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Combinatorial aspects of jump codes

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Abstract

We develop an approach to jump codes concentrating on their combinatorial and symmetry properties. The main result is a generalization of a theorem previously proved in the context of isodual codes. We show that several previously constructed jump codes are instances of this theorem.

Keywords: Quantum error correction; Block designs; Orbits

1. Introduction

In this paper, we develop an approach to a recently introduced class of quantum error correcting codes called *jump codes* which were previously studied in [1,2,4] and elsewhere. We concentrate here on the combinatorial and symmetry aspects of these codes. Our main result is Theorem 4 which is a generalization of a theorem previously proved in a restricted setting in [4]. We demonstrate that a number of previously constructed jump codes are instances of this theorem or its variant.

2. Jump operators

The physical intuition underlying quantum error correction where the error is caused by the spontaneous jumps of energy levels was developed in [1,2], while the combinatorial

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approach to these codes was considered in [4]. We refer the reader to these sources for an explanation of unfamiliar notions, as well as the book by Nielsen and Chuang [11].

The physical intuition behind jump codes is the following. As for classical codes, further side information, for example, about the position of an error, might aid in the process of error correction [1,2]. We consider the quantum error correction model in which the errors are due to quantum jumps, i.e., the (excited) state $|1\rangle$ may spontaneously decay into the (ground) state $|0\rangle$; (see [1,2]). Furthermore, we assume that the decay rate is equal for all subsystems, i.e., independent of the position. Then the corresponding error operator is given by

$$a := |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If the operator acts only on the ith subsystem, the following notation will be used:

$$J_i := a_i := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes |0\rangle\langle 1| \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}. \tag{1}$$

The operators J_i are up to scalars just the Lindblad operators of quantum optics (see for example [1]).

3. Symmetries

The combinatorial prototypes of t-detecting jump codes $\mathscr{C} = (v, l, t)_k$ with l states on v qubits are known as SEED'S and were introduced in [3,4]. The connection between jump codes and SEED'S is by means of the encoding—*superposition of states*

$$|\mathscr{B}^{(i)}\rangle \longrightarrow \frac{1}{\sqrt{|\mathscr{B}^{(i)}|}} \sum_{B \in \mathscr{B}^{(i)}} |B\rangle, \tag{2}$$

where $|B\rangle := |\chi_B(1), \ldots, \chi_B(v)\rangle$ (χ_B is the characteristic function of B). Most of the known quantum error correcting codes and jump codes have been constructed using group representations, for example [6], or group actions. We will describe one such construction in some detail in this paper.

In what follows

$$|c_0\rangle, \ldots, |c_{l-1}\rangle$$

denotes the orthogonal basis of a t-detecting $\mathscr{C} = (v, l, t)_k$ jump code, where the parameter k refers to size k subsets of $[1 \ldots v]$. We write $|c_i\rangle$ instead of the $|\mathscr{B}^{(i)}\rangle$ above (cf. [4, Theorem 15]).

We will assume that some nontrivial subgroup of the symmetric group

$$G \leqslant S_v$$

acts transitively on the position coordinates of a jump code $\mathscr{C} = (v, l, t)_k$. The transitivity assumption reflects the physical requirement that a jump code must protect against jump errors in every coordinate position, i.e. all the position coordinates should be equivalent with respect to the action of G. We will frequently consider the induced action of G on the $\binom{v}{k}$ subsets of size k of the position coordinates.

Each code vector $|c_0\rangle, \ldots, |c_{l-1}\rangle$ of the Hilbert space $(C^2)^{\otimes l}$, is a superposition of length v binary vectors. The *supports* of these vectors are the k-subsets of the position coordinates of $\mathscr{C}$. If the k-subsets of the position coordinates corresponding to the $|c_i\rangle$ are distinct orbits of the group G, then $\{|c_0\rangle, \ldots, |c_{l-1}\rangle\}$ is a set of mutually orthogonal vectors. So the first requirement of a quantum code is met. (Knill and Laflamme [9] determined the general conditions which any quantum error-correcting code must satisfy.)

Lemma 1. *Suppose that the group $G \leqslant S_v$ acts transitively on the position coordinates of a jump code $C = (v, l, t)_k$, and G acts transitively on the k-subsets of the position coordinates encoding the $|c_0\rangle, \ldots, |c_{l-1}\rangle$. Then the k-subsets corresponding to each logical state $|c_i\rangle$ are the blocks of a 1-design.*

Proof. Let $C = (v, l, t)_k$ be a jump code and $B_1, B_2, \ldots,$ the blocks encoding a logical state $|c_i\rangle$. G acts on these blocks in a single orbit and transitively on the v coordinates. So for any two coordinates i_i and i_j, there is some element g in G so that $i_i^g = i_j$. Moreover the blocks $B_1^g, B_2^g, \ldots,$ belong to the same block-orbit. This ensures that i_i^g and i_j occur with the same frequency r_i in the blocks of the orbit; in other words the set of coordinate positions and the blocks $B_1, B_2, \ldots,$ form a 1-design.

From Lemma 1, we obtain the condition

$$vr_i = b_i k,$$

where the number of blocks of the 1-design corresponding to $|c_i\rangle$ is

$$b_i = |G|/|\mathrm{Stab}(B_i)|. \quad \square$$

4. A group-theoretic construction

In this section, we prove a result, Theorem 4, which provides a group-theoretic setting for the construction of jump codes. Unlike the setting for quantum codes which protect against Pauli errors (e.g. [6,7]), our construction is based on non-abelian groups. Several of the previously discovered jump codes can be recast in this setting or its variant which we develop in Section 5.

By a *block system* $C := (S, B)$ we mean a finite set $S = \{1, \ldots, n\}$ and a collection of subsets $B = \{B_1, \ldots, B_k\}$ of S all having the same cardinality. The automorphism group of C, $G = \mathrm{Aut}(C)$, consists of all the one-to-one mappings of S which preserve the blocks of B; i.e. for all $i \in S$ and $g \in G$, then $i \in B_k$ if and only if $i^g \in B_k^g$. The *dual* block system $C^\perp$ has the same underlying set S, but the blocks of $C^\perp$ are the images under a one-to-one mapping σ of the blocks of B.

A familiar example of these notions is that of 1-designs [5]. In these (regular) block systems every element of S occurs in the same number λ of blocks $B_{i_1}, \ldots, B_{i_l}$. The automorphism group which acts on the *points* and the blocks of S preserves the *incidence* of point-block pairs.

Lemma 2. *Suppose that C and $C^\perp$ are dual block systems with respect to the map $\sigma : C \to C^\perp$ and that $G = \mathrm{Aut}(C)$ is the automorphism group of C. Then $G = \mathrm{Aut}(C^\perp)$ if and only if*

$$\sigma G = G\sigma. \tag{3}$$

Proof. Suppose that $\sigma G = G\sigma$, then every element of the underlying point set of C^- can be written as i^σ where $i \in C$. Let $g \in \mathrm{Aut}(C)$ then $(i^\sigma)^g = (i^h)^\sigma$ for some $h \in \mathrm{Aut}(C)$. But $i^h \in C$ hence $(i^h)^\sigma \in C^\perp$ and thus $G \leqslant \mathrm{Aut}(C^\perp)$. Since $\mathrm{Aut}(C)$ and $\mathrm{Aut}(C^\perp)$ are isomorphic and finite, we have that $G = \mathrm{Aut}(C^\perp)$.

Assume now that $G = \mathrm{Aut}(C^\perp)$. For any $g \in \mathrm{Aut}(C)$ we have that $\sigma^{-1}g^{-1}\sigma \in \mathrm{Aut}(C^\perp)$, and by our hypothesis there is some $h \in G$ so that $\sigma^{-1}g^{-1}\sigma = h$, giving $g^{-1}\sigma = \sigma h$. But this implies that $\sigma G = G\sigma$. □

Lemma 3. *If C and $C^\perp$ are dual block systems then the orbits of G on the blocks $B^\perp$ are the images under σ of the orbits of G on the blocks B.*

Proof. Let $O = \{B_{i_1}^\sigma, \ldots, B_{i_n}^\sigma\}$ be the image under σ of some block-orbit of G acting on C. For any $g \in G$, $O^g = \{B_{i_1}^{\sigma g}, \ldots, B_{i_n}^{\sigma g}\}$. Since σ normalizes G, $O^g = \{B_{i_1}^{h\sigma}, \ldots, B_{i_n}^{h\sigma}\}$ for some $h \in G$. Since $\{B_{i_1}, \ldots, B_{i_n}\}$ is a G orbit, it follows that $O^g = O$ and that O is a G orbit. □

Theorem 4. *Suppose C and $C^\perp$ are dual block systems of a set $S := \{1, \ldots, n\}$ with non-trivial automorphism group $G = \mathrm{Aut}(C)$. Let G act on C, and O_{i_j} and $O_{i_j}^\sigma$ be disjoint orbits of blocks of cardinality i in C and $C^\perp$, respectively. Suppose that $t < i$. For each integer s $1 \leqslant s \leqslant t$ decompose the s-subsets of $\{1, \ldots, n\}$ into orbits $\Theta_s = \{\Theta_{s_1}, \ldots, \Theta_{s_m}\}$ with respect to the induced action of G on the s-subsets of S. Suppose that for each s the constituent orbits of Θ_s are preserved by σ, i.e., $\Theta_{s_i}^\sigma = \Theta_{s_i}$ (as sets) for $s_1, \ldots, s_m$. Then the i-blocks in O_{i_j} and $O_{i_j}^\sigma$ give a t-SEED$(n, i; 2)$.*

Proof. The orbits O_{i_j} and $O_{i_k}^\sigma$ are disjoint, so we have only to establish the generalized t-design property.

Let $\{p_1, \ldots, p_s\}$ be an s-subset of each of the blocks $B_1, \ldots, B_m$ of O_{i_j}, where $m \geqslant 1$ is maximal. Then $\{p_1^\sigma, \ldots, p_s^\sigma\}$ is a subset of each of the blocks $B_1^\sigma, \ldots, B_m^\sigma$ of $O_{i_j}^\sigma$. Now $\{p_1, \ldots, p_s\} \in \Theta_{s_j}$ for some index s_j and by σ-invariance $\{p_1^\sigma, \ldots, p_s^\sigma\} \in \Theta_{s_j}$. Since Θ_{s_j} is a G orbit there is an element $h \in G$ such that $\{p_1, \ldots, p_s\} = \{p_1^{\sigma h}, \ldots, p_s^{\sigma h}\}$. This implies that $\{p_1, \ldots, p_s\}$ is a subset of each of the m blocks $B_1^{\sigma h}, \ldots, B_m^{\sigma h}$ of $O_{i_j}^\sigma$. Now using $\sigma G = G\sigma$, we have that $\{B_1^{\sigma h}, \ldots, B_m^{\sigma h}\} = \{B_1^{g\sigma}, \ldots, B_m^{g\sigma}\}$ for some $g \in G$. Since the blocks $B_1, \ldots, B_m$ belong to a G orbit, so do the m blocks $B_1^g, \ldots, B_m^g$.Hence $\{p_1, \ldots, p_s\}$ is a subset of m blocks of the orbit $O_{i_j}^\sigma$.

The case where $\{p_1, \ldots, p_s\}$ is not covered by any blocks is handled by a similar argument by noting that if $\{p_1, \ldots, p_s\} \not\subset B_i$, then $\{p_1^\sigma, \ldots, p_s^\sigma\} \not\subset B_i^\sigma$. In this way, we have shown that the *normalized multiplicity* condition (Definition 12 of [4]) is satisfied. By Theorem 15 of [4], the orbits O_{i_j} and $O_{i_j}^\sigma$ give a t-SEED$(n, i; 2)$. □

Remark 5. It is clear that Lemmas 2 and 3 can be extended to block systems $C_0, \ldots, C_m$ defined on a common set S for more than two logical states. In this situation the map σ,

$$C_0 \xrightarrow{\sigma} C_1 \rightarrow \cdots \xrightarrow{\sigma} C_m$$

satisfies the normalizing condition: $\sigma G = G\sigma$, where $G = \mathrm{Aut}(C_{i-1}) = \mathrm{Aut}(C_i)$ for all i. Under this hypothesis the argument in Theorem 4 can be modified and yields an m-state t-SEED$(n, i; m)$.

Examples of jump codes in this setting are given in Section 6.

5. A variant of the construction

A family of optimal even length jump codes protecting against single jumps was described in [2,4]. We will show that a variant of Theorem 4 provides a setting for these jump codes.

Theorem 6 (*Alber et al. [2] and Beth et al. [4]*). *For even length n, the*

$$\left(n, \frac{1}{2}\binom{n}{n/2}, 1\right)_{n/2}$$

jump code with basis states $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ (where $\overline{x}$ denotes the binary complement of the bitstring x) is optimal.

To motivate the further development of Theorem 4, we consider the stabilizers of the basis states of these jump codes. We call the basis state $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ corresponding to the pair $([1, 2, \ldots, n/2], [n/2 + 1, n/2 + 2, \ldots, n])$, the *standard* basis state.

Theorem 7. *The stabilizer $G \leqslant S_n$ of each basis state $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ of the jump codes described in Theorem 6 is isomorphic to $(S_{n/2} \times S_{n/2})S_2$. The stabilizer of the standard basis state is generated by the involutions*: $\{(1, 2), (2, 3), \ldots, (n/2 - 1, n/2)\}$, $\{(n/2 + 1, n/2 + 2), (n/2 + 2, n/2 + 3), \ldots, (n - 1, n)\}$, *and* $(1, n/2 + 1)(2, n/2 + 2) \cdots (n/2, n)$. *$G$ is a self-normalizing subgroup of S_n. The stabilizers of each basis state form a single conjugacy class of subgroups of S_n.*

Proof. It is apparent that stabilizer G of any basis state $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ has a subgroup which is isomorphic to $(S_{n/2} \times S_{n/2})S_2$. This subgroup acts on any basis state as follows. The first factor $S_{n/2}$ permutes the ones coordinates of $|x\rangle$ and the zeroes of $|\overline{x}\rangle$, the second factor $S_{n/2}$ plays the same role with the zeroes instead of the ones. The S_2 factor swaps the ones and zero coordinates, thus $|x\rangle$ is mapped to a $|\overline{x}\rangle$ by the action of this S_2.

Let $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ and $1/\sqrt{2}(|x'\rangle + |\overline{x'}\rangle)$ be two distinct basis states, and G and H, respectively, their stabilizers in S_n. The symmetric group S_n acts m-transitively on m-tuples for all $m \leqslant n$. This action induces a mapping on the logical states, i.e. there is some $g \in S_n$ mapping $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ to $1/\sqrt{2}(|x'\rangle + |\overline{x'}\rangle)$. Hence $g^{-1}Hg = G$, and the stabilizers of the basis states form a single conjugacy class in S_n. The size of this conjugacy class is the index $|S_n : N_{S_n}(G)|$, and this coincides with the number of basis states in the code. Since S_n permutes the basis states transitively, the number of basis states is $|S_n : G|$. Hence $|G| = |N_{S_n}(G)|$ and thus $G = N_{S_n}(G)$. This establishes the last two assertions.

Next we show that the full stabilizer of every basis state is isomorphic to $(S_{n/2} \times S_{n/2})S_2$. Let G be the stabilizer of a basis state. Then G contains a subgroup isomorphic to $(S_{n/2} \times S_{n/2})S_2$, and hence

$$\frac{1}{2}\binom{n}{n/2} = |S_n : G| \leqslant |S_n : (S_{n/2} \times S_{n/2})S_2| = \frac{1}{2}\binom{n}{n/2},$$

or

$$|S_n : (S_{n/2} \times S_{n/2})S_2| = |S_n : G|.$$

Thus $|G| = |(S_{n/2} \times S_{n/2})S_2|$ and we conclude that $G = (S_{n/2} \times S_{n/2})S_2$.

To prove the generational part, we note that with respect to a standard basis, the involutions $\{(1,2), (2,3), \ldots, (n/2-1, n/2)\}$, generate the first factor of $(S_{n/2} \times S_{n/2})S_2$. The second factor is generated by the involutions $\{(n/2+1, n/2+2), (n/2+2, n/2+3), \ldots, (n-1, n)\}$. The generators in the first set commute with all the generators in the second set, and the intersection of the groups generated by the two sets of generators is the identity. Thus collectively the involutions generate a direct product $H := S_{n/2} \times S_{n/2}$, which has index two in the full stabilizer. The swapping involution $(1, n/2+1)(2, n/2+2)\cdots(n/2, n)$ belongs to the full stabilizer but is not in H. Hence this involution and the previous generators must generate the full stabilizer. □

Remark 8. In Theorem 7 we have listed the generators for the stabilizer of a basis state which is in standard form. The generators of the stabilizers of the other states of the code can be obtained from the standard basis by conjugation.

By Theorem 7 the normalizing condition (3) cannot be satisfied for more than one state. So we must drop the assumption that G is the common automorphism group of the different states. Instead a weaker condition holds, namely that the stabilizers of the different states of the jump codes in Theorem 6 form a single conjugacy class, and the stabilizers act *equivalently* on the blocks underlying the basis states.

Recall that, if the permutation groups G and H act on sets Ω_1 and Ω_2, then (G, Ω_1) and (H, Ω_2) are equivalent, if there exists a one-to-one mapping $\varepsilon : \Omega_1 \to \Omega_2$ and an isomorphism $\varphi : G \to H$ so that

$$(i^g)^\varepsilon = (i^\varepsilon)^{g^\varphi} \tag{4}$$

holds for all $i \in \Omega_1$ and all $g \in G$.

Lemma 9. *Let G and H be the stabilizers of two basis states $|c_i\rangle := 1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ and $|c_j\rangle := 1/\sqrt{2}(|x'\rangle + |\overline{x'}\rangle)$. Then G and H are equivalent permutation groups on the block sets underlying $|c_i\rangle$ and $|c_j\rangle$.*

Proof. The map ε between the block sets arises as follows. There is some $g \in S_n$ which maps the support set of $|x\rangle$ to the support set of $|x'\rangle$, and hence maps the support sets of $|\overline{x}\rangle$ and $|\overline{x'}\rangle$ to each other. Since G and H are conjugate subgroups of S_n there is an isomorphism φ from G to H. Condition (3) holds because conjugacy is a special case of equivalence.

We can state the variant of Theorem 4. □

Theorem 10. *The basis states of the jump codes described in Theorem 6 can be partitioned into orbits of lengths at least two with respect to a local frequency preserving permutation σ of the coordinate positions.*

Proof. Let σ be a permutation of the code coordinates $\sigma := (1)(2, \ldots, n)$; i.e. the first coordinate is fixed and the remaining coordinates are permuted in a cycle of length $n - 1$. For any basis state $|c_i\rangle := 1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$, σ cannot swap $|x\rangle$ and $|\overline{x}\rangle$ because it fixes the first coordinate. Nor can σ fix $|x\rangle$ or $|\overline{x}\rangle$; it does not belong to the $S_{n/2}$ factors of the stabilizer of a state. Hence σ acts fixed-point freely on the basis states, and splits them into orbits of lengths at least two. We show that the local frequency condition is preserved by σ.

Let $|c_i\rangle := 1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ be a basis state and $|c_i\rangle^\sigma := 1/\sqrt{2}(|x\rangle^\sigma + |\overline{x}\rangle^\sigma)$ its image under σ. Then B_x and $B_{\overline{x}}$—the support sets of $|x\rangle$ and $|\overline{x}\rangle$, partition the coordinate positions, i.e. any coordinate i belongs to either B_x or $B_{\overline{x}}$, but not to both sets. Since σ is a permutation of $[1 \ldots n]$, $|B_x^\sigma| = |B_x|$ and $|B_{\overline{x}}^\sigma| = |B_{\overline{x}}|$. Moreover B_x^σ is disjoint from$B_{\overline{x}}^\sigma$. For if there was some $i^\sigma \in B_x^\sigma \cap B_{\overline{x}}^\sigma$, then i would belong to $B_x \cap B_{\overline{x}}$, a contradiction. Hence B_x^σ and $B_{\overline{x}}^\sigma$ is another partition of $[1 \ldots n]$. We conclude that any coordinate $i \in [1 \ldots n]$ occurs with frequency one in $|c_i\rangle$ and in $|c_i\rangle^\sigma$. Thus at least two code states are accounted for by the σ-cycle $|c_i\rangle \to |c_i\rangle^\sigma \ldots$. Starting from a code state $|c_j\rangle$ not belonging to the previous σ-cycle, we can form another local frequency preserving σ-cycle $|c_j\rangle \to |c_j\rangle^\sigma \ldots$. This process can be continued until all the states of the code appear in the σ-cycles.

By the transitivity of S_n on the coordinate positions, we are free to choose any other coordinate and cyclicly permute the remaining coordinates. Applying this transformation gives an equivalent decomposition of the basis states into σ-cycles, i.e. the resulting cycles will have the same lengths as those induced by $(1)(2, \ldots, n)$. □

We provided a detailed analysis of the optimal jump codes of Theorem 7. It is an open question whether the same codes are optimal in the case of an odd number of coordinates. In [8] Grassl conjectures that they are also optimal for an odd number of coordinates. Our analysis could have some bearing on this question.

6. Further examples

Theorem 4 was originally formulated in the context of isodual codes [4], so the associated jump codes had just two logical states $|c_0\rangle$, $|c_1\rangle$, which were encoded by the vectors of

Hamming weight w in the code and by the corresponding vectors of its isodual partner. (An isodual code is a binary linear code which is *equivalent* to its dual code; see [10].) We now consider more diverse examples of jump codes which illustrate the methods developed in this paper.

Consider the six qubit code $(6, 2, 2)_3$ which protects against two jump errors [4]. This code can be realized in two distinct ways. The two logical states $|c_0\rangle, |c_1\rangle$ of this code can be encoded by the orbits of two distinct groups of order 24 which act on the six coordinate positions. We call the first group G_{24}, the second is the familiar symmetric group S_4.

In the G_{24} representation the encoding of the logical states expressed as support sets is

$$|c_0\rangle \to \{\{1,2,3\},\{1,2,4\},\{1,5,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\},$$

$$|c_1\rangle \to \{\{1,2,5\},\{1,2,6\},\{1,3,4\},\{2,3,4\},\{3,5,6\},\{4,5,6\}\}.$$

The block sets in the G_{24} representation form a pair of disjoint $1-(6,3,3)$ designs which are unique up to equivalence. The group G_{24} has a centre of order two generated by $(1,2)(3,4)(5,6)$; the derived subgroup is the Klein 4-group. A permutation of the coordinates $\sigma := (3,5)(4,6)$ maps $|c_0\rangle$ to $|c_1\rangle$. The group generated by G_{24} and σ has order 48, and the two orbits of G_{24} are fused into a single orbit.

The *tetrahedral* representation provides a combinatorial description of this jump code. We label the six edges of a tetrahedron. Then $|c_0\rangle$ corresponds to the four faces, and $|c_1\rangle$ to the four claws which complete each face to a tetrahedron. The four faces and claws are orbits of a symmetric group S_4 acting on the six edges of the tetrahedron. The generators of this S_4 correspond to the rotations and the reflections of the tetrahedron. The associated transformation σ mapping $|c_0\rangle$ to $|c_1\rangle$ is induced by the geometric self-duality of the tetrahedron.

Next we look at the two $(8,3,3)_4$ jump codes of [1,8].

The first code is associated with the group G of order 48

$$G := \langle (1,2,3)(5,7,8), (1,2,3)(5,8,6), (1,2,4)(6,7,8)\rangle$$

which has three orbits $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2$ of length 12

$$\{1,2,5,6\}^G, \quad \{1,2,5,7\}^G, \quad \{1,2,5,8\}^G$$

on the 4-subsets of eight coordinate positions. The permutation $\sigma = (2,3,4)$ acts on the basis states as

$$\mathcal{O}_0 \xrightarrow{\sigma} \mathcal{O}_1 \xrightarrow{\sigma} \mathcal{O}_2$$

and satisfies the condition $\sigma G = G\sigma$. The orbits of G on the 3-subsets, 2-subsets, and 1-subsets of the eight coordinates are preserved by σ. Thus the conditions of Theorem 4 are satisfied.

The three error correcting $(8,3,3)_4$ jump code of [8] provides an hybrid setting for Theorem 4. This time the group is $G \simeq PSL_2(7)$, the simple group of order 168

$$G := \langle (1,2,3,4)(5,6,7,8), (1,3)(2,8)(4,5)(6,7)\rangle.$$

G has three orbits $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2$

$$\{1, 2, 5, 6\}^G, \quad \{1, 2, 5, 8\}^G, \quad \{1, 4, 5, 6\}^G$$

of lengths 14, 14 and 42, respectively, on the 4-subsets of eight coordinate positions. The permutation $\sigma := (1, 2, 5, 6, 3, 8)$ acts on two orbits as

$$\mathcal{O}_0 \xrightarrow{\sigma} \mathcal{O}_1$$

and fixes the third orbit $\mathcal{O}_2^\sigma = \mathcal{O}_2$.

The normalizing condition (3) holds. The orbits of G on the 3-subsets, 2-subsets and 1-subsets of the eight coordinates are preserved by σ. Thus Theorem 4 applies to orbits $\mathcal{O}_0$ and $\mathcal{O}_1$.

To complete the argument, we note that G acts transitively on the 3-subsets, 2-subsets, and 1-subsets of the eight coordinates. Thus the frequency of occurrence the s-subsets in the blocks of $\mathcal{O}_3$ depends only on the s-value. Now $|\mathcal{O}_2| = 3|\mathcal{O}_1| = 3|\mathcal{O}_0|$, and we can check that the frequency of occurrence of each s-subset in $\mathcal{O}_2$ is three times that of $\mathcal{O}_1$ and $\mathcal{O}_0$. Thus the normalized multiplicity condition is satisfied. The orthonormal basis vectors $|c_0\rangle$, $|c_1\rangle$, $|c_2\rangle$ of this jump code are encoded by the blocks of the orbits $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2$.

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References

[1] G. Alber, T. Beth, C. Charnes, A. Delgado, M. Grassl, M. Mussinger, Detected jump-error correcting quantum codes, quantum error designs and quantum computation, Phys. Rev. A 68 (2003) 012316.

[2] G. Alber, M. Mussinger, A. Delgado, Quantum information processing and error correction with jump codes, in: Leuchs, Gerd, Beth (Eds.), Quantum Technology, Wiley-VCH, Berlin, 2003.

[3] T. Beth, A class of designs protecting against quantum jumps, in: A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel, J.A. Thas (Eds.), Finite Geometries, Report no. 52/2001, Mathematisches Forschungsinstitut, Oberwolfach, p. 4.

[4] T. Beth, C. Charnes, M. Grassl, G. Alber, A. Delgado, M. Mussinger, A new class of designs which protect against quantum jumps, Des. Codes Cryptogr. 29 (2003) 51–70.

[5] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Encyclopedia of Mathematics, second ed., Cambridge University Press, Cambridge, 1999.

[6] A.R. Calderbank, E.M. Rains, P.W. Shor, N.J.A. Sloane, Quantum error correction and orthogonal geometry, Phys. Rev. Lett. 78 (3) (1997) 405–408.

[7] A.R. Calderbank, E.M. Rains, P.W. Shor, N.J.A. Sloane, Quantum error correction via codes over GF(4), IEEE Trans. Inform. Theory 44 (4) (1998) 1369–1387.

[8] M. Grassl, Quantum codes for detected quantum jumps, Notes IAKS, 2003.

[9] E. Knill, R. Laflamme, Theory of quantum error-correcting codes, Phys. Rev. A 55 (2) (1997) 900–911.

[10] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes, North-Holland, Amsterdam, 1977.

[11] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.