)	36(2)	3; 2
CUT_6	15	31(3)	210(4)	1; 3
MET_6	15	296(7)	60(1)	2; 2
P_{6}^{3}	15	65(2)	4065(16)	2; 3
NHM_6^3	15	287(5)	45(2)	3; 2
P_{6}^{2}	20	90(3)	$\geq 2095154 (\geq 3086)$	2; ?
NHM_6^2	20	12492(41)	80(2)	3; 2
$OMCUT_5$	20	540(10)	35320(194)	2; 3
$QMET_5$	20	43590(229)	80(2)	3; 2
P_{7}^{4}	21	140(2)	474390(153)	2; 3
NHM_7^4	21	3692(8)	63(2)	3; 2
CUT_7	21	63(3)	38780(36)	1; 3
MET_7	21	55226(46)	105(1)	3; 2
CUT_8	28	127(4)	$\geq 49604520 (\geq 2169)$	1; ?
P_{8}^{5}	28	266(2)	$\geq 322416108 (\geq 8792)$?; ?
NHM_8^5	28	55898(13)	84(2)	3; 2

Table 12 Some parameters of cones for small n

9 Conjectures for general m, n

Conjecture 5 The two partition *m*-hemimetrics $\alpha(S_1, \ldots, S_{m+1})$ and $\alpha(T_1, \ldots, T_{m+1})$ on V_n are nonadjacent in the skeleton of P_n^m if and only if there exist six different subsets S_i, S_j, S_k and $T_{i'}, T_{j'}, T_{k'}$, such that $S_i \cup S_j = T_{k'}$ and $S_k = T_{i'} \cup T_{j'}$.

The conjecture holds for m = 1: all cut semimetrics are adjacent. It holds for n - m = 2: we have the graph J(m + 2, 2). It also holds for (m, n) = (2, 5) and (3, 6).

Conjecture 6 The ridge graphs of HM_n^m and of NHM_n^m are induced subgraphs of the ridge graphs of NHM_n^m and P_n^m , respectively.

Recall that the ridge graph of NHM_n^m has two orbits of vertices: F_1, F_2 , consisting of $(n-m-1)\binom{n}{m+1}$ simplex and $\binom{n}{m+1}$ nonnegativity inequalities.

Conjecture 7 The ridge graph NHM_n^m satisfies:

(i) The (m + 1)-simplex facet $T_{i_1...i_{m+1},i_{m+2}}$ is adjacent to all other facets, except the following m + 2 facets:

all other (m+1)-simplex facets with the same support and $N_{i_1...i_{m+1}}$;

(*ii*) $G(F_2) = \overline{J(n,3)}$ for m = 2 and $G(F_2) = K_{\binom{n}{m+1}}$ for $m \ge 3$.

Clearly, (i) implies that the restriction of the ridge graph on F_1 is $G(F_1) = K_{m+2,...,m+2}$. It is easy to see that Conjecture 3 would imply that the diameter of the ridge graph of NHM_n^m is 2 (it was proved in [DeDe94] that the diameter of the ridge graph of $NHM_n^1 = MET_n$ is 2). In fact, to see it for m = 2 consider all 3 types of pairs of nonadjacent vertices:

(i) let $x, y \in F_1$ have the same support, say, 1234. Suppose that $x_{124} = y_{124} = -1$. Then N_{123} is a common neighbor for x and y.

(ii) for N_{123} and N_{124} , any tetrahedron facet $T_{134,2}$ is their common neighbor.

(iii) for N_{123} and $T_{123,4}$, the facet N_{345} is a common neighbor.

Conjecture 8 The extreme rays of NHM_n^m include:

- (i) any ray whose R-graph is an R-graph of an extreme ray of NHM_{n-1}^{m-1} ;
- (ii) every (0,1)-valued extreme ray of NHM_{m+3}^m with R-graph C_i $(3 \le i \le m+3)$.

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