# Small cones of $m$-hemimetrics 

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#### Abstract

We introduce polyhedral cones associated with $m$-hemimetrics on $n$ points, and, in particular, with $m$-hemimetrics coming from partitions of an $n$-set into $m+1$ blocks. We compute generators and facets of the cones for small values of $m, n$ and study their skeleton graphs.


Key words: polyhedral cones; m-partitions; metrics, m-hemimetrics

## 1 Introduction

The notions of $m$-hemimetrics and $m$-partition hemimetrics are generalizations of the notions of metrics and cuts, which are well-known and central objects in Graph Theory, Combinatorial Optimization and, more generally, Discrete Mathematics.

Obviously the $m$-hemimetrics on an $n$-element set $E$ form a cone. What are its facets and extreme rays? The calculations are rather complex due to the high dimension, $\binom{n}{m+1}$, of the cone even for small $n$; moreover, matrices can no longer be used.

Recall that a metric or a metric space is a pair $(E, d)$ where $E$ is a nonvoid set and $d: E^{2} \longrightarrow \mathbb{R}_{+}$(the set of nonnegative reals) satisfies for all $x, y, z \in E$ :
(d1) $d(x, y)=0 \Longleftrightarrow x=y$,
(d2) $d(x, y)=d(y, x) \quad$ (symmetry),
(d3) $d(x, y) \leq d(x, z)+d(z, y) \quad$ (the triangle inequality).
A basic example is $\left(\mathbb{R}^{2}, d\right)$, where $d$ is the Euclidean distance of $x$ and $y$; i.e., the length of the segment joining $x$ and $y$. An immediate extension is $\left(\mathbb{R}^{3}, d\right)$, where $d(x, y, z)$ is
the area of the triangle with vertices $x, y$ and $z$. This leads to the following definition (see $[\mathrm{Me} 28, \mathrm{Bl} 53, \operatorname{Fr} 58, \mathrm{Gä} 63]$ ). A 2-metric is a pair $(E, d)$, where $E$ is a nonempty set and $d: E^{3} \longrightarrow \mathbb{R}_{+}$satisfies for all $x, y, z, t \in E$
$\left(\mathrm{d} 1^{\prime}\right) d(x, x, y)=0$,
(d1") $x \neq y \Longrightarrow d(x, y, u)>0$ for some $u \in E$,
(d2) $d(x, y, z)$ is totally symmetric,
(d3) $d(x, y, z) \leq d(t, y, z)+d(x, t, z)+d(x, y, t) \quad$ (the tetrahedron inequality).
The axiom ( d 2 ) means that the value of $d(x, y, z)$ is independent of the order of $x, y$ and $z$. The axiom (d3) captures that fact that in $\mathbb{R}^{3}$ the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces. A 2-metric allows the introduction of several geometrical and topological concepts - e.g. the betweenness, convexity, line and neighborhood - which lead to interesting results.

For finite 2-metrics (the object of this study), for their polyhedral aspects and for applications, the axiom ( $\mathrm{d} 1^{\prime}$ ) and ( $\mathrm{d} 1^{\prime \prime}$ ) seem to be too restrictive and so we drop them. The definition given below is formulated for an arbitrary positive integer $m$. A map $d: E^{m+1} \longrightarrow \mathbb{R}$ is totally symmetric if for all $x_{1}, \ldots, x_{m+1} \in E$ and every permutation $\pi$ of $\{1, \ldots, m+1\}$

$$
d\left(x_{\pi(1)}, \ldots, x_{\pi(m+1)}\right)=d\left(x_{1}, \ldots, x_{m+1}\right)
$$

Definition. Let $m>0$. An $m$-hemimetric is a pair $(E, d)$, where $d: E^{m+1} \longrightarrow \mathbb{R}$ is totally symmetric and satisfies the simplex inequality: for all $x_{1}, \ldots, x_{m+2} \in E$

$$
\begin{equation*}
d\left(x_{1}, \ldots, x_{m+1}\right) \leq \sum_{i=1}^{m+1} d\left(x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}\right) \tag{1}
\end{equation*}
$$

Call a $m$-hemimetric $d$ nonnegative if $d$ takes only nonnegative values. The notion of a $m$-hemimetric is new, but it is closely related to the notion of a $m$-metric, considered in about 200 references collected in [Gä90]. The study of $m$-hemimetrics is motivated (apart, of course, that it represents an extension of metrics) by applications in Statistics and Data Analysis (see [DeRo99] for some relevant references). Notice the following immediate:

Fact 1. If $(E, d)$ and $\left(E, d^{\prime}\right)$ are $m$-hemimetrics and $a, b \in \mathbb{R}_{+}$then $\left(E, a d+b d^{\prime}\right)$ is an m-hemimetric.

Here, as usual, for all $x_{1}, \ldots, x_{m+1} \in E$

$$
\left(a d+b d^{\prime}\right)\left(x_{1}, \ldots, x_{m+1}\right):=a d\left(x_{1}, \ldots, x_{m+1}\right)+b d^{\prime}\left(x_{1}, \ldots, x_{m+1}\right)
$$

Given an $(m+1)$-partition $S_{1}, \ldots, S_{m+1}$ of $V_{n}:=\{1,2, \ldots, n\}$, a partition m-hemimetric $\alpha\left(S_{1}, \ldots, S_{m+1}\right)$ is defined by setting $\alpha\left(S_{1}, \ldots, S_{m+1}\right)\left(i_{1}, \ldots, i_{m+1}\right)$ is equal to 1 if for no $1 \leq j<l \leq m+1$ both $i_{j}$ and $i_{l}$ belong to the same $S_{k}$, and 0 otherwise. It is easy to
see that $\alpha\left(S_{1}, \ldots, S_{m+1}\right)$ is a nonnegative $m$-hemimetric and for $m=1$ it is the usual cut semimetric (see, for example, [DeLa97]).

For small values of $n$ and $m$ we consider the cone of all $m$-hemimetrics, the cone of all nonnegative $m$-hemimetrics and the cone, generated by all partition $m$-hemimetrics on $V_{n}$. Using computer search we list facets and generators for these cones and tables of their adjacencies and incidences. The different orbits were determined manually, using symmetries. We study two graphs, the 1-skeleton and the ridge graph, of these polyhedra: the number of their nodes and edges, their diameters, conditions of adjacency, inclusions among the graphs and their restrictions on some orbits of nodes. In fact, we would like to describe two graphs $G(C), G\left(C^{*}\right)$ for our three cones as fully as possible, but in the cases, when it is too difficult, we will give some partial information on adjacencies in those graphs. Especially we are interested in the diameters of the graphs, in a good criterion of adjacency, in their local graphs (i.e. in the subgraphs induced by all neighbors of a given vertex) and in their restrictions on some orbits. Finally, we compare obtained results with similar results for metric case (see [DeDe95,DDFu96,DeLa97]) and quasi-metric case (see [DePa99]). All computation was done using the programs $c d d$ of [Fu95].

The following notation will be used below:

- the ( $m+1$ )-simplex inequality (1) and, in particular, for $m=2$, the tetrahedron inequality

$$
T_{i j k, l}: x_{i j l}+x_{i k l}+x_{j k l}-x_{i j k} \geq 0
$$

- the nonnegativity inequality $N_{i_{1}, \ldots, i_{m+1}}: x_{i_{1}, \ldots, i_{m+1}} \geq 0$;
- the cone $P_{n}^{m}$ of partition m-hemimetrics, generated by all $(m+1)$-partitions of $V_{n}$;
- the cone $N H M_{n}^{m}$ of nonnegative m-hemimetrics, defined by all
$(n-m-1)\binom{n}{m+1} \quad(m+1)$-simplex inequalities and all $\binom{n}{m+1}$ nonnegativity inequalities on $V_{n}$;
- the cone of all m-hemimetrics, $H M_{n}^{m}$, defined by all $(m+1)$-simplex inequalities on $V_{n}$.

Clearly, $P_{n}^{m} \subseteq N H M_{n}^{m} \subseteq H M_{n}^{m}$ and these 3 cones are of full dimension $\binom{n}{m+1}$ each. For $m=1$ the last two cones coincide and the first two cones are the cut cone $C U T_{n}$ and the semimetric cone $M E T_{n}$, considered in detail in [DeLa97] and in the references listed there. The cone $P_{n}^{1}=C U T_{n}$ has $\left\lfloor\frac{n}{2}\right\rfloor$ orbits of extreme rays (represented by the cuts $\alpha(1 \ldots i,(i+1) \ldots n))$.

To simplify the notation we keep $n$ fixed and denote by $E_{k}$ the family of all $k$-element subsets of $V_{n}(k=1, \ldots, n)$. Let $d$ be a semimetric on the set $V_{n}$. Because of symmetry (d2) and since $d(i, i)=0$ for all $i \in V_{n}$, we can view the semimetric $d$ as a vector $\left(d_{i_{1}, i_{2}}\right)_{\in} \mathbb{R}^{E_{2}}$. In the same way, we can view the $m$-hemimetric $d$ on the set $V_{n}$ as a vector $\left(d_{i_{1}, \ldots, i_{m+1}}\right) \in \mathbb{R}^{E_{m+1}}$. In particular, each extreme ray of the cones, considered below, will be represented by an integer vector on the ray with relatively prime coordinates. We also will represent the facets of cones by such vectors.

Any such vector $v=\left(v_{i_{1}, \ldots, i_{m+1}}\right) \in \mathbb{R}^{E_{m+1}}$ can be represented by the following vertexlabeled induced subgraph $R(v)$ of the Johnson graph $J(n, m+1)$. The vertices of $R(v)$ are all unordered $(m+1)$-tuples $\left(i_{1}, \ldots, i_{m+1}\right)$ such that $\left(v_{i_{1}, \ldots, i_{m+1}}\right)$ is not zero. This value will be the label of the vertex; we will omit the label when it is 1 . Two vertices of the Johnson graph (and also of its induced subgraph $R(v)$ ) are adjacent if the corresponding ( $m+1$ )-tuples have $m$ common elements.

For example, $R\left(\alpha\left(S_{1}, \ldots, S_{m+1}\right)\right)$ is the complement to the Hamming graph $H\left(\left|S_{1}\right|, \ldots,\left|S_{m+1}\right|\right)$, i.e. the direct (Cartesian) product of the cliques $K_{\left|S_{i}\right|}, 1 \leq i \leq m+1$. Another example: the graph $R$ of the vector defining a nonnegativity facet is a vertex and, for a $(m+1)$ simplex facet, it is the complete graph $K_{m+2}$ with one vertex labeled -1 .

## 2 Partition $m$-hemimetrics and related polyhedra

Recall that for a partition $S_{1}, S_{2}$ of $V_{n}$ the cut semimetric $\alpha\left(S_{1}, S_{2}\right)$ satisfies $\alpha\left(S_{1}, S_{2}\right)_{i j}=1$ if $\{i, j\} \cap S_{1}$ is a singleton and $\alpha\left(S_{1}, S_{2}\right)_{i j}=0$ otherwise. We extend it as follows. Let $q \geq 2$ be an integer and let $S_{1}, \ldots, S_{q}$ be pairwise disjoint nonvoid subsets of $V_{n}$, forming a partition of $V_{n}$. The multicut semimetric $\delta\left(S_{1}, \ldots, S_{q}\right)$ is the vector in $\mathbb{R}^{E_{2}}$, defined by $\delta\left(S_{1}, \ldots, S_{q}\right)_{i j}=0$, if $i, j \in S_{h}$ for some $h, 1 \leq h \leq q$, and $\delta\left(S_{1}, \ldots, S_{q}\right)_{i j}=1$, otherwise.

The connection between $\delta\left(S_{1}, \ldots, S_{q}\right)$ and $\alpha\left(S_{1}, \ldots, S_{q}\right)$ from Section 1 is given by
$\alpha\left(S_{1}, \ldots, S_{q}\right)\left(i_{1}, \ldots, i_{q}\right)=\prod_{1 \leq s<t \leq q} \delta\left(S_{1}, \ldots, S_{q}\right)\left(i_{s}, i_{t}\right)=$
$\left\lfloor\frac{\sum_{1 \leq s<t \leq q} \delta\left(S_{1}, \ldots, S_{q}\right)\left(i_{s}, i_{t}\right)}{\binom{q}{2}}\right\rfloor$; compare it with the half-perimeter $m$-semimetric from [DeRo99].
The cone generated by all multicut semimetrics $\delta\left(S_{1}, \ldots, S_{q}\right)(q \geq 2)$ on $V_{n}$, is called the multicut cone and denoted by $M C U T_{n}$; it coincides with $C U T_{n}$ (see [DeLa97], Proposition 4.2.9). The convex hull of the cut semimetrics (multicut semimetrics) on $V_{n}$, is called the cut polytope ( multicut polytope) and is denoted by $C U T_{n}^{\square}\left(M C U T_{n}^{\square}\right)$; the two polytopes not coincide.

## 3 Facets, extreme rays and their orbits in polyhedra

We recall some terminology. Let $C$ be a polyhedral cone in $\mathbb{R}^{n}$. Given $v \in \mathbb{R}^{n}$, the inequality $v^{T} x \leq 0$ is said to be valid for $C$, if it holds for all $x \in C$. Then the set $\left\{x \in C \mid v^{T} x=0\right\}$ is called the face of $C$, induced by the valid inequality $v^{T} x \leq 0$. A face of dimension $\operatorname{dim}(C)-1$ is called a facet of $C$; a face of dimension 1 is called an extreme ray of $C$. A face of dimension $\operatorname{dim}(C)-2$ is called a ridge.

Two vertices $x, y$ of $C$ are said to be adjacent, if they generate a face of dimension 2 of $C$. Two facets of $C$ are said to be adjacent, if their intersection has dimension $\operatorname{dim}(C)-2$. The 1-skeleton graph of $C$ is the graph $G(C)$ whose nodes are the extreme rays of $C$ and whose edges are the pairs of adjacent nodes. Denote by $C^{*}$ the dual cone of $C$. The ridge graph of $C$ is the graph whose nodes are the facets of $C$ and with an edge between two facets if they are adjacent on $C$. So, the ridge graph of a cone $C$ is the 1-skeleton $G\left(C^{*}\right)$ of its dual cone.

A mapping $f: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$ is called a symmetry of a cone $C$ (or a polytope $P$ ), if it is an isometry, satisfying $f(C)=C$ (or $f(P)=P)$. (An isometry of $\mathbb{R}^{n}$ is a linear mapping preserving the Euclidean distance.) Given a face $F$, the orbit $\Omega(F)$ of $F$ consists of all faces, that can be obtained from $F$ by the group of all symmetries of $C$.

Clearly, all the faces of $C U T_{n}$ and $C U T_{n}^{\square}$ are preserved by any permutation of $V_{n}$.
For $m>1$ all orbits of faces of $m$-hemimetric cones $P_{n}^{m}, N H M_{n}^{m}, H M_{n}^{m}$ on $V_{n}$ are also preserved under any permutation of the set $V_{n}=\{1, \ldots, n\}$. We conjecture that the symmetry group consists only of permutations of $V_{n}$, i.e. it is the group $\operatorname{Sym}(n)$ of all permutations on $V_{n}$ ( see Theorem 3.3 in [DGLu91] stating that the symmetry group of a truncated multicut polytope is $\operatorname{Sym}(n)$ ).

## $4 \quad$ The case of $n=m+2$

The minimal $n$ for which the three cones are nontrivial is $m+2$; the dimension of the cones is also $m+2$ for $n=m+2$.

First, we present a complete linear description for case $(m, n)=(2,4)$.
It turns out that $P_{4}^{2}=N H M_{4}^{2}$. This cone has 6 extreme rays (all in the same orbit under $\operatorname{Sym}(4)): \alpha\left(S_{1}, S_{2}, S_{3}\right)$ for the 3-partitions
$(1,2,34),(1,3,24),(1,4,23),(2,3,14),(2,4,13),(3,4,12)$.
There are 8 facets, which form 2 orbits: the orbit $F_{1}$ of all 4 tetrahedron facets and the orbit $F_{2}$ of all 4 nonnegativity facets.

The edge graph $G(C)$ is $K_{6}-3 K_{2}$ (the octahedron); the 3 pairs of nonadjacent rays are of the form $\alpha(a, b, c d), \alpha(c, d, a b)$. Each extreme ray (say, $\alpha(1,2,34)$ ) is incident to 2 tetrahedron and to 2 nonnegativity facets (namely, to $T_{123,4}, T_{124,3}$ and $N_{134}, N_{234}$ ).

The ridge graph $G\left(C^{*}\right)$ is the cube. Adjacencies of facets of $N H M_{4}^{2}$ are shown in Table 1. For each orbit a representative and the number of adjacent facets from other orbits are given, as well as the total number of adjacent ones, the number of incident extreme rays

Table 1
The adjacencies of facets in the cone $N H M_{4}^{2}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{123,4}$ | 0 | 3 | 3 | 3 | 4 |
| $F_{2}$ | $N_{123}$ | 3 | 0 | 3 | 3 | 4 |

and the cardinality of orbits.
More precisely, for the ridge graph of $N H M_{4}^{2}$ it holds:
(i) The tetrahedron facet $T_{i j k, l}$ is adjacent only to the facets $N_{i j l}, N_{i k l}, N_{j k l}$;
(ii) The nonnegativity facet $N_{i j k}$ is adjacent only to the facets $T_{i j l, k}, T_{i k l, j}, T_{j k l, i}$.

The cone $H M_{m+2}^{m}$ is a simplex $(m+2)$-dimensional cone; so $G(C)=G\left(C^{*}\right)=K_{m+2}$. Its facets are all $(1,-1)$-valued $(m+2)$-vectors with only one -1 , its generators are all $(1-m, 1)$-valued $(m+2)$-vectors with only one $1-m$. Notice that $P_{3}^{1}=H M_{3}^{1}=C U T_{3}=$ $M E T_{3}$.

In general, $P_{m+2}^{m}=N H M_{m+2}^{m}$ for any $m \geq 2$. This cone has $\binom{m+2}{2}$ extreme rays, all in the same orbit, represented by $\alpha(12,3, \ldots, m+2)$, i.e. by any vector of length $m+2$, consisting of two ones and $m$ zeros. The skeleton of $P_{m+2}^{m}$ is the Johnson graph $J(m+2,2)$, called also the triangular graph $T(m+2)$, which is is the line graph $L\left(K_{m+2}\right)$. It is also the skeleton of the $(m+1)$-polytope (called ambo $-\alpha_{m+1}$ ), obtained from the $(m+1)$-simplex as the convex hull of the mid-points of all its edges; e.g. $T(4)$ is the skeleton of the octahedron, $T(5)$ is the complement of the Petersen graph. In general, $T(m), m \geq 2$, has diameter 2 ; moreover, it is a strongly regular graph.

The cone $P_{m+2}^{m}$ has two orbits, $F_{1}$ and $F_{2}$, of facets, containing $m+2$ facets each and represented by the ( $m+1$ )-simplex facet $T_{1 \ldots(m+1),(m+2)}$ and by the nonnegativity facet $N_{1 \ldots(m+1)}$. The orbit $F_{1}$ consists of simplex cones, i.e. facets from this orbit are incident to $m+1$ linearly independent extreme rays. Any nonnegativity inequality $N$ defines the cone $P_{m+1}^{m-1}=N H M_{m+1}^{m-1}$, i.e. it becomes equality on this smaller cone. So, $N$ is non-facet only for $m=1$ and it is a simplex cone only for $m=2$; in general, $N$ is incident to $\binom{m+1}{2}$ extreme rays. The ridge graph is $\overline{K_{m+2}}$ on $F_{1}$; on $F_{2}$ it is $\overline{K_{4}}$ for $m=2$ and $K_{m+2}$ for $m \geq 3$. Finally, the $m+2$ pairs $\left(T_{\bar{i}, i}, N_{\bar{i}}\right)$ (of $(m+1)$-simplex and nonnegativity facets) are the only non-edges for pairs of facets from different orbits.

## 5 Small 2-hemimetrics

We present here the complete linear description of $P_{5}^{2}, N H M_{5}^{2}$ and $H M_{5}^{2}$. The cone $P_{5}^{2}$ has 25 extreme rays, which form 2 orbits with representatives $\alpha\left(1,2,345\right.$ ) (orbit $O_{1}$ ) and $\alpha(1,23,45)$ (orbit $O_{2}$ ). The skeleton and the ridge graph of $P_{5}^{2}$ has 270 and 1185 edges, respectively. The cone $P_{5}^{2}$ has 120 facets divided into 4 orbits, induced by the 20 tetrahedron inequalities (orbit $F_{1}$ ), the 10 nonnegativity inequalities (orbit $F_{2}$ ), the 60 inequalities (orbit $F_{3}$ ), represented by

$$
A: 2 x_{123}-\left(x_{124}+x_{135}\right)+\left(x_{134}+x_{125}+x_{245}+x_{345}\right) \geq 0
$$

and the 30 inequalities (orbit $F_{4}$ ), represented by

$$
B: 2\left(x_{123}+x_{145}+x_{245}-x_{345}\right)+\left(x_{134}+x_{135}+x_{234}+x_{235}-x_{124}-x_{125}\right) \geq 0
$$

The above two inequalities are the 2-hemimetric analogs of the following 5-gonal inequality (the simplest inequality, different from the triangle inequality), appearing in the cone $C U T_{n}$ for $n \geq 5$ :

$$
\left(x_{13}+x_{14}+x_{15}+x_{23}+x_{24}+x_{25}\right)-\left(x_{12}+x_{34}+x_{35}+x_{45}\right) \geq 0
$$

This facet and $B$ have both the Petersen graph as their $\bar{R}$ (i.e. the complement of their graph $R$ ). Clearly, the graphs $R$ for partition 2-hemimetrics $\alpha(1,2,34), \alpha(1,2,345)$, $\alpha(1,23,45)$ are the cycles $C_{2}, C_{3}, C_{4}$.

The graphs $\overline{R(A)}, \overline{R(B)}$ are given on the Figure 1.


Figure 1: $\overline{R(A)}, \overline{R(B)}$ in the cone $P_{5}^{2}$

Table 2
The adjacencies of extreme rays in the cone $P_{5}^{2}$

| Orbit | Representative | $O_{1}$ | $O_{2}$ | Adj. | Inc. | $\left\|O_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | $\alpha(1,2,345)$ | 9 | 12 | 21 | 54 | 10 |
| $O_{2}$ | $\alpha(1,23,45)$ | 8 | 14 | 22 | 54 | 15 |

Table 3
The adjacencies of facets in the cone $P_{5}^{2}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | $F_{3}$ | $F_{4}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{123,4}$ | 16 | 9 | 18 | 6 | 49 | 16 | 20 |
| $F_{2}$ | $N_{123}$ | 18 | 3 | 18 | 3 | 42 | 16 | 10 |
| $F_{3}$ | $A$ | 6 | 3 | 4 | 2 | 15 | 10 | 60 |
| $F_{4}$ | $B$ | 4 | 1 | 4 | 0 | 9 | 9 | 30 |

Table 4
The adjacencies of extreme rays in the cone $N H M_{5}^{2}$

| Orbit | Representative | $O_{1}$ | $O_{2}$ | $O_{3}$ | Adj. | Inc. | $\left\|O_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | $\alpha(1,2,345)$ | 9 | 12 | 6 | 27 | 21 | 10 |
| $O_{2}$ | $\alpha(1,23,45)$ | 8 | 6 | 8 | 22 | 18 | 15 |
| $O_{3}$ | $v_{3}$ | 5 | 10 | 5 | 20 | 15 | 12 |

Table 5
The adjacencies of facets in the cone $N H M_{5}^{2}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{123,4}$ | 16 | 9 | 25 | 22 | 20 |
| $F_{2}$ | $N_{123}$ | 18 | 3 | 21 | 22 | 10 |

The facets from the orbit $F_{4}$ are simplex-cones, i.e. the extreme rays on them are linearly independent. Among the 9 neighbors of $B, 4$ are from the orbit $F_{1}, 4$ are from the orbit $F_{3}$ and exactly one (actually, $N_{123}$ ) from the orbit $F_{2}$. The local graph of a facet from $F_{4}$ (i.e. the subgraph of the ridge graph of $P_{5}^{2}$, induced by all neighbors of $B$ ) is $K_{9}-C_{4}$. In fact, all nonadjacencies in this local graph are the four edges of the 4 -cycle of the 4 neighbors of $B$ from the orbit $F_{3}$.

Facets from orbits $F_{1}, F_{2}, F_{3}, F_{4}$ are incident, respectively, to 7,$9 ; 7,9 ; 4,6 ; 3,6$ extreme rays from orbits $O_{1}, O_{2}$ of $P_{5}^{2}$.

The skeleton and the ridge graph of $N H M_{5}^{2}$ have 420 and 355 edges, respectively. The adjacencies of 37 extreme rays and of 30 facets of this cone are given in Tables 4,5 . The extreme rays are divided into 3 orbits $O_{1}, O_{2}, O_{3}$, represented by $(0,1)$-valued vectors $v_{1}, v_{2}, v_{3}$ below; their $R$-graphs are $C_{3}, C_{4}, C_{5}$, respectively.

Table 6
The adjacencies of extreme rays in the cone $H M_{5}^{2}$

| Orbit | Representative | $O_{1}$ | $O_{2}$ | $O_{3}$ | $O_{4}$ | $O_{5}$ | $O_{6}$ | Adj. | Inc. | $\left\|O_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | $v_{1}=\alpha(1,2,345)$ | 6 | 12 | 6 | 6 | 9 | 9 | 48 | 14 | 10 |
| $O_{2}$ | $v_{2}=\alpha(1,23,45)$ | 8 | 2 | 4 | 2 | 4 | 8 | 28 | 12 | 15 |
| $O_{3}$ | $v_{3}$ | 5 | 5 | 0 | 5 | 5 | 5 | 25 | 10 | 12 |
| $O_{4}$ | $v_{4}$ | 6 | 3 | 6 | 0 | 3 | 6 | 24 | 12 | 10 |
| $O_{5}$ | $v_{5}$ | 6 | 4 | 4 | 2 | 0 | 4 | 20 | 12 | 15 |
| $O_{6}$ | $v_{6}$ | 3 | 4 | 2 | 2 | 2 | 0 | 13 | 10 | 30 |

Each facet (from both orbits) of $N H M_{5}^{2}$ is incident to $7,9,6$ extreme rays from orbits $O_{1}, O_{2}, O_{3}$, respectively. Each (tetrahedron) facet of $H M_{5}^{2}$ is incident to $7,9,6,6,9,15$ extreme rays from orbits $O_{1}, \ldots, O_{6}$.

For any cone, let $I_{O_{i}, F_{j}}$ and $I_{F_{j}, O_{i}}$ denote the number of facets from the orbit $F_{j}$, incident to an extreme ray of the orbit $O_{i}$, and, respectively, the number of extreme rays from $O_{i}$, incident to a facet from $F_{j}$. Clearly, $\left|O_{i}\right| I_{O_{i}, F_{j}}=\left|F_{j}\right| I_{F_{j}, O_{i}}$.

The cone $H M_{5}^{2}$ has 92 extreme rays divided into 6 orbits. Below we give some representatives $v_{1}, \ldots, v_{6}$ of those orbits $O_{1}, \ldots, O_{6}$. The first two represent both orbits of $P_{5}^{2}$, the first 3 represent the 3 orbits of $N H M_{5}^{2}$.
$x=\left(x_{123}, x_{124}, x_{125}, x_{134}, x_{135}, x_{145}, x_{234}, x_{235}, x_{245}, x_{345}\right):$
$v_{1}=(1,1,1,0,0,0,0,0,0,0) ;$
$v_{2}=(0,1,1,1,1,0,0,0,0,0) ;$
$v_{3}=(1,0,1,0,0,1,1,0,0,1) ;$
$v_{4}=(1,1,1,-1,0,0,1,0,1,1) ;$
$v_{5}=(1,1,1,-1,-1,1,1,1,1,1) ;$
$v_{6}=(1,0,1,0,1,-1,1,1,2,1)$.
Proposition 1 The diameters of the skeleton graphs of $P_{5}^{2}$ and of $N H M_{5}^{2}$ are 2.

In fact, each of the orbits $O_{1}, O_{2}$ of $P_{5}^{2}$ is a dominating clique. There is only one type of a non-edge, represented by $\alpha(1,23,45), \alpha(2,3,145)$ ), but $\alpha(1,3,245)$ is one of common neighbors. The complement of the skeleton of $P_{5}^{2}$ turns out to be the Petersen graph with a new vertex (corresponding to a member of the orbit $O_{2}$ ) on each of 15 edges. The result

Table 7
The adjacencies of facets in the cone $N H M_{6}^{2}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{123,4}$ | 56 | 19 | 75 | 4001 | 60 |
| $F_{2}$ | $N_{123}$ | 57 | 10 | 67 | 3939 | 20 |

for $N H M_{5}^{2}$ comes also by finding out a common neighbor to each possible non-edge.
Proposition 2 For the ridge graphs of $N H M_{5}^{2}$ and $H M_{5}^{2}$ it holds:
(i) The diameter of the ridge graph of $N H M_{5}^{2}$ is 2;
(ii) Its restriction on the orbits $F_{1}$ and $F_{2}$ is $K_{4,4,4,4,4}$ and the Petersen graph, respectively;
(iii) The ridge graph of $H M_{5}^{2}$ is $K_{4,4,4,4,4}$ (of diameter 2).

### 5.2 The case of 6 points

$N H M_{6}^{2}$ has exactly 12492 extreme rays, with (adjacency,incidence) pairs being, respectively, $(2278,64),(1321,56),(1030,40),(818,48),(731,48),(358,40),(270,36),(93,28)$, $(66,28),(51,28),(47,28),(46,39),(37,31),(32,28),(30,27),(29,26),(27,23),(26,24),(26,23)$, $(25,25),(23,22),(22,21),(21,21)$.

Three of the above pairs (1st, 2nd and 4th) are realized by orbits (say, $0_{1}, O_{2}$ and $O_{4}$ ), which are represented by 3-partition 2-hemimetrics $\alpha(1,2,3456), \alpha(1,23,456), \alpha(12,34,56)$ and have size $15,60,15$, respectively. The graphs $R$ of members of orbits $O_{1}, O_{2}$ and $O_{4}$ are $K_{4}, K_{6}-C_{6}=K_{3} \times K_{2}$ (the skeletons of the tetrahedron and 3-prism) and the skeleton of the cube. Three other orbits consist also of ( 0,1 )-valued extreme rays: $O_{3}$ (with $R$ being the Petersen graph), $O_{5}$ (with $R$-graph being the skeleton of the simple polyhedron with $p$-vector $\left.p=\left(p_{3}=2, p_{4}=2, p_{5}=2\right)\right)$ and $O_{7}$ with graph $R$ (non-planar, non-regular), given on Figure 2, together with one for $O_{5}$. The extreme rays of the remaining orbits are $(0,1,2)$-valued and ( $0,1,2,3$ )-valued vectors.


Figure 2 : The graphs R of extreme rays from orbits $O_{7}, O_{5}$ of the cone $N H M_{6}^{2}$

The cone $P_{6}^{2}$ has more than 950.000 facets (computer stopped, by lack of memory, after 72 , out of 90 , iterations). Here are two examples of a ( $0,1,-1$ )-valued facets of $P_{6}^{2}$; see also Figure 3 (for the facet W).
$W:\left(-x_{145}+x_{146}+x_{136}+x_{123}+x_{125}\right)+\left(x_{245}+x_{234}+x_{346}+x_{356}+x_{256}\right)-\left(x_{235}+x_{236}\right) \geq 0$.


Figure $3: R(W)$ in the cone $P_{6}^{2}$
$Z: \sum x_{i j k}-\left(x_{124}+x_{125}+x_{145}\right)-\left(x_{234}+x_{235}+x_{345}\right)-2\left(x_{146}+x_{156}+x_{456}\right)-2 x_{236} \geq 0$.
Remark that the triples with coefficients zero, in $W$ and $Z$, form the skeleton of 1- and 2-truncated tetrahedron, respectively; the triples with coefficient -1 form $K_{3}+K_{1}$ and $K_{2}+K_{1}$, respectively.

## 6 Small 3-hemimetrics

The cone $N H M_{6}^{3}$ has 287 extreme rays divided into 5 orbits. Below we give representatives $u_{1}, \ldots, u_{5}$ of the orbits $O_{1}, \ldots, O_{5}$. These vectors are indexed by 4 -subsets of the set $\{1, \ldots, 6\}$; the 4 -subsets are given as the complements of 2 -subsets. The first four are $(0,1)$-valued; their $R$-graphs (in the Johnson graph $J(6,4)$ of all 4 -tuples) are the cycles $C_{3}, C_{4}, C_{5}, C_{6}$, respectively. The first two are partition 3-hemimetrics; they represent both orbits of $P_{6}^{3}$. The graphs $R\left(u_{4}\right)$ and $R\left(u_{5}\right)$ are on Figure 4.
$x=\left(x_{\overline{12}}, x_{\overline{13}}, x_{\overline{14}}, x_{\overline{15}}, x_{\overline{16}}, x_{\overline{23}}, x_{\overline{24}}, x_{\overline{25}}, x_{\overline{26}}, x_{\overline{34}}, x_{\overline{35}}, x_{\overline{36}}, x_{\overline{45}}, x_{\overline{46}}, x_{\overline{56}}\right):$
$u_{1}=(0,0,1,1,0,0,0,0,0,0,0,0,1,0,0) ;$
$u_{2}=(0,0,1,1,0,0,0,0,0,1,1,0,0,0,0) ;$
$u_{3}=(0,0,1,1,0,0,0,0,0,1,0,1,0,0,1) ;$
$u_{4}=(0,0,1,1,0,0,0,1,1,1,0,1,0,0,0) ;$
$u_{5}=(0,0,1,1,0,1,0,0,1,0,0,1,1,2,0) ;$


Figure $4: R\left(u_{4}\right), R\left(u_{5}\right)$ in the cone $N H M_{6}^{3}$
The cone $P_{6}^{3}$ has 4065 facets divided into at least 11 orbits. Below we give representatives $f_{1}, \ldots, f_{11}$ of the orbits $F_{1}, \ldots, F_{11}$. Their (adjacency, incidence) pairs are, respectively, $(1526,49),(703,41),(100,23),(37,19),(31,18),(30,18),(23,17),(23,15),(22,18),(18,16)$, $(14,14)$. The facets $f_{1}, f_{2}$ are nonnegativity and 4 -simplex facets; $f_{11}$ is a simplex cone. The $R$-graphs of the facets $f_{3}$ and $f_{4}$ are on Figure 5 .


Figure $5: \overline{R\left(f_{3}\right)}, \overline{R\left(f_{4}\right)}$ in the cone $P_{6}^{3}$
$f_{1}=(0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) ;$
$f_{2}=(0,1,0,0,0,1,0,0,0,-1,1,1,0,0,0) ;$
$f_{3}=(-1,1,0,1,1,2,1,0,0,-1,0,0,1,1,0) ;$
$f_{4}=(-1,1,0,1,0,2,1,0,1,-1,0,1,1,0,1) ;$
$f_{5}=(-1,1,1,2,1,2,2,-1,0,-2,1,0,1,2,1) ;$
$f_{6}=(-1,1,0,2,2,2,1,-1,1,-1,1,-1,2,2,0) ;$
$f_{7}=(-1,1,1,3,2,2,2,-2,1,-2,2,-1,2,3,1) ;$
$f_{8}=(1,-1,3,1,4,2,2,-2,1,-2,2,3,2,-1,1) ;$

Table 8
The adjacencies of extreme rays in the cone $N H M_{6}^{3}$

| Orbit | Representative | $O_{1}$ | $O_{2}$ | $O_{3}$ | $O_{4}$ | $O_{5}$ | Adj. | Inc. | $\left\|O_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | $u_{1}=\alpha(1,2,3,456)$ | 19 | 36 | 36 | 18 | 27 | 136 | 21,12 | 20 |
| $O_{2}$ | $u_{2}=\alpha(1,2,34,56)$ | 16 | 18 | 24 | 20 | 16 | 94 | 18,11 | 45 |
| $O_{3}$ | $u_{3}$ | 10 | 15 | 20 | 15 | 10 | 70 | 15,10 | 72 |
| $O_{4}$ | $u_{4}$ | 6 | 15 | 18 | 9 | 6 | 54 | 12,9 | 60 |
| $O_{5}$ | $u_{5}$ | 6 | 8 | 8 | 4 | 0 | 26 | 10,8 | 90 |

$f_{9}=(-1,1,1,2,2,2,2,-1,-1,-2,1,1,1,1,2) ;$
$f_{10}=(-1,1,1,1,2,1,1,1,-1,-1,-1,1,2,1,1) ;$
$f_{11}=(-1,1,1,2,0,2,2,-1,1,-2,1,1,1,1,2)$.
Proposition 3 The diameter of the skeleton graph of $P_{6}^{3}$ is 2. Moreover:
(i) $G\left(O_{1}\right)=K_{20}, G\left(O_{2}\right)=K_{45}-15 K_{3}$;
(ii) all non-edges are represented by $\alpha(12,34,5,6)$ that are nonadjacent to
$\alpha(12,3,4,56), \alpha(1,2,34,56)$ (from the same orbit $O_{2}$ ) and to
$\alpha(125,3,4,6), \alpha(126,3,4,5), \alpha(345,1,2,6), \alpha(346,1,2,5)$.
In fact, both non-neighbors of $\alpha(12,34,5,6)$ are in $O_{2}$. For both types of non-edges $\alpha(12,34,5,6)$ with $\alpha(12,3,4,56)$ and $\alpha(125,3,4,6)$ - the ray $\alpha(13,24,5,6)$ is a common neighbor. Also, all 9 non-neighbors of a ray from $O_{1}$, form $K_{9}$ in the skeleton graph.

Notice that the skeleton of $P_{6}^{3}$ is not an induced subgraph of the skeleton of $N H M_{6}^{3}$; the only difference is in their restriction $G\left(O_{2}\right)$ to the orbit of rays, represented by $u_{2}$. One can check that all neighbors of a partition hemimetric $\alpha\left(a_{1}, a_{2}, b_{1} b_{2}, c_{1} c_{2}\right)$ from the same orbit $O_{2}$ of $N H M_{6}^{3}$ are the 10 rays obtained by a transposition $(x y)$ and the 8 rays obtained by a product $\left(a_{1} b_{i}\right)\left(a_{2} c_{j}\right)$ or $\left(a_{1} c_{i}\right)\left(a_{2} b_{j}\right)$ of two transpositions. But in the skeleton of $P_{6}^{3}$, the ray $\alpha\left(a_{1}, a_{2}, b_{1} b_{2}, c_{1} c_{2}\right)$ is adjacent to all other members of $O_{2}$, except for the two rays, obtained from it by $\left(a_{1} b_{1}\right)\left(a_{2} b_{2}\right)$ or $\left(a_{1} c_{1}\right)\left(a_{2} c_{2}\right)$. The complement of the graph, induced by all 18 neighbors of the ray $\alpha\left(a_{1}, a_{2}, b_{1} b_{2}, c_{1} c_{2}\right)$ from the same orbit $O_{2}$ of $N H M_{6}^{3}$, is $C_{4}+C_{4}$ on 8 rays, obtained by a product $\left(a_{1} b_{i}\right)\left(a_{2} c_{j}\right)$ or a product $\left(a_{1} c_{i}\right)\left(a_{2} b_{j}\right)$, the skeleton of the cube on 8 rays, obtained by $\left(a_{i} b_{j}\right)$ or $\left(a_{i} c_{j}\right)$, and it is $\overline{K_{2}}$ on two rays obtained by $\left(b_{i} c_{j}\right)$.

The skeletons of $P_{6}^{3}$ and $N H M_{6}^{3}$ both contain a dominating clique $O_{1}$; so their diameters are 2 or 3 . In order to see closer the skeleton of $N H M_{6}^{3}$, we now describe the local graph, denoted by $H$, of the ray $u_{5}$. All 26 neighbors are in orbits $O_{1}, O_{2}, O_{3}, O_{4}$ only. It will be easier to describe $\bar{H}$. The restrictions of $\bar{H}$ on them are $\overline{K_{6}}, C_{8}$, the skeleton of the cube

Table 9
The adjacencies of extreme rays in the cone $P_{6}^{3}$

| Orbit | Representative | $O_{1}$ | $O_{2}$ | Adj. | Inc. | $\left\|O_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $O_{1}$ | $\alpha(1,2,3,456)$ | 19 | 36 | 55 | 1113 | 20 |
| $O_{2}$ | $\alpha(1,2,34,56)$ | 16 | 42 | 58 | 993 | 45 |

Table 10
The adjacencies of facets in the cone $N H M_{6}^{3}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{1234,5}$ | 25 | 14 | 39 | 131 | 30 |
| $F_{2}$ | $N_{1234}$ | 28 | 14 | 42 | 181 | 15 |

and $2 K_{2}$, respectively. Two vertices from $O_{1}$ (say 15 and 16) are isolated; so the diameter of $H$ is 2 . Here we denote by $i j$ the $j$-th member of the orbit $O_{i}$ in $H$. All edges of $\bar{H}$ (without isolated vertices 15 and 16) are presented on Figure 6. On the right picture the members of $O_{1}$ are excluded while on the left one the members of $O_{2}$ are excluded. $\bar{H}$ does not contain cross-edges among orbits $O_{1}$ and $O_{2}$.


Figure 6 : A presentation of the local graph of a ray of the orbit $O_{5}$ of the cone $N H M_{6}^{3}$
Proposition 4 The ridge graph of $N H M_{6}^{3}$ has diameter 2. Moreover:
(i) any 4-simplex facet $T_{i j k l, m}$ is adjacent to all but 5 facets: $N_{i j k l}$ and all 4 other 4-simplex facets with the same support;
(ii) the restrictions of the ridge graph to the orbits $F_{1}$ and $F_{2}$ are
$K_{5,5,5,5,5,5}$ and $K_{15}$, respectively.

## 7 Small 4-hemimetrics

The cone $\mathrm{NH}_{7}^{4}$ has 3692 extreme rays divided into 8 orbits. We give below representatives $w_{1}, \ldots, w_{8}$ of their orbits $O_{1}, \ldots, O_{8}$. These vectors are indexed by 5 -subsets of the set

Table 11
The adjacencies of facets in the cone $N H M_{7}^{4}$

| Orbit | Representative | $F_{1}$ | $F_{2}$ | Adj. | Inc. | $\left\|F_{i}\right\|$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $F_{1}$ | $T_{12345,6}$ | 36 | 20 | 56 | 1302 | 42 |
| $F_{2}$ | $N_{12345}$ | 40 | 20 | 60 | 2437 | 21 |

$\{1, \ldots, 7\}$; the 5 -subsets are given as the complements of 2 -subsets. The (adjacency, incidence) pairs of those rays are, respectively, $(985,48),(535,43),(315,38),(192,33),(126,28),(67,30)$, $(43,25),(42,25)$. The first five vectors are $(0,1)$-valued; their graphs $R$ are $C_{3}, C_{4}, C_{5}, C_{6}, C_{7}$, respectively. The first two are partition 4 -hemimetrics; they represent both orbits of $P_{7}^{4}$. The vectors $w_{i}, 1 \leq i \leq 4$, and $w_{6}$ have same $R$-graphs as the members of orbits $O_{i}, 1 \leq i \leq 5$, of $N H M_{6}^{3}$, respectively; so, the graphs of Figure 5 represent also $w_{4}$ and $w_{6}$. The graphs $R\left(w_{7}\right)$ and $R\left(w_{8}\right)$ are on Figure 7.
$(\overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}, \overline{23}, \overline{24}, \overline{25}, \overline{26}, \overline{27}, \overline{34}, \overline{35}, \overline{36}, \overline{37}, \overline{45}, \overline{46}, \overline{47}, \overline{56}, \overline{57}, \overline{67}):$
$w_{1}=(0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0) ;$
$w_{2}=(0,0,0,1,1,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0) ;$
$w_{3}=(0,0,0,1,1,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0) ;$
$w_{4}=(0,0,0,1,1,0,1,0,0,1,0,1,0,0,0,1,0,0,0,0,0) ;$
$w_{5}=(0,0,0,1,1,0,1,0,0,1,0,1,0,0,0,0,0,1,0,1,0) ;$
$w_{6}=(0,0,0,1,1,0,0,0,0,0,0,1,0,2,1,0,0,1,1,0,0) ;$
$w_{7}=(0,0,0,2,2,0,0,0,0,1,1,1,1,0,0,1,0,0,0,0,1) ;$
$w_{8}=(0,0,0,1,1,0,0,1,0,0,1,2,1,1,0,0,0,1,0,0,0)$.


Figure $7: R\left(W_{6}\right), R\left(W_{7}\right), R\left(W_{8}\right)$ in the cone $N H M_{7}^{4}$
It is easy to check that the ridge graph of $N H M_{7}^{4}$ is $K_{6,6,6,6,6,6,6}$ on $F_{1}$ and $K_{21}$ on $F_{2}$. All non-edges among $F_{1}, F_{2}$ are of the form $T_{i_{1} \ldots i_{5}, i_{6}}$ and $N_{i_{1} \ldots i_{5}}$.

## 8 Comparison of the small cones

Now we compare some semimetric and m-hemimetric cones on $n$ points for small $n$. The triangle inequalities suffice to describe the cut cones for $n \leq 4$, but $C U T_{n} \subset M E T_{n}$ (strictly) for $n \geq 5$. The complete description of all the facets of the cut cone $C U T_{n}$ is known for $n \leq 8$, the complete description of the semimetric cone $M E T_{n}$ is known for $n \leq$ 7 (see, for example, the linear description of $\mathrm{MET}_{7}$ in [Gr92]). Here the "combinatorial explosion" starts from $n=8$. The number of orbits of facets and of extreme rays of those and other cones, when it is known, is given in Table 12.

In fact, $P_{n}^{2}=N H M_{n}^{2}$ holds only for the smallest value $n=4$. For $n=4,5$ we computed all facets, extreme rays and their adjacencies and incidences for three cones $P_{n}^{2}, N H M_{n}^{2}, H M_{n}^{2}$. For 2-hemimetrics the "combinatorial explosion" (in terms of the amount of computation and memory) starts already for the cone $P_{6}^{2}$.

In the Table 12 we compare the small 2-hemimetric cones $P_{n}^{2}, N H M_{n}^{2}$ with the 1-hemimetric cones $C U T_{n}, M E T_{n}$ and their generalization in another direction: the cones $O M C U T_{n}, Q M E T_{n}$. Last two cones consist of all quasi-semimetrics on $V_{n}$ and of those obtained from oriented multicuts; see [DePa99] for the notions and results for them given in the Table 12. The cones $N H M_{n}^{2}$ and $Q M E T_{n}$ have, besides of generalizations of the usual triangle inequality, only nonnegativity facets. In the Table 12 , columns 3 and 4 give the number of extreme rays and facets, respectively; in parenthesis are given the numbers of their orbits. In column 5 are given the diameters of the skeleton and the ridge graphs of the cone specified in the row. In the Table 12, the number of orbits of extreme rays and the diameter for cones $Q M E T_{5}, N H M_{6}^{2}, P_{6}^{2}, P_{7}^{4}$ and dual $P_{6}^{3}, O M C U T_{5}$ are taken from recent work [DuDe01], as well as the exact value of the diameter for $N H M_{6}^{3}, N H M_{7}^{4}$ and for the duals of $P_{5}^{2}$, $\mathrm{Cut}_{7}$.

Incidences (to the extreme rays) of facets $T_{i j k, l}$ and $N_{i j k}$ on the cones $P_{4}^{2}=N H M_{4}^{2}$, $P_{5}^{2}, N H M_{5}^{2}$ amount to 3,14 and 22 , respectively, but they are different (4001 and 3939) on $H M_{6}^{2}$. Incidences of similar facets $T_{i j, k}$ (oriented triangular inequality, i.e. $d(x, y) \leq$ $d(x, z)+d(z, y)$ for a quasimetric $d), N_{i j}$ (nonnegativity inequality) are equal (to 7,43 ) on cones $O M C U T_{3}=Q M E T_{3}, O M C U T_{4}$, but they are different (78 and 80) on $Q M E T_{4}$.

For $n=4,5$ we observe that the ridge graphs of $H M_{n}^{2}$ and $N H M_{n}^{2}$ are induced subgraphs of the ridge graphs of $N H M_{n}^{2}$ and $P_{n}^{2}$, respectively. The similar property does not hold for the 1 -skeletons of those cones. For example, any extreme ray of the orbit $O_{2}$ is adjacent to $14,6,2$ members of the same orbit in the cones $P_{n}^{2}, N H M_{n}^{2}, H M_{n}^{2}$, respectively. Also, the ridge graph of $Q M E T_{4}$ is an induced subgraph of the ridge graph of $O M C U T_{4}$, but the skeleton of $\mathrm{OMCUT}_{4}$ is not an induced subgraph of the skeleton of $Q M E T_{4}$ (see [DePa99]). On the other hand, the ridge graph of $M E T_{n}$ and the skeleton of $C U T_{n}$ (for any $n$ ) have diameters 2 and 1 , respectively, and those graphs are induced subgraphs of the ridge graph of $C U T_{n}$ and of the skeleton of $M E T_{n}$, respectively (see Lemma 2.1 and Theorem 3.5 in [DeDe94])

Table 12
Some parameters of cones for small $n$

| cone | dimension | ext. rays (orbits) | facets (orbits) | diameters |
| :---: | :---: | :---: | :---: | :---: |
| $P_{m+2}^{m}=N H M_{m+2}^{m}$ | $\mathrm{~m}+2$ | $\binom{m+2}{2}(1)$ | $2 m+4(2)$ | $2 ; 2$ |
| $m \geq 3$ |  |  |  |  |
| $C U T_{3}=M E T_{3}$ | 3 | $3(1)$ | $3(1)$ | $1 ; 1$ |
| $P_{4}^{2}=N H M_{4}^{2}$ | 4 | $6(1)$ | $8(2)$ | $2 ; 3$ |
| $C U T_{4}=M E T_{4}$ | 6 | $7(2)$ | $12(1)$ | $1 ; 2$ |
| $O M C U T_{3}=Q M E T_{3}$ | 6 | $12(2)$ | $12(2)$ | $2 ; 2$ |
| $C U T_{5}$ | 10 | $15(2)$ | $40(2)$ | $1 ; 2$ |
| $M E T_{5}$ | 10 | $25(3)$ | $30(1)$ | $2 ; 2$ |
| $P_{5}^{2}$ | 10 | $25(2)$ | $120(4)$ | $2 ; 3$ |
| $N H M_{5}^{2}$ | 10 | $37(3)$ | $30(2)$ | $2 ; 2$ |
| $O M C U T_{4}$ | 12 | $74(5)$ | $72(4)$ | $2 ; 2$ |
| $Q M E T_{4}$ | 12 | $164(10)$ | $36(2)$ | $3 ; 2$ |
| $C U T_{6}$ | 15 | $31(3)$ | $210(4)$ | $1 ; 3$ |
| $M E T_{6}$ | 15 | $296(7)$ | $60(1)$ | $2 ; 2$ |
| $P_{6}^{3}$ | 15 | $65(2)$ | $4065(16)$ | $2 ; 3$ |
| $N H M_{6}^{3}$ | 15 | $287(5)$ | $45(2)$ | $3 ; 2$ |
| $P_{6}^{2}$ | 20 | $90(3)$ | $\geq 2095154(\geq 3086)$ | $2 ; ?$ |
| $N H M_{6}^{2}$ | 20 | $12492(41)$ | $80(2)$ | $3 ; 2$ |
| $O M C U T_{5}$ | 20 | $540(10)$ | $35320(194)$ | $2 ; 3$ |
| $Q M E T_{5}$ | 20 | $43590(229)$ | $80(2)$ | $3 ; 2$ |
| $P_{7}^{4}$ | 21 | $140(2)$ | $474390(153)$ | $2 ; 3$ |
| $N H M_{7}^{4}$ | $3692(8)$ | $63(2)$ | $3 ; 2$ |  |
| $C U T_{7}$ | 21 | $63(3)$ | $38780(36)$ | $1 ; 3$ |
| $M E T_{7}$ | 21 | $25226(46)$ | $105(1)$ | $3 ; 2$ |
| $C U T_{8}$ | 21 | $27(4)$ | $\geq 49604520(\geq 2169)$ | $1 ; ?$ |
| $P_{8}^{5}$ | 28 | $266(2)$ | $\geq 322416108(\geq 8792)$ | $? ; ?$ |
| $N H M_{8}^{5}$ | 28 | $8589(13)$ | $84(2)$ | $3 ; 2$ |

## 9 Conjectures for general $m, n$

Conjecture 5 The two partition m-hemimetrics $\alpha\left(S_{1}, \ldots, S_{m+1}\right)$ and $\alpha\left(T_{1}, \ldots, T_{m+1}\right)$ on $V_{n}$ are nonadjacent in the skeleton of $P_{n}^{m}$ if and only if there exist six different subsets $S_{i}, S_{j}, S_{k}$ and $T_{i^{\prime}}, T_{j^{\prime}}, T_{k^{\prime}}$, such that $S_{i} \cup S_{j}=T_{k^{\prime}}$ and $S_{k}=T_{i^{\prime}} \cup T_{j^{\prime}}$.

The conjecture holds for $m=1$ : all cut semimetrics are adjacent. It holds for $n-m=2$ : we have the graph $J(m+2,2)$. It also holds for $(m, n)=(2,5)$ and $(3,6)$.

Conjecture 6 The ridge graphs of $H M_{n}^{m}$ and of $N H M_{n}^{m}$ are induced subgraphs of the ridge graphs of $N H M_{n}^{m}$ and $P_{n}^{m}$, respectively.

Recall that the ridge graph of $N H M_{n}^{m}$ has two orbits of vertices: $F_{1}, F_{2}$, consisting of $(n-m-1)\binom{n}{m+1}$ simplex and $\binom{n}{m+1}$ nonnegativity inequalities.

Conjecture 7 The ridge graph $N H M_{n}^{m}$ satisfies:
(i) The $(m+1)$-simplex facet $T_{i_{1} \ldots i_{m+1}, i_{m+2}}$ is adjacent to all other facets, except the following $m+2$ facets:
all other $(m+1)$-simplex facets with the same support and $N_{i_{1} \ldots i_{m+1}}$;
(ii) $G\left(F_{2}\right)=\overline{J(n, 3)}$ for $m=2$ and $G\left(F_{2}\right)=K_{\binom{n}{m+1}}$ for $m \geq 3$.

Clearly, (i) implies that the restriction of the ridge graph on $F_{1}$ is $G\left(F_{1}\right)=K_{m+2, \ldots, m+2}$. It is easy to see that Conjecture 3 would imply that the diameter of the ridge graph of NHM $M_{n}^{m}$ is 2 (it was proved in [DeDe94] that the diameter of the ridge graph of $N H M_{n}^{1}=M E T_{n}$ is 2). In fact, to see it for $m=2$ consider all 3 types of pairs of nonadjacent vertices:
(i) let $x, y \in F_{1}$ have the same support, say, 1234. Suppose that $x_{124}=y_{124}=-1$. Then $N_{123}$ is a common neighbor for $x$ and $y$.
(ii) for $N_{123}$ and $N_{124}$, any tetrahedron facet $T_{134,2}$ is their common neighbor.
(iii) for $N_{123}$ and $T_{123,4}$, the facet $N_{345}$ is a common neighbor.

Conjecture 8 The extreme rays of $N H M_{n}^{m}$ include:
(i) any ray whose $R$-graph is an $R$-graph of an extreme ray of $N H M_{n-1}^{m-1}$;
(ii) every $(0,1)$-valued extreme ray of $N H M_{m+3}^{m}$ with $R$-graph $C_{i}(3 \leq i \leq m+3)$.

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