# ASYMPTOTICS OF CHARACTERS OF SYMMETRIC GROUPS, GENUS EXPANSION AND FREE PROBABILITY 

PIOTR ŚNIADY


#### Abstract

The convolution of indicators of two conjugacy classes on the symmetric group $S_{q}$ is usually a complicated linear combination of indicators of many conjugacy classes. Similarly, a product of the moments of the Jucys-Murphy element involves many conjugacy classes with complicated coefficients. In this article we consider a combinatorial setup which allows us to manipulate such products easily: to each conjugacy class we associate a two-dimensional surface and the asymptotic properties of the conjugacy class depend only on the genus of the resulting surface. This construction closely resembles the genus expansion from the random matrix theory. As the main application we study irreducible representations of symmetric groups $S_{q}$ for large $q$. We find the asymptotic behavior of characters when the corresponding Young diagram rescaled by a factor $\mathrm{q}^{-1 / 2}$ converge to a prescribed shape. The character formula (known as the Kerov polynomial) can be viewed as a power series, the terms of which correspond to two-dimensional surfaces with prescribed genus and we compute explicitly the first two terms, thus we prove a conjecture of Biane.


## 1. Introduction

1.1. Irreducible representations of large symmetric groups. Irreducible representations of symmetric groups are in the one-to-one correspondence with Young diagrams and due to algorithms such as Murnaghan-Nakayama rule or Robinson-Littlewood rule the essential questions concerning representations and characters of symmetric groups can be answered by a combinatorial study of the corresponding Young diagrams [Ful97]. However, when one studies the asymptotic properties of large symmetric groups, the work with Young tableaux becomes cumbersome and one is in the need to find another object which would encode the same information in a more convenient way. Alternatively, one can state the above problem as follows: a typical partition of a large number $q$ (or equivalently, a Young diagram with $q$ boxes) is a collection of at least $\sqrt{q}$ numbers, hence it contains a lot of information. Nevertheless, we can expect that one does not need to know all this information to extract (with a reasonable accuracy) the properties of characters and representations. Therefore the question arises how
to compress the information about the Young diagrams in the most efficient way.

It turns out that the right tool for development of the above program is the notion of the transition measure of a Young diagram which was introduced by Kerov Ker93, Ker99]. The transition measure of a Young diagram is a probability measure on the real line-therefore it has a much more analytic nature than a combinatorial notion of a Young diagram and for this reason it is very appealing for our purposes. Yet another advantage of this object is that it is possible to characterize it in many ways [Bia98, Oko00].

Of course, a question arises how to relate the transition measure of a Young diagram with the values of the corresponding characters. In this article we find the appropriate formula in a form of an asymptotic series the terms of which correspond to two-dimensional surfaces with a prescribed genus.

The first result in this direction was obtained by Biane [Bia98]: he showed that if the sequence of Young diagrams (after appropriate scaling) converges to some limit shape then the leading term corresponds to surfaces with genus zero and therefore this leading term can be computed by the means of Voiculescu's free probability theory [VDN92, HP00]. Okounkov [Oko00] was studying the distribution of the length of the first rows of a large random Young diagram sampled according to the Plancherel measure; he showed that the limit distribution coincides with the limit distribution of the biggest eigenvalues of a random matrix in the Gaussian Unitary Ensemble (GUE). The main idea of his proof was the observation that the both the formula for the moments of a GUE random matrix and the formula for the moments of the transition measure of a random Young diagram can be viewed as series the terms of which correspond to two-dimensional surfaces with prescribed genus. One can view results of this article as an attempt to simplify some of the arguments of Biane [Bia98] and simultaneously to provide better asymptotic expansion. Our results are closely related to those of Okounkov [Oko00] with the difference that we do not study the connection with random matrices but on the bright side we do not restrict ourselves to the case of the Plancherel measure.

In the remaining part of the introduction we will present a more detailed view of the methods and the results of this article.
1.2. The main technical problem: convolution in $S_{q}$. Convolution of central functions $f, g \in \mathbb{C}\left(S_{q}\right)$ has a very simple structure if we write them as linear combinations of characters: if

$$
f(\pi)=\sum_{\lambda \vdash q} a_{\lambda} \frac{\chi^{\lambda}(e) \chi^{\lambda}(\pi)}{q!}, \quad g(\pi)=\sum_{\lambda \vdash q} b_{\lambda} \frac{\chi^{\lambda}(e) \chi^{\lambda}(\pi)}{q!}
$$

then

$$
(f g)(\pi)=\sum_{\lambda \vdash q} a_{\lambda} b_{\lambda} \frac{\chi^{\lambda}(e) \chi^{\lambda}(\pi)}{q!}
$$

where the product $f g$ denotes a product of elements of the group algebra, i.e. the convolution of functions. However, in many cases we cannot afford the luxury of using the character expansion, for example in the case when we actually try to find the asymptotics of characters. For this reason we should find some other families of central functions on $S_{q}$ for which the convolution would have a relatively simple form.

In Section 2.1.10 we shall define such a family $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(S_{q}\right)$ which has a particularly simple structure: $\Sigma_{k_{1}, \ldots, k_{m}}$ is (up to a normalizing factor) an indicator of the conjugacy class of permutations with a prescribed cycle decomposition $k_{1}, \ldots, k_{m}$. This great simplicity has very appealing consequences: it will be very easy to evaluate functions $\Sigma$ on permutations and also it will be very simple to write any central function on $S_{q}$ as a linear combination of functions $\Sigma$ (operations which are somewhat cumbersome for characters). This object was introduced and studied by Ivanov and Kerov [IK99].

A question arises how to express a product $\Sigma_{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{m}}} \cdot \Sigma_{\mathrm{l}_{1}, \ldots, \mathrm{l}_{\mathrm{n}}} \in \mathbb{C}\left(\mathrm{S}_{\mathrm{q}}\right)$ as a linear combination of some other normalized conjugacy class indicators $\Sigma$. This problem is very closely related to calculation of the, so called, connection coefficients [GJ96, GJL01, Gou90, Gou94, GS98]. The latter problem asks for the number of solutions of the equation $\pi_{1} \pi_{2} \pi_{3}=e$ where $\pi_{1}, \pi_{2}, \pi_{3} \in S_{q}$ must have a prescribed cycle structure. The formulas for the connection coefficients are available in many concrete cases, however they are not satisfactory for the purpose of this article.
1.2.1. The first main result: calculus of partitions and genus expansion. It turns out that the solution to the above problem can be obtained by considering some more general objects. In Section 3.1.1 we shall define normalized conjugacy class indicators $\Sigma_{\pi}$ which are indexed no longer by sequences of integers but by partitions of finite ordered sets. We also equip partitions with an explicit multiplicative structure in such a way that $\Sigma$ becomes a homomorphism. For this reason we can in fact forget in applications about the symmetric group $S_{q}$ and perform all calculations in the partition language.

In applications we are interested in the asymptotic behavior of the contribution of various conjugacy classes when $\mathrm{q} \rightarrow \infty$. It turns out that the order of such a contribution of $\Sigma_{\pi}$ can be read directly from the genus of a twodimensional surface associated to the partition $\pi$; in this way our description is very similar to the genus expansion for random matrices [Zvo97]. Even more striking is that the degree $q$ of the symmetric group $S_{q}$ does not enter into the multiplicative structure of partitions and therefore the calculus of
partitions is able to provide statements about symmetric groups which do not depend on q. Another great advantage is that the moments of the JucysMurphy element (which are closely related to the moments of the transition measure) can be easily expressed within our calculus. All these features make the calculus of partition and its genus expansion a perfect tool for the study of symmetric groups.

Unfortunately, one of our ultimate goals-formulas which relate characters and the moments of the transition measure-turn out to be quite involved and we need also some other tools to solve this problem. As we shall see in the following, these tools are provided by the free probability theory.
1.3. Free probability and free cumulants. In this paper we study free cumulants of the transition measure of a given Young diagram. The notion of free cumulants plays a fundamental role in the free probability, a theory which was initiated by Voiculescu in order to answer some old questions in the theory of operator algebras but it soon evolved into an exciting self-standing theory with many links to other fields, see [HP00, VDN92, Voi95, Voi00]. This theory can be viewed as a highly non-commutative probability theory in which the notion of independence of random variables was replaced by a non-commutative notion of freeness. For the purpose of this article we shall concentrate on the combinatorial aspect of this theory connected with non-crossing partitions [Kre72] and mentioned above free cumulants, which were introduced by Speicher [Spe94, Spe97, Spe98].
1.3.1. Free convolution and free cumulants. For probability measures $\mu$, $\nu$ on the real line one can define their free convolution $\mu \boxplus \nu$ which is also a probability measure on the real line. One of the first problems of free probability was to study this convolution. The simplest approach is to consider the sequence of moments

$$
\begin{equation*}
M_{i}(\mu)=\int_{\mathbb{R}} x^{i} d \mu(x) \tag{1.1}
\end{equation*}
$$

of a given measure $\mu$ and ask what is the relation between the moments of $\mu \boxplus \nu$ and the moments of $\mu$ and $\nu$. Unfortunately, in turns out that the answer is given by a sequence of quite complicated polynomials. The solution to the above problem of finding a nice description of the free convolution is given by free cumulants. To the measure $\mu$ we assign a sequence of its free cumulants $R_{1}(\mu), R_{2}(\mu), \ldots$; every free cumulant $R_{i}(\mu)$ is a certain polynomial in the moments of $\mu$, and conversely, every moment of $\mu$ can be expressed as a certain polynomial in free cumulants; therefore the sequence of moments and the sequence of free cumulants carry the same
information. The advantage of the notion of free cumulants over the notion of moments is the simplicity of the relation between free cumulants: $R_{n}(\mu \boxplus v)=R_{n}(\mu)+R_{n}(v)$. In this article we are not interested in the study of free convolution; our point is that free cumulants have a miraculous property of simplifying certain complicated non-commutative relations.
1.3.2. Free cumulants of the transition measure. In this article we study the relation between the transition measure $\mu^{\lambda}$ of a Young diagram $\lambda$ and the characters of the corresponding irreducible representation. The simplest idea would be to describe the transition measure in terms of its moments $M_{i}\left(\mu^{\lambda}\right)$, however-as we already mentioned-the relation between these moments and characters turns out to be quite complicated. Similarly as in the example from the above Section 1.3 .1 free cumulants can simplify dramatically the complexity of the formulas: it was shown by Biane [Bia98] that for each $n$ the value $\Sigma_{n}$ of the normalized character on the $n$-cycle can be expressed as a certain polynomial in free cumulants $R_{2}, R_{3}, \ldots$ of the transition measure of the corresponding Young diagram. Furthermore, the leading term is particularly simple, namely

$$
\begin{equation*}
\Sigma_{n}=R_{n+1}+\text { lower degree terms. } \tag{1.2}
\end{equation*}
$$

The fundamental property of this polynomial, called Kerov polynomial, is that it is universal for all Young diagrams.

Kerov polynomials seem to have very interesting combinatorial properties but not too much about them is known [Bia03, Sta02a, Sta02b]. Kerov conjectured that all coefficients of Kerov polynomials are non-negative integers; this conjecture seems to be quite difficult. Unfortunately, Kerov's conjecture does not seem to have interesting applications, but on the other hand its proof might be much more interesting than the conjecture itself: Biane [Bia03] suggested that the coefficients of Kerov polynomials might be interpreted as the number of certain intervals in the decomposition of the Cayley graph of the symmetric group and it would be very interesting to state Biane's conjecture in a more concrete form.
1.3.3. The second main result: second-order expansion for Kerov polynomials. One of the main results of this paper is a more precise asymptotic expansion of characters

$$
\begin{align*}
& \text { 1.3) } \quad \Sigma_{n}=R_{n+1}+  \tag{1.3}\\
& \sum_{\substack{m_{2}, m_{3}, \ldots \geq 0 \\
2 m_{2}+3 m_{3}+4 m_{4}+\cdots=n-1}} \frac{1}{4}\binom{n+1}{3}\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots} \prod_{s \geq 2}\left((s-1) R_{s}\right)^{m_{s}}+
\end{align*}
$$

which was conjectured by Biane [Bia03]. In other words: we calculate explicitly the coefficients of Kerov polynomials corresponding to the two highest-degree terms. We also outline an algorithm which can provide such an expansion of any order.

After the first version of this article was made public SŚni03a, Śni03b Goulden and Rattan [GR05] using different methods found an explicit formula for all coefficients of Kerov polynomials. The Kerov's positivity conjecture remains open until now.
1.4. Applications: Fluctuations of random Young diagrams. The methods presented in this article are very useful in the study of the asymptotic properties of symmetric groups. An example of such an application is presented in our subsequent work [Śni05] where we study the distribution of a random Young diagram contributing to a given reducible representation of the symmetric group $S_{q}$ in the limit $q \rightarrow \infty$. We prove that for a large class of such representations the fluctuations of the Young diagram around the limit shape are asymptotically Gaussian. Our main tool in [Śni05] is the calculus of partitions introduced in this article.
1.5. Overview of the article. In Section 2 we introduce the main actors: the normalized indicators of conjugacy classes $\Sigma_{k_{1}, \ldots, k_{1}}$ and the JucysMurphy element J. We also outline briefly how important properties of Young diagrams and representations of $S_{q}$ are encoded by the distribution of J. It will be convenient to use in this article the language of the probability theory ('distribution' and 'moments' of 'random variables') therefore we also introduce the necessary conventions, however a reader might easily translate all statements into her/his favorite language. We also recall briefly some notions connected with partitions of finite ordered sets: fat partitions and non-crossing partitions.

In the central part of this paper, Section 3, we introduce the calculus of partitions and study its properties. This part is written as a 'user-friendly user guide': we postponed all technical and boring proofs to Section 5 in order to allow the readers to use the calculus of partitions without troubling why the machinery works.

In Section 4 we study general properties of free cumulants of the JucysMurphy element; in particular in Section 4.4 we find explicitly the secondorder asymptotic expansion of these cumulants.

Section 5 is devoted to proofs of some technical results.
In Section 6 we present some final remarks. Especially interesting are Sections 6.1 and 6.2 where we present connections with the work of Biane [Bia98] and Okounkov [Oko00]. Section 6.2 provides a natural geometric
interpretation of pushing partitions and it can be used to rephrase the results of Okounkov in a more combinatorial language.

## 2. Preliminaries

2.1. Symmetric group. There are many equivalent definitions of the transition measure of a Young diagram VK85, Ker93, Ker99, OV96, Bia98, Oko00] and for the sake of completeness we shall recall them in the following. Nevertheless, we shall use in this article only the description from Section 2.1.8
2.1.1. Transition measure of a Young diagram—the eigenvalues approach. The following description of the transition measure is due to Biane [Bia98] and probably it is the simplest one.

Consider an element $\Gamma \in \mathcal{M}_{q+1}\left(\mathbb{C}\left(S_{q}\right)\right)$ given by

$$
\Gamma=\left[\begin{array}{cccccc}
0 & (1,2) & (1,3) & \ldots & (1, q) & 1  \tag{2.1}\\
(1,2) & 0 & (2,3) & \ldots & (2, q) & 1 \\
(1,3) & (2,3) & 0 & \ldots & (3, q) & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1, q) & (2, q) & (3, q) & \ldots & 0 & 1 \\
1 & 1 & 1 & \ldots & 1 & 0
\end{array}\right],
$$

where $(i, j) \in S_{q}$ denotes the transposition exchanging $i$ and $j$.
Let $\rho^{\lambda}: \mathbb{C}\left(S_{q}\right) \rightarrow \mathcal{M}_{k}(\mathbb{C})$ be an irreducible representation of $S_{q}$ corresponding to the Young diagram $\lambda$. We apply map $\rho^{\lambda}$ to every entry of the matrix $\Gamma \in \mathcal{M}_{q+1}\left(S_{q}\right)$ and denote the outcome by $\rho^{\lambda}(\Gamma) \in$ $\mathcal{M}_{q+1}\left(\mathcal{M}_{k}(\mathbb{C})\right)=\mathcal{M}_{(q+1) k}(\mathbb{C})$. Alternatively, if we treat $\Gamma$ as an element of $\mathcal{M}_{q+1}(\mathbb{C}) \otimes \mathbb{C}\left(S_{q}\right)$ then $\rho^{\lambda}(\Gamma)=\left(1 \otimes \rho^{\lambda}\right) \Gamma \in \mathcal{M}_{q+1}(\mathbb{C}) \otimes \mathcal{M}_{k}(\mathbb{C})=$ $\mathcal{M}_{(q+1) k}(\mathbb{C})$.

Let $\zeta_{1}, \ldots, \zeta_{(q+1) k} \in \mathbb{R}$ be the eigenvalues of the matrix $\rho^{\lambda}(\Gamma) \in$ $\mathcal{M}_{(q+1) k}(\mathbb{C})$; then the transition measure of the Young diagram $\lambda$ is the probability measure on $\mathbb{R}$ which (up to a normalization) is the counting measure of eigenvalues of $\rho^{\lambda}(\Gamma)$ :

$$
\mu^{\lambda}=\frac{\delta_{\zeta_{1}}+\cdots+\delta_{\zeta_{(q+1) k}}}{(q+1) k} .
$$

2.1.2. Generalized Young diagrams. Let $\lambda$ be a Young diagram. We assign to it a piecewise affine function $\omega^{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$ with slopes $\pm 1$, such that $\omega^{\lambda}(x)=|x|$ for large $|x|$ as it can be seen on the example from Figure 2.2 By comparing Figure 2.1 and Figure 2.2 one can easily see that the graph of $\omega^{\lambda}$ can be obtained from the graphical representation of the Young diagram by an appropriate mirror image, rotation and scaling by the factor $\sqrt{2}$. We


Figure 2.1. Young diagram associated with a partition $8=$ $4+3+1$.


Figure 2.2. Generalized Young diagram associated with a partition $8=4+3+1$.
call $\omega^{\lambda}$ the generalized Young diagram associated with the Young diagram $\lambda$ Ker93, Ker98, Ker99]. Alternatively, we can encode the Young diagram $\lambda$ using the sequence of local minima of $\omega^{\lambda}$ (denoted by $x_{1}, \ldots, x_{m}$ ) and the sequence of local maxima of $\omega^{\lambda}$ (denoted by $y_{1}, \ldots, y_{m-1}$ ), which form two interlacing sequences of integers [Ker98].

The class of generalized Young diagrams consists of all functions $\omega$ : $\mathbb{R} \rightarrow \mathbb{R}$ which are Lipschitz with constant 1 and such that $\omega(x)=|x|$ for large $|x|$ and of course not every generalized Young diagram can be obtained by the above construction from some Young diagram $\lambda$.

The setup of generalized Young diagrams is very useful in the study of the asymptotic properties since it allows us to define easily various notions of convergence of the Young diagram shapes.
2.1.3. Transition measure-the analytic approach. To any generalized Young diagram $\omega$ we can assign the unique probability measure $\mu^{\omega}$ on
$\mathbb{R}$, called transition measure of $\omega$, which fulfills

$$
\begin{equation*}
\log \int_{\mathbb{R}} \frac{1}{z-x} d \mu^{\omega}(x)=-\frac{1}{2} \int_{\mathbb{R}} \log (z-x) \omega^{\prime \prime}(x) d x \tag{2.2}
\end{equation*}
$$

for every $z \notin \mathbb{R}$ [OV96, Bia98 Oko00]. A great advantage of this definition is that after applying integration by parts one can easily see that the map $\omega \mapsto \mu^{\omega}$ is continuous in many reasonable topologies.

The generalized Young diagram $\omega^{p \lambda}: \chi \mapsto p \omega^{\lambda}\left(\frac{x}{p}\right)$ corresponds to the Young diagram $\lambda$ geometrically scaled by factor $p>0$; it is easy to see that (2.2) implies that the corresponding transition measure $\mu^{\mathrm{p} \mathrm{\lambda}}$ is a dilation of $\mu^{\lambda}$ :

$$
\begin{equation*}
\mu^{\mathrm{p} \mathrm{\lambda}}=\mathrm{D}_{\mathrm{p}} \mu^{\lambda} . \tag{2.3}
\end{equation*}
$$

The above definition (2.2) becomes simpler in the case of the (usual) Young diagrams

$$
\int_{\mathbb{R}} \frac{1}{z-x} d \mu^{\lambda}(x)=\frac{\prod_{1 \leq i \leq m-1}\left(z-y_{i}\right)}{\prod_{1 \leq i \leq m}\left(z-x_{i}\right)}
$$

and implies that the transition measure is explicitly given by

$$
\mu^{\lambda}=\sum_{1 \leq k \leq n} \frac{\prod_{1 \leq i \leq n-1}\left(x_{k}-y_{i}\right)}{\prod_{i \neq k}\left(x_{k}-x_{i}\right)} \delta_{x_{k}} .
$$

2.1.4. Transition measure-the representation theoretic approach. It is possible to find another interpretation of transition measure in the language of representation theory [OV96, Bia98, Oko00]: let us consider the representation $[\lambda] \uparrow_{S_{q}}^{S_{q+1}}$ of $S_{q+1}$ induced from the representation $[\lambda]$ of $S_{q}$. By the branching rule $[\lambda] \uparrow_{S_{q}}^{S_{S_{+1}}}$ decomposes as a direct sum of representations corresponding to the Young diagrams obtained from $\lambda$ by adding one box. It is possible to add a box exactly at the minima $x_{k}$. The measure $\mu_{\lambda}$ assigns a mass to each point $\chi_{k}$ which is proportional to the dimension of the irreducible representation of $S_{q+1}$ corresponding to the diagram $\lambda$ augmented in point $\chi_{k}$.
2.1.5. Partial permutations. The following notion was introduced and studied by Ivanov and Kerov [IK99]. A partial permutation of the set $A$ is a pair $\alpha=(d, w)$, where $d \subseteq A$ and $w: A \rightarrow A$ is any bijection such that $d(x)=x$ for every $x \in A \backslash d$. Set $d$ is called the support of $\alpha$. The set of partial permutations of $A$ will be denoted by $\widetilde{S_{A}}$. Given two partial permutations $\left(d_{1}, w_{1}\right),\left(d_{2}, w_{2}\right)$ we consider their product $\left(d_{1}, w_{1}\right) \cdot\left(d_{2}, w_{2}\right):=\left(d_{1} \cup d_{2}, w_{1} w_{2}\right)$; given this multiplication the set $\widetilde{S_{A}}$ of partial permutations becomes a semigroup. By $S_{A}$ we denote the permutation group of the set $A$. There is an important homomorphism of
semigroups $\widetilde{S_{A}} \rightarrow S_{\text {A }}$ given by forgetting the support $(\mathrm{d}, w) \mapsto w$; therefore every partial permutation can be regarded as a (usual) permutation as well.

For $A \subseteq B$ there is a homomorphism $\theta_{A}^{B}: \mathbb{C}\left(\widetilde{S_{B}}\right) \rightarrow \mathbb{C}\left(\widetilde{S_{A}}\right)$ of partial permutation algebras given by

$$
\theta_{A}^{B}(d, w)= \begin{cases}(d, w) & \text { if } d \subseteq A \\ 0 & \text { otherwise }\end{cases}
$$

By $\mathbb{C}\left(\widetilde{S_{\infty}}\right)$ we denote the projective limit of algebras $\mathbb{C}\left(\widetilde{S_{\{1, \ldots, \mathrm{q}\}}}\right)$ with respect to the morphisms $\theta$.
2.1.6. Abelian algebras in the language of probability theory. Usually as a primary object of the probability theory one considers a Kolmogorov space $(\Omega, \mathcal{M}, P)$-where $\Omega$ is a set, $\mathcal{M}$ is a $\sigma$-field of measurable sets and $P$ is a probability measure- but equally well we may consider an Abelian algebra $\mathfrak{A}$ of all random variables on $\Omega$ with all moments finite and the expected value $\mathbb{E}_{\mathbb{C}}: \mathfrak{A} \rightarrow \mathbb{C}$. More generally, if $\mathcal{M}^{\prime}$ is a $\sigma$-subfield of $\mathcal{M}$ we may consider an algebra $\mathfrak{B}$ of $\mathcal{M}^{\prime}$-measurable random variables and a conditional expectation $\mathbb{E}_{\mathfrak{B}}: \mathfrak{A} \rightarrow \mathfrak{B}$. In this way the probability theory becomes a theory of Abelian algebras $\mathfrak{A}$ equipped with a linear functional $\mathbb{E}_{\mathbb{C}}$ or, more generally, a theory of pairs of Abelian algebras $\mathfrak{B} \subset \mathfrak{A}$ and maps $\mathbb{E}_{\mathfrak{B}}: \mathfrak{A} \rightarrow \mathfrak{B}$.

By turning the picture around we may regard any Abelian algebra $\mathfrak{A}$ equipped with a linear map $\mathbb{E}$ as an algebra of random variables even if $\mathfrak{A}$ does not arise from any Kolmogorov space (this observation was a starting point of the non-commutative probability theory [Mey93]). Now we can use the probability theoretic language when speaking about $\mathfrak{A}$ : elements of $\mathfrak{A}$ can be called random variables and the numbers $\mathbb{E}\left(X^{k}\right)$ can be called moments of the random variable $X \in \mathfrak{A}$. Similarly we shall interpret pairs of Abelian algebras $\mathfrak{B} \subset \mathfrak{A}$ equipped with a map $\mathbb{E}: \mathfrak{A} \rightarrow \mathfrak{B}$.

In the classical probability theory any real-valued random variable $X$ can be alternatively viewed as a selfadjoint multiplication operator on the algebra $\mathfrak{A}$ and according to the spectral theorem can be written as an operatorvalued integral $X=\int_{\mathbb{R}} z \mathrm{dQ}(z)$, where $Q$ denotes the spectral measure of $X$. It is easy to see that the distribution $\mu$ of the random variable $X$ and the spectral measure of the operator $X$ are related by equality $\mu(F)=\mathbb{E}_{\mathbb{C}}[Q(F)]$ for any Borel set $F \subseteq \mathbb{R}$. This simple observation can be used in our new setup to define a distribution $\mu$ of an element $X=X^{\star} \in \mathfrak{A}$ to be a $\mathfrak{B}$-valued measure on $\mathbb{R}$ such that $\mu(F)=\mathbb{E}_{\mathfrak{B}}[Q(F)]$ for any Borel set $F \subseteq \mathbb{R}$. If $X$ is bounded then its distribution can be alternatively described by the moment
formula

$$
\int_{\mathbb{R}} x^{n} d \mu(x)=\mathbb{E}_{\mathfrak{B}}\left(X^{n}\right)
$$

2.1.7. Symmetric group in the language of probability theory. Let $A$ be a finite set and let $*$ be an extra distinguished element such that $* \notin A$. In the following, if not stated otherwise, we set $A=\{1,2, \ldots, q\}$. Group $S_{\text {A }}$ will be regarded as a subgroup of $S_{\mathcal{A \cup \{ * \}}}$. In our case the role of the algebra $\mathfrak{A}$ of random variables will be played by some Abelian subalgebra of $\mathbb{C}\left(S_{A \cup\{*\}}\right)$ and the role of the smaller algebra $\mathfrak{B}$ will be played by the center of $\mathbb{C}\left(S_{A}\right)$. The conditional expectation $\mathbb{E}_{\mathbb{C}\left(S_{A}\right)}: \mathbb{C}\left(S_{A \cup\{*\}}\right) \rightarrow \mathbb{C}\left(S_{A}\right)$ will be given by the orthogonal projection, i.e.

$$
\mathbb{E}_{\mathbb{C}\left(S_{\mathcal{A}}\right)}(\sigma)=\left\{\begin{array}{ll}
\sigma & \text { if } \sigma \in S_{\mathcal{A}} \\
0 & \text { if } \sigma \notin S_{\mathcal{A}}
\end{array}= \begin{cases}\sigma & \text { if } \sigma(*)=* \\
0 & \text { if } \sigma(*) \neq *\end{cases}\right.
$$

In the rest of this article, when it does not lead to confusions, instead of $\mathbb{E}_{\mathbb{C}\left(S_{\mathcal{A}}\right)}$ we shall simply write $\mathbb{E}$.

Suppose that some finite-dimensional representation $\rho: S_{\mathcal{A}} \rightarrow \mathcal{M}_{k}(\mathbb{C})$ is given. This allows us to define a scalar-valued expectation $\mathbb{E}_{\mathbb{C}}$ : $\mathbb{C}\left(S_{\text {AU\{*\} }}\right) \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
\mathbb{E}_{\mathbb{C}}(\mathrm{X})=\frac{1}{\mathrm{k}} \operatorname{Tr} \rho\left(\mathbb{E}_{\mathbb{C}\left(\mathrm{S}_{\mathcal{A}}\right)}(\mathrm{X})\right) \tag{2.4}
\end{equation*}
$$

and to consider the distribution $\mu^{\rho}$ of a random variable $X \in \mathbb{C}\left(S_{A \cup\{*\}}\right)$ with respect to this new expectation; in this new setup $\mu^{\rho}$ is a usual (scalarvalued) probability measure on $\mathbb{R}$.
2.1.8. Jucys-Murphy element. Transition measure-the algebraic approach. The algebra $\mathfrak{P}$. In the group algebra $\mathbb{C}\left(S_{\mathcal{A} \cup *\}}\right)$ we consider the Jucys-Murphy element

$$
J=\sum_{a \in A}(a *)
$$

where ( $\mathfrak{i j}$ ) denotes the transposition exchanging $i$ and $j$. We define moments of the Jucys-Murphy element by

$$
\begin{equation*}
M_{k}^{\mathrm{JM}}=\mathbb{E}\left(\mathrm{J}^{\mathrm{k}}\right)=\sum_{a_{1}, \ldots, a_{k} \in \mathcal{A}} \mathbb{E}\left[\left(a_{1} *\right) \cdots\left(a_{k} *\right)\right] \in \mathbb{C}\left(S_{A}\right) \tag{2.5}
\end{equation*}
$$

In Section 4.2.2 of we shall consider an extension of this concept.
For a given Young diagram $\lambda \vdash|A|$ we consider $\rho=\rho^{\lambda}$ to be an irreducible representation of $S_{\text {A }}$ corresponding to $\lambda$ and consider the distribution $\mu^{\lambda}:=\mu^{\rho^{\lambda}}$ of the element J with respect to the expected value (2.4). We call $\mu^{\lambda}$ the transition measure of $\lambda$ [VK85, Ker93, OV96, Bia98, Oko00].

This definition is very algebraic and for this reason it will be our favorite definition of the transition measure.

We can treat the moments $M_{\mathrm{k}}^{\mathrm{JM}}$ of the Jucys-Murphy element as elements of the partial permutations algebra $\mathbb{C}\left(\widetilde{S_{A}}\right)$; in order to do this we treat every non-zero summand on the right-hand side of (2.5) as a partial permutation with the support $\left\{a_{1}, \ldots, a_{k}\right\}$. It is easy to check that for $A \subseteq B$ the morphism $\theta_{A}^{\mathrm{B}}$ maps $M_{\mathrm{k}}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{\mathrm{S}_{\mathrm{B}}}\right)$ to $M_{\mathrm{k}}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{S_{A}}\right)$ hence the projective limit of the elements $M_{k}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{\mathrm{S}_{\mathrm{q}}}\right)$ exists and will be denoted by the same symbol $M_{k}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$.

We denote by $\mathfrak{P}$ the algebra generated by elements $M_{k}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$. It is easy to check that $\mathfrak{P}$ is commutative. Elements of $\mathfrak{P}$ can be also regarded as elements of $\mathbb{C}\left(\widetilde{S_{q}}\right)$ and $\mathbb{C}\left(S_{q}\right)$.
2.1.9. Gradation on $\mathfrak{P}$. Since we would like to study asymptotic properties of Young diagrams, we need to specify what kind of scaling we are interested in.

Let a sequence of Young diagrams ( $\lambda_{N}$ ) be given, $\lambda_{N} \vdash N$. It is natural to consider generalized Young diagrams which correspond to geometrically scaled diagrams $\frac{1}{\sqrt{\mathrm{~N}}} \lambda_{N}$ (observe that all such scaled diagrams have the same area equal to 2 ). Suppose that the shape of the scaled diagrams converges in some sense toward a generalized Young diagram $\lambda$. In many natural topologies this implies the convergence of scaled moments of the transition measure, cf (2.3)

$$
\begin{aligned}
\lim _{N \rightarrow \infty} \frac{1}{N^{k / 2}} \mathbb{E}_{\mathbb{C}}\left(M_{k}^{\mathrm{JM}}\right)= & \lim _{N \rightarrow \infty} \mathbb{E}_{\mathbb{C}}\left[\left(\frac{1}{N^{1 / 2}} J\right)^{k}\right]= \\
& \lim _{N \rightarrow \infty} \frac{1}{N^{k / 2}} \int_{\mathbb{R}} x^{k} d \mu^{\lambda_{N}}(x)=\int_{\mathbb{R}} x^{k} d \mu^{\lambda}(x) .
\end{aligned}
$$

In the study of asymptotic properties of some elements of the group algebra $\mathbb{C}\left(S_{N}\right)$ we would like to group summands which have asymptotically the same growth for large $N$ and for this reason $M_{k}^{\mathrm{JM}}$ can be treated as a monomial in $\sqrt{\mathrm{N}}$ of degree $k$.

More formally, we consider a gradation on $\mathfrak{P}$ by setting

$$
\begin{equation*}
\operatorname{deg} M_{k}^{\mathrm{JM}}=\mathrm{k} \tag{2.6}
\end{equation*}
$$

In Corollary 4.8 we will show that $M_{2}^{\mathrm{JM}}, M_{3}^{\mathrm{JM}}, \cdots \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ are algebraically free and therefore this gradation is well defined. This gradation coincides with the weight gradation considered by Ivanov and Olshanski [IO02].
2.1.10. Normalized conjugacy class indicators. Let integer numbers $k_{1}, \ldots, k_{m} \geq 1$ be given. We define the normalized conjugacy class indicator $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(S_{A}\right)$ as follows [KO94 Bia03]:

$$
\begin{equation*}
\Sigma_{k_{1}, \ldots, k_{m}}=\sum_{a}\left(a_{1,1}, a_{1,2}, \ldots, a_{1, k_{1}}\right) \cdots\left(a_{m, 1}, a_{m, 2}, \ldots, a_{m, k_{m}}\right), \tag{2.7}
\end{equation*}
$$

where the sum runs over all one-to-one functions

$$
a:\left\{\{r, s\}: 1 \leq r \leq m, 1 \leq s \leq k_{r}\right\} \rightarrow A
$$

and $\left(a_{1,1}, a_{1,2}, \ldots, a_{1, k_{1}}\right) \cdots\left(a_{m, 1}, a_{m, 2}, \ldots, a_{m, k_{m}}\right)$ is a product of disjoint cycles. Of course, if $|\mathcal{A}|<\mathrm{k}_{1}+\cdots+\mathrm{k}_{\mathrm{m}}$ then the above sum runs over an empty set and $\Sigma_{k_{1}, \ldots, k_{m}}=0$. If any of the numbers $k_{1}, \ldots, k_{m}$ is equal to 0 we set $\Sigma_{k_{1}, \ldots, k_{m}}=0$.

In other words, let $k_{1}^{\prime} \geq \cdots \geq k_{m}^{\prime}$ be the sequence $k_{1}, \ldots, k_{m}$ sorted decreasingly; we consider a Young diagram ( $k_{1}^{\prime}, \ldots, k_{m}^{\prime}$ ) and all ways of filling it with the elements of the set $A$ in such a way that no element appears more than once. Each such a filling can be interpreted as a permutation when we treat rows of the Young tableau as disjoint cycles.

We can also treat $\Sigma_{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{m}}}$ as a an element of the partial permutations algebra $\mathbb{C}\left(\widetilde{S_{A}}\right)$; in order to do this, we treat every summand $\left(a_{1,1}, a_{1,2}, \ldots, a_{1, k_{1}}\right) \cdots\left(a_{m, 1}, a_{m, 2}, \ldots, a_{m, k_{m}}\right)$ as a partial permutation with support $\left\{a_{i j}: 1 \leq i \leq m, 1 \leq j \leq k_{i}\right\}$. It is easy to check that for $A \subseteq B$ the morphism $\theta_{A}^{B}$ maps $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{B}}\right)$ to $\Sigma_{k_{1}}, \ldots, k_{m} \in \mathbb{C}\left(\widetilde{S_{A}}\right)$ hence the projective limit of the elements $\Sigma_{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{m}}} \in \mathbb{C}\left(\widetilde{S}_{\mathrm{q}}\right)$ exists and will be denoted by the same symbol $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$. It is easy to check that elements $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ commute.
2.1.11. Filtration. We consider a filtration on the commutative algebra generated by $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ by setting

$$
\begin{equation*}
\operatorname{deg} \Sigma_{k_{1}, \ldots, k_{m}}=\left(k_{1}+1\right)+\left(k_{2}+1\right)+\cdots+\left(k_{m}+1\right) \tag{2.8}
\end{equation*}
$$

It is easy to check that the elements from the family $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ indexed by integer $m \geq 0$ and $k_{1} \geq \cdots \geq k_{m} \geq 1$ are linearly independent and therefore this definition makes sense but it is not clear that this formula indeed defines a filtration; we shall prove it in Corollary 3.8. The reason for studying this filtration is that-as we shall see in Theorem 4.9-elements $M_{k}^{\mathrm{JM}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ and $\Sigma_{k_{1}, \ldots, k_{m}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ generate the same commutative algebra denoted by $\mathfrak{P}$ and filtration (2.8) is induced by the gradation (2.6).

### 2.2. Partitions.



Figure 2.3. Graphical representation of a partition $\{\{1,3\},\{2,5,7\},\{4\},\{6\}\}$.
2.2.1. Partitions. We recall that $\pi=\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ is a partition of a finite ordered set $X$ if sets $\pi_{1}, \ldots, \pi_{r}$ are nonempty and disjoint and if $\pi_{1} \cup \cdots \cup$ $\pi_{r}=X$. We denote the set of all partitions of a set $X$ by $\mathrm{P}(X)$. We call sets $\pi_{1}, \ldots, \pi_{r}$ blocks of the partition $\pi$. We say that elements $a, b \in X$ are connected by the partition $\pi$ if they are the elements of the same block. We will call elements of the set $X$ the labels of the vertices of the partition $\pi$. As usually, the numbers $\left|\pi_{1}\right|, \ldots,\left|\pi_{r}\right|$ denote the numbers of elements in consecutive blocks of $\pi$.

We say that a block $\pi_{s}$ is trivial if it contains only one element and a trivial partitions has all blocks trivial. We say that a partition is a pair partition if all its blocks contain exactly two elements.

We say that a partition $\pi$ is finer than a partition $\rho$ of the same set if every block of $\pi$ is a subset of some block of $\rho$ and we denote it by $\pi \leq \rho$. The restriction of a partition $\pi$ to a set $Y \subseteq X$ is a partition of $Y$ which connects elements $a, b \in Y$ if and only if $a, b$ are connected by $\pi$.

In the following we present some constructions on partitions of the set $X=\{1,2, \ldots, n\}$. However, it should be understood that by a change of labels these constructions can be performed for any finite ordered set $X$.

It is very useful to represent partitions graphically by arranging the elements of the set $X$ counterclockwise on a circle and joining elements of the same block by a line, as it can be seen on Figure 2.3 Note that we do not need to write labels on the vertices in order to recover the partition from its graphical representation if we mark the first element as the 'starting point'.
2.2.2. Fat partitions. Let $\pi=\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ be a partition of the set $\{1, \ldots, n\}$. For every $1 \leq s \leq r$ let $\pi_{s}=\left\{\pi_{s, 1}, \ldots, \pi_{s, l_{s}}\right\}$ with $\pi_{s, 1}<$ $\cdots<\pi_{s, l_{s}}$. We define $\pi_{\mathrm{fat}}$, called fat partition of $\pi$, to be a pair partition of the $2 n$-element ordered set $\left\{1,1^{\prime}, 2,2^{\prime}, \ldots, n, n^{\prime}\right\}$ given by

$$
\pi_{\mathrm{fat}}=\left\{\left\{\pi_{\mathrm{s}, \mathrm{t}}^{\prime}, \pi_{\mathrm{s}, \mathrm{t}+1}\right\}: 1 \leq \mathrm{s} \leq \mathrm{r} \text { and } 1 \leq \mathrm{t} \leq \mathrm{l}_{\mathrm{s}}\right\}
$$



Figure 2.4. Fat partition $\pi_{\text {fat }}$ corresponding to the partition $\pi$ from Figure 2.3
where it should be understood that $\pi_{s, l_{s}+1}=\pi_{s, 1}$.
This operation can be easily described graphically as follows: we draw blocks of a partition with a fat pen and take the boundary of each block, as it can be seen on Figure 2.4 This boundary is a collection of lines hence it is a pair partition. However, every vertex $k \in\{1, \ldots, n\}$ of the original partition $\pi$ has to be replaced by its 'right' and 'left' copy (denoted respectively by $k$ and $k^{\prime}$ ). Please note that in the graphical representation of $\pi_{\text {fat }}$ we mark the space between 1 and 1 ' as the 'starting point'.
2.2.3. Non-crossing partitions. We say that a partition $\pi$ is non-crossing [Kre72, SU91, Spe94, Spe97, Spe98] if for every $i \neq j$ and $a, c \in \pi_{i}$ and $b, d \in \pi_{j}$ it cannot happen that $a<b<c<d$. For example, the partition from Figure 2.3 is crossing (as it can be easily seen from its graphical representation) while a partition from Figure 2.5 is non-crossing. The set of all non-crossing partitions of a set $X$ will be denoted by $\mathrm{NC}(X)$ and the set of all non-crossing pair partitions of a set $X$ will be denoted by $\mathrm{NC}_{2}(\mathrm{X})$.
2.2.4. Kreweras complementation map. The function $\pi \mapsto \pi_{\text {fat }}$ which maps partitions of a $n$-element set to pair partitions of a $2 n$-element set is one to one but in general is not a bijection. However, it is a bijection between $\mathrm{NC}(1,2, \ldots, n)$ and $\mathrm{NC}_{2}\left(1,1^{\prime}, \ldots, \mathrm{n}, \mathrm{n}^{\prime}\right)$ [Kre72].

If $\rho$ is a pair partition of the set $\left\{1,1^{\prime}, 2,2^{\prime}, \ldots, n, n^{\prime}\right\}$, we denote by $r(\rho)$ a pair partition of the same set $\left\{1,1^{\prime}, \ldots, n, n^{\prime}\right\}$ which is obtained by cyclic change of labels $\cdots \rightarrow 2^{\prime} \rightarrow 2 \rightarrow 1^{\prime} \rightarrow 1 \rightarrow \mathrm{n}^{\prime} \rightarrow \mathrm{n} \rightarrow \cdots$ carried by the vertices. In the graphical representation of a partition this corresponds


Figure 2.5. Graphical representation of a non-crossing partition $\{\{1\},\{2,5,6\},\{3,4\}\}$.


FIGURE 2.6. Fat partition $\pi_{\text {fat }}$ for a non-crossing partition $\pi$ from Figure 2.5


Figure 2.7. Graphical representation of $r(\rho)$ where $\rho$ is given by Figure 2.6
to a shift of the starting point by one counterclockwise (cf Figure 2.7). Of course $r$ is a bijection of the set $\mathrm{NC}_{2}\left(1,1^{\prime}, \ldots, n, n^{\prime}\right)$.

It follows that the map $\pi \mapsto \pi_{\text {comp }}$, called Kreweras complementation map, given by $\left(\pi_{\text {comp }}\right)_{\text {fat }}=r\left(\pi_{\text {fat }}\right)$ is well-defined and is a permutation of


Figure 2.8. Partition $\pi$ from Figure 2.5 in a solid line and its Kreweras complement in a dashed line.
$\mathrm{NC}(1,2, \ldots, n)$. Due to the canonical bijection between $\{1,2, \ldots, n\}$ and $\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ we can identify $\operatorname{NC}(1,2, \ldots, n)$ with $\operatorname{NC}\left(1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right)$ and in the future we shall sometimes regard $\pi_{\text {comp }}$ as an element of $N C(1,2, \ldots, n)$ and sometimes as an element of $N C\left(1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right)$. It will be clear from the context which of the options we choose.

One can also state the above definition as follows: $\pi_{\text {comp }}$ is the biggest non-crossing partition of the set $\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ with a property that $\pi \cup$ $\pi_{\text {comp }}$ is a non-crossing partition of the set $\left\{1,1^{\prime}, 2,2^{\prime}, \ldots, n, n^{\prime}\right\}$ (cf Figure 2.8).

We shall also consider the inverse of the Kreweras complementation map: for $\pi, \rho \in \operatorname{NC}(1,2, \ldots, n)$ we have $\pi_{\text {comp }^{-1}}=\rho$ if and only if $\pi=\rho_{\text {comp }}$. One can observe that $\rho_{\text {comp }^{-1}}$ and $\rho_{\text {comp }}$ are always cyclic rotations of each other.

## 3. The first main result: Calculus of partitions

### 3.1. Partition-indexed conjugacy class indicator $\Sigma_{\pi}$.

3.1.1. Definition of $\Sigma_{\pi}$. Let $\pi$ be a partition of the set $\{1, \ldots, n\}$. Since the fat partition $\pi_{\text {fat }}$ connects every element of the set $\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ with exactly one element of the set $\{1,2, \ldots, n\}$, we can view $\pi_{\text {fat }}$ as a bijection $\pi_{\text {fat }}:\left\{1^{\prime}, 2^{\prime}, \ldots, \mathfrak{n}^{\prime}\right\} \rightarrow\{1,2, \ldots, n\}$. We also consider a bijection $c:\{1,2, \ldots, n\} \rightarrow\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ given by $\ldots, 3 \mapsto 2^{\prime}, 2 \mapsto 1^{\prime}, 1 \mapsto$ $n^{\prime}, \mathrm{n} \mapsto(\mathrm{n}-1)^{\prime}, \ldots$. Finally, we consider a permutation $\pi_{\text {fat }} \circ \mathrm{c}$ of the set $\{1,2, \ldots, n\}$.

For example, for the partition $\pi$ given by Figure 2.3 the composition $\pi_{\text {fat }} \circ \mathrm{c}$ has a cycle decomposition $(1,2,3,5,4)(6,7)$, as it can be seen from Figure 3.1.

Alternatively, one can identify the set $\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ with the set $\{1,2, \ldots, n\}$; then $\pi_{\text {fat }}$ becomes a permutation which coincides with the


Figure 3.1. Bijection corresponding to the partition $\pi_{\text {fat }}$ from Figure 2.4 plotted with a solid line and the bijection c plotted with a dashed line. Lines corresponding to two different cycles were plotted in different colors (gray and black). The meaning of the additional decoration of some of the vertices will be explained in Section 5.2.1.
construction from Kre72, Bia97, Bia98 and c is equal to the full cycle ( $n, n-1, \ldots, 2,1$ ).

We decompose the permutation

$$
\pi_{\mathrm{fat}} \circ c=\left(\mathrm{b}_{1,1}, \mathrm{~b}_{1,2}, \ldots, \mathrm{~b}_{1, \mathrm{j}_{1}}\right) \cdots\left(\mathrm{b}_{\mathrm{t}, 1}, \ldots, \mathrm{~b}_{\mathrm{t}, \mathrm{j}_{\mathrm{t}}}\right)
$$

as a product of disjoint cycles. Every cycle $b_{s}=\left(b_{s, 1}, \ldots, b_{s, j_{s}}\right)$ can be viewed as a closed clockwise path on a circle and therefore one can compute how many times it winds around the circle. It might be useful to draw a line between the central disc and the starting point (cf Figure 3.2); the number of winds is equal to the number of times a given cycle crosses this line and therefore is equal to the number of indices $1 \leq i \leq j_{s}$ such that $b_{s, i} \leq b_{s, i+1}$, where we use the convention that $b_{s, j_{s}+1}=b_{s, 1}$.

To a cycle $b_{s}$ we assign the number
(3.1) $k_{s}=\left(\right.$ number of elements in a cycle $\left.b_{s}\right)-$
(number of clockwise winds of $b_{s}$ ).
In the above example we have $b_{1}=(1,2,3,5,4), b_{2}=(6,7)$ and $k_{1}=$ $2, k_{2}=1$, as it can be seen from Figure 3.2, where all lines clockwise wind around the central disc.

Alternatively, we can treat every cycle $b_{s}$ of $\pi_{\text {fat }} \circ c$ as a closed counterclockwise path on a circle (cf Figure 3.3). Equation (3.1) implies that

$$
\begin{equation*}
k_{s}=\left(\text { number of counterclockwise winds of the cycle } b_{s}\right) . \tag{3.2}
\end{equation*}
$$



Figure 3.2. A version of Figure 3.1 in which all lines wind clockwise around the central disc.


Figure 3.3. A version of Figure 3.1 in which all lines of $\pi_{\text {fat }}$ wind counterclockwise around the central disc.

Let a positive integer $q$ be given. In Section 5 we shall define the most important tool of this article: a certain map $\Phi_{\mathbb{C}\left(\widetilde{\left.s_{\infty}\right)}\right.}^{\mathrm{P}}$ from the set of partitions to $\mathbb{C}\left(\widetilde{S_{\infty}}\right)$. The following Claim states that $\Phi_{\mathbb{C}\left(\widetilde{\left.S_{\infty}\right)}\right.}^{\mathrm{P}}(\pi)$ is closely related with the normalized conjugacy class indicators from Section 2.1.10 of Section 2 and hence $\Sigma_{\pi}:=\Phi_{\mathbb{C}\left(\widetilde{\left.S_{\infty}\right)}\right.}^{\mathrm{P}}(\pi)$ can be viewed as a normalized conjugacy class indicator which is indexed by a partition $\pi$ instead of a tuple of integers ( $k_{1}, \ldots, k_{t}$ ).
Claim 3.1. Let $\pi$ be a partition of the set $\{1,2, \ldots, n\}$ and let numbers $k_{1}, \ldots, k_{t}$ be given by the above construction. Then

$$
\Sigma_{\pi}:=\Phi_{\mathbb{C}\left(\widetilde{S_{\infty}}\right)}^{\mathrm{P}}(\pi)=\Sigma_{\mathrm{k}_{1}, \ldots, k_{\mathrm{t}}}
$$

where $\Sigma_{k_{1}, \ldots, k_{t}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ on the right-hand side should be understood as in Section 2.1.10

This Claim will follow from Theorem 5.7. Reader not interested in too technical details may even take this Claim as a very simple alternative definition of $\Sigma_{\pi}=\Phi_{\mathbb{C}\left(\widetilde{\left.S_{\infty}\right)}\right.}^{\mathrm{P}}(\pi)$.
3.1.2. Basic properties of $\Sigma_{\pi}$. We say that partitions $\pi$ and $\pi^{\prime}$ are cyclic rotations of each other if one can obtain $\pi^{\prime}$ from $\pi$ by a cyclic shift of labels of vertices. Graphically, this corresponds to a change of a marked starting point on a circle.

Proposition 3.2. Let $\pi$ and $\pi^{\prime}$ be two partitions of the same set which are cyclic rotations of each other. Then

$$
\Sigma_{\pi}=\Sigma_{\pi^{\prime}} .
$$

Proof. It is enough to observe that the winding numbers considered in the algorithm from Section 3.1.1 do not depend on the choice of the starting point.

Proposition 3.3. If partition $\pi$ connects some neighbor elements then $\Sigma_{\pi}=$ 0 .

Proof. If partition $\pi$ connects some $r$ with $r+1$ then $\pi_{\text {fat }}$ connects $r^{\prime}$ with $\mathrm{r}+1$, hence $\pi_{\text {fat }}\left(\mathrm{r}^{\prime}\right)=\mathrm{r}+1$ and $\left(\pi_{\text {fat }} \circ \mathrm{c}\right)(\mathrm{r}+1)=\mathrm{r}+1$. One of the cycles of $\pi_{\text {fat }} \circ \mathrm{c}$ is a 1 -cycle $\mathrm{b}_{\mathrm{s}}=(\mathrm{r}+1)$ and one can easily check that $\mathrm{k}_{\mathrm{s}}=0$. Claim 3.1 implies that $\Sigma_{\pi}=\Sigma_{k_{1}, \ldots, k_{t}}=0$.
3.1.3. Degree of $\Sigma_{\pi}$ and its geometric interpretation. We keep the notation from the previous Section; in particular $t$ denotes the number of cycles of $\pi_{\text {fat }} \circ \mathrm{c}$ and r the number of blocks of $\pi$. Observe that

$$
\sum_{1 \leq s \leq t}\left(\text { number of elements in a cycle } b_{s}\right)=n .
$$

Furthermore $\sum_{1 \leq s \leq t}$ (number of clockwise winds of $b_{s}$ ) is equal to the total number of clockwise winds of $\pi_{\text {fat }} \circ \mathrm{c}$. When we identify sets $\{1,2, \ldots, n\}$ and $\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ then $c$ is a full clockwise cycle and it contributes with a one wind. It is easy to observe that every block $\pi_{s}$ of partition $\pi$ is a counterclockwise cycle of the permutation $\pi_{\text {fat }}$, hence when we treat it as a collection of clockwise steps it contributes with $\left|\pi_{s}\right|-1$ clockwise


Figure 3.4. The first collection of discs for partition $\pi$ from Figure 2.3.
winds. It follows that

$$
\begin{aligned}
& \sum_{1 \leq s \leq t}\left(\text { number of clockwise winds of } b_{s}\right)= \\
& \qquad 1+\sum_{1 \leq s \leq r}\left(\left|\pi_{s}\right|-1\right)=n-r+1
\end{aligned}
$$

since $\sum_{1 \leq s \leq r}\left|\pi_{s}\right|=n$. Therefore (3.1) gives us

$$
\begin{equation*}
\operatorname{deg} \Sigma_{\pi}=\mathrm{k}_{1}+\cdots+\mathrm{k}_{\mathrm{t}}+\mathrm{t}=\mathrm{r}+\mathrm{t}-1 \tag{3.3}
\end{equation*}
$$

where the degree is taken with respect to the filtration (2.8).
We can find a natural geometric interpretation of (3.3): consider a large sphere with a small circular hole. The boundary of this hole is the circle that we consider in the graphical representations of partitions. Some pairs of points on this circle are connected by lines: the blocks of the partition $\pi_{\text {fat }}$. To the arcs on the boundary of the circle between points $1,1^{\prime}, 2,2^{\prime}, \ldots, n, n^{\prime}$ and to the lines of $\pi_{\text {fat }}$ we shall glue two collections of discs. Every disc from the first collection corresponds to one of the blocks of $\pi$ (cf Figure 3.4).

After gluing the first collection of discs, our sphere becomes a surface with a number of holes. The boundary of each hole corresponds to one of the cycles of $\pi_{\text {fat }} \circ \mathrm{c}$ and we shall glue this hole with a disc from the second collection.

Thus we obtained an orientable surface without a boundary. We define the genus of the partition $\pi$ to be the genus of this surface. The numbers of vertices, edges and faces of the polyhedron constructed above are the following: $\mathrm{V}=2 \mathrm{n}, \mathrm{E}=3 \mathrm{n}, \mathrm{F}=1+\mathrm{r}+\mathrm{t}$ therefore the genus of this
surface is equal to

$$
\begin{equation*}
\text { genus }_{\pi}=\frac{2-\mathrm{V}+\mathrm{E}-\mathrm{F}}{2}=\frac{\mathrm{n}+1-\mathrm{r}-\mathrm{t}}{2} . \tag{3.4}
\end{equation*}
$$

We proved therefore the following result.
Proposition 3.4. For any partition $\pi$ of an $n$-element set

$$
\begin{equation*}
\operatorname{deg} \Sigma_{\pi}=\mathrm{n}-2 \text { genus }_{\pi} \tag{3.5}
\end{equation*}
$$

where the degree is taken with respect to the filtration (2.8).
We leave the proof of the following simple result to the reader.
Proposition 3.5. A partition is non-crossing if and only if its genus is equal to zero.

### 3.2. Multiplication of partitions.

3.2.1. Definition of multiplication. We will introduce an algebraic structure on the partitions. For any finite ordered set $X$ one can consider a linear space spanned by all partitions of $X$. We will define a 'multiplicative' structure as follows: let $\rho=\left\{\rho_{1}, \ldots, \rho_{\mathrm{r}}\right\}$ be a non-crossing partition of a finite ordered set $X$ and for every $1 \leq s \leq r$ let $\pi^{s}$ be a partition of the set $\rho_{s}$. We define the ' $\rho$-ordered product' of partitions $\pi^{s}$ by

$$
\begin{equation*}
\prod_{s} \pi^{s}:=\sum_{\sigma} \sigma \tag{3.6}
\end{equation*}
$$

where the sum denotes a formal linear combination and it runs over all partitions $\sigma$ of the set $X$ such that
(1) for any $a, b \in \rho_{s}, 1 \leq s \leq r$ we have that $a$ and $b$ are connected by $\sigma$ if and only if they are connected by $\pi^{s}$,
(2) $\sigma \geq \rho_{\text {comp }^{-1}}$,
where $\geq$ denotes the order on partitions we introduced in Section 2.2.1.
Above we defined the product of the elements of the basis of the linear space; by requirement that multiplication is distributive the definition extends uniquely to general vectors.

For example, for

$$
\begin{align*}
\rho_{1}=\{1,2,7,8\}, \rho_{2}= & \{3,4,5,6\},  \tag{3.7}\\
& \pi^{1}=\{\{1,7\},\{2\},\{8\}\}, \pi^{2}=\{\{3,5\},\{4,6\}\}
\end{align*}
$$

we have

$$
\begin{aligned}
\pi^{1} \cdot \pi^{2}=\{\{1,7\}, & \{2,4,6\},\{3,5\},\{8\}\}+ \\
& \{\{1,7\},\{2,4,6\},\{3,5,8\}\}+\{\{1,3,5,7\},\{2,4,6\},\{8\}\} .
\end{aligned}
$$



Figure 3.5. Graphical representation of example (3.7).
3.2.2. Geometric interpretation of partitions multiplication. Suppose for simplicity that $X=\{1,2, \ldots, n\}$. On the surface of a large sphere we draw a small circle on which we mark counterclockwise points $1,2, \ldots, n$. Inside the circle we cut $r$ holes; for any $1 \leq s \leq r$ the corresponding hole has a shape of a disc, the boundary of which passes through the points from the block $\rho_{s}$ (it is possible to perform this operation since $\rho$ is non-crossing). For every $1 \leq s \leq r$ the partition $\pi^{s}$ connects some points on the boundary of the hole $\rho_{s}$ and this situation corresponds exactly to the case we considered in Section 3.1.3. We shall glue to the hole $\rho_{s}$ only the first collection of discs that we considered in Section 3.1.3 i.e. the discs which correspond to the blocks of the partition $\pi^{s}$. Thus we obtained a number of holes with a collection of glued discs (cf Figure 3.5).

When we inflate the original small holes inside the circle we may think about this picture alternatively: instead of $r$ small holes we have a big one (in the shape of the circle) but some arcs on its boundary are glued by extra discs (on Figure 3.6 drawn in black) given by the partition $\rho_{\text {comp }^{-1}} \in$ NC( $1,2, \ldots, n$ ). Furthermore we still have all discs (on Figure 3.6 drawn in gray) corresponding to partitions $\pi^{s}$. We merge discs from these two collections if they touch the same vertex. After this merging the collection of discs corresponds to the partition $\rho_{\text {comp }^{-1}} \vee\left(\pi^{1} \cup \cdots \cup \pi^{\mathrm{t}}\right)$, i.e. the smallest partition which is bigger than both $\rho_{\text {comp }^{-1}}$ and $\pi^{1} \cup \cdots \cup \pi^{\mathrm{t}}$.

The last step is to consider all ways of merging of the discs (or equivalently: all partitions $\sigma \geq\left(\rho_{\text {comp }^{-1}} \vee\left(\pi^{1} \cup \cdots \cup \pi^{\mathrm{t}}\right)\right)$ ) with the property that any two vertices that were lying on the boundary of the same small hole $\rho_{s}$ if were not connected by a disc from the collection $\pi^{s}$ then they also cannot be connected after all mergings.


Figure 3.6. Figure 3.5 after inflating small holes.
3.3. Map $\Sigma$ is a homomorphism. Since the map $\Sigma=\Phi_{\mathbb{C}\left(\widetilde{\left.S_{\infty}\right)}\right.}^{P}$ was defined on the basis of the linear space of partitions, we can extend it linearly in a unique way. It turns out that $\Phi_{\mathbb{C}\left(\widetilde{s_{\infty}}\right)}^{\mathrm{P}}$ preserves also the multiplication hence it is a homomorphism. However, since we consider only ' $\rho$-ordered' products of partitions, we have to state it in a bit unusual way.
Claim 3.6. Let $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ be a non-crossing partition of the set $\{1,2, \ldots, n\}$ and for every $1 \leq s \leq r$ let $\pi^{s}$ be a partition of the set $\rho_{s}$. Then

$$
\Sigma\left(\prod_{s} \pi^{s}\right)=\prod_{1 \leq s \leq r} \Sigma_{\pi^{s}}
$$

where the multiplication on the left hand side should be understood as the $\rho$-ordered product of partitions and on the right hand side it should be understood as the usual product of commuting elements in $\mathbb{C}\left(\widetilde{S_{\infty}}\right)$.

This Claim will follow from Theorem 5.11 Due to distributivity of multiplication the above Claim remains true if we allow $\pi^{s}$ to be a linear combination of partitions. It would be very interesting to prove Claim 3.6 directly if one treats Claim 3.1 as a definition of $\Sigma_{\pi}$.

The following result was proved in [K094, [K99, LT01, Bia03].
Corollary 3.7. For any choice of tuples of positive integers $\left(k_{1}, \ldots, k_{m}\right)$ and $\left(k_{1}^{\prime}, \ldots, k_{n}^{\prime}\right)$ there exists a function $f$ with a finite support defined on the set of the tuples of integers such that

$$
\begin{equation*}
\Sigma_{k_{1}, \ldots, k_{m}} \cdot \Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}=\sum_{p_{1}, \ldots, p_{r}} f_{p_{1}, \ldots, p_{r}} \Sigma_{p_{1}, \ldots, p_{r}} \tag{3.8}
\end{equation*}
$$

Proof. It is enough to find partitions $\pi, \pi^{\prime}$ such that $\Sigma_{\pi}=\Sigma_{k_{1}, \ldots, k_{m}}, \Sigma_{\pi^{\prime}}=$ $\Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}$ and use the fact that $\Sigma_{\pi} \Sigma_{\pi^{\prime}}=\Sigma_{\pi \cdot \pi^{\prime}}$.

More detailed information about the product $\Sigma_{k_{1}, \ldots, k_{m}} \cdot \Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}$ is provided by the following result which was proved by Ivanov and Olshanski [IO02].

Corollary 3.8. For any choice of integers $k_{1}, \ldots, k_{m}, k_{1}^{\prime}, \ldots, k_{n}^{\prime} \geq 1$ let us express the product $\Sigma_{k_{1}, \ldots, k_{m}} \cdot \Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ as a linear combination of normalized conjugacy class indicators $\Sigma$. Then

$$
\begin{aligned}
& \Sigma_{k_{1}, \ldots, k_{m}} \cdot \Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}=\Sigma_{k_{1}, \ldots, k_{m}, k_{k}^{\prime}, \ldots, k_{n}^{\prime}}+ \\
& \text { (terms of degree at most } k_{1}+\cdots+k_{m}+k_{1}^{\prime}+\cdots+k_{n}^{\prime}+m+n-2 \text { ) }
\end{aligned}
$$

and therefore (2.8) indeed defines a filtration.
Proof. Define $l_{i}=k_{i}+1$ and $l_{i}^{\prime}=k_{i}^{\prime}+1$. We set $\pi^{1}$ to be a partition of the set of consecutive integers $\rho_{1}=\left\{1, \ldots, l_{1}+\cdots+l_{m}\right\}$ which has only one non-trivial block $\pi_{1}^{1}=\left\{l_{1}, l_{1}+l_{2}, l_{1}+l_{2}+l_{3}, \ldots, l_{1}+\cdots+l_{m}\right\}$ and let $\pi^{2}$ be a partition of the set of consecutive integers $\rho_{2}=\left\{l_{1}+\cdots+l_{m}+1, l_{1}+\cdots+\right.$ $\left.l_{m}+2, \ldots, l_{1}+\cdots+l_{m}+l_{1}^{\prime}+\cdots+l_{n}^{\prime}\right\}$ which has only one non-trivial block $\pi_{1}^{2}=\left\{l_{1}+\cdots+l_{m}+l_{1}^{\prime}, l_{1}+\cdots+l_{m}+l_{1}^{\prime}+l_{2}^{\prime}, \ldots, l_{1}+\cdots+l_{m}+l_{1}^{\prime}+\cdots+l_{n}^{\prime}\right\} ;$ all the other blocks of these partitions consist of single elements. One can easily check that $\Sigma_{\pi^{1}}=\Sigma_{k_{1}, \ldots, k_{m}}$ and $\Sigma_{\pi^{2}}=\Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}$. We set $\rho=\left\{\rho_{1}, \rho_{2}\right\}$; let us compute the $\rho$-ordered product $\pi^{1} \cdot \pi^{2}$.

Equation (3.5) implies that the terms of the maximal degree $l_{1}+\cdots+$ $l_{m}+l_{1}^{\prime}+\cdots+l_{n}^{\prime}$ will correspond to non-crossing partitions (genus equal to zero) and the degree corresponding to any crossing partition cannot exceed $l_{1}+\cdots+l_{m}+l_{1}^{\prime}+\cdots+l_{n}^{\prime}-2$. It is enough to show that there is only one non-crossing partition $\sigma$ which contributes to $\pi^{1} \cdot \pi^{2}$, namely the partition $\tau=\left(\pi^{1} \cup \pi^{2}\right) \vee \rho_{\text {comp }^{-1}}$ which has only one non-trivial block $\pi_{1}^{1} \cup \pi_{1}^{2}$. This statement is pretty obvious from the geometric interpretation; we provide a more detailed proof below.

Let us consider a partition $\sigma \neq \tau$ which contributes to $\pi^{1} \cdot \pi^{2}$. Of course we have $\sigma \geq \tau$. Let $\mathrm{a}, \mathrm{c}$ be a pair of elements which are connected by $\sigma$ and are not connected by $\tau$; by the definition of the product of partitions a and $c$ cannot be both elements of $\rho_{1}$ or both elements of $\rho_{2}$ so let us assume that $a \in \rho_{1}$ and $c \in \rho_{2}$.

Suppose that $\mathrm{a} \in \pi_{1}^{1}$; then $\mathrm{c} \notin \pi_{1}^{2}$ (otherwise a and c would be connected by $\tau$ ). It follows that there is an element of $\rho_{2} \backslash \pi_{1}^{2}$, namely c , which is connected by $\sigma$ with $a \in \pi_{1}^{1}$, and the latter element must be connected by $\sigma$ with the elements of $\pi_{1}^{2}$. This contradicts the definition of $\pi^{1} \cdot \pi^{2}$. Therefore $\mathrm{a} \notin \pi_{1}^{1}$ and similarly we show that $\mathrm{c} \notin \pi_{1}^{2}$.

It follows that a tuple $a, b, c, d$ with $b=l_{1}+\cdots+l_{m}, d=l_{1}+\cdots+$ $l_{m}+l_{1}^{\prime}+\cdots+l_{n}^{\prime}$ is the one required for $\sigma$ to be crossing.

## 4. Free cumulants of partitions

4.1. Free cumulants. Let us fix some finite ordered set $X$, a linear space $V$ and a map $M: N C(X) \rightarrow V$ called the moment map. Usually, the space V carries some kind of multiplicative structure and there exists a sequence $M_{1}, M_{2}, \ldots$, called moment sequence, such that for every non-crossing partition $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ the moment map is given by a multiplicative extension

$$
\begin{equation*}
M_{\rho}=\prod_{1 \leq s \leq r} M_{\left|\rho_{s}\right|} . \tag{4.1}
\end{equation*}
$$

For every integer $n \geq 1$ the Speicher's free cumulant $R_{n}$ [Spe94, Spe97, Spe98] is defined by

$$
\begin{equation*}
R_{n}=\sum_{\rho \in \operatorname{NC}(1,2, \ldots, n)} \operatorname{Moeb}_{\rho_{\text {comp }}} M_{\rho} \tag{4.2}
\end{equation*}
$$

where the sum runs over all non-crossing partitions of the set $\{1,2, \ldots, n\}$ and Moeb is the Möbius function on the lattice of non-crossing partitions given explicitly by

$$
\operatorname{Moeb}_{\sigma}=\prod_{1 \leq s \leq r}(-1)^{\left|\sigma_{s}\right|-1} c_{\left|\sigma_{s}\right|-1}
$$

for $\sigma=\left\{\sigma_{1}, \ldots, \sigma_{r}\right\}$ and where $c_{k}=\frac{(2 k)!}{(k+1)!k!}$ denotes the Catalan number.
We are going to use in this article the following defining property of the Möbius function [Spe94].

Lemma 4.1. For every $\rho \in \operatorname{NC}(1,2, \ldots, n)$

$$
\sum_{\substack{\pi \in \mathrm{NC}(1, \ldots, n) \\ \pi \leq \rho}} \operatorname{Moeb}_{\pi}= \begin{cases}1 & \text { if } \rho \text { is trivial }, \\ 0 & \text { otherwise } .\end{cases}
$$

One can show [Spe94] that if (4.1) holds then the moments can be computed from the corresponding cumulants by the following simple formula

$$
\begin{equation*}
M_{n}=\sum_{\rho \in \operatorname{NC}(1, \ldots, n)} R_{\rho} \tag{4.3}
\end{equation*}
$$

where similarly as in (4.1) for every non-crossing partition $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ we set

$$
R_{\rho}=\prod_{1 \leq s \leq r} R_{\left|\rho_{s}\right|} \cdot
$$

In a typical application the moments $M_{n}$ are real numbers given by

$$
M_{n}=\int_{-\infty}^{\infty} x^{n} d \mu(x)
$$

where $\mu$ is a probability measure on $\mathbb{R}$ with all moments finite or given by $M_{n}=\mathbb{E}\left(X^{n}\right)$, where $X$ is some random variable. It is easy to check that in this case the free cumulant $R_{n}$ behaves like a homogeneous polynomial of degree $n$ in a sense that the free cumulant $R_{n}\left(D^{p} \mu\right)$ of the dilated measure is related to the free cumulant $R_{n}(\mu)$ of the original measure by

$$
R_{n}\left(D^{p} \mu\right)=p^{n} R_{n}(\mu)
$$

This very simple scaling property is very useful in the study of asymptotic properties. Another advantage of free cumulants is that the generating function (called Voiculescu's R-transform [VDN92]) of the sequence of free cumulants of a given measure $\mu$ can be computed directly from the Cauchy transform of $\mu$ which is very useful in practical applications.

The definition of the free cumulants itself does not involve the multiplicative structure of V ; this structure is accessed only implicitly through the moment map $M$. Therefore a question arises if there are some reasonable generalizations of (4.1). The key point is that we need to define the moments $M_{\rho}$ only for non-crossing partitions $\rho$ and it is well-known that non-crossing partitions of $n$-element set are in correspondence with certain ways of writing brackets in a product of $n$ factors and hence recursive definitions of $M_{\rho}$ can be successfully applied. Canonical examples are provided by operator-valued free probability [Spe98]. In the following we will use this weakness of requirements for the moment map $M$ in order to define and study the partition-valued free cumulants.

### 4.2. Free cumulants of partitions and of the Jucys-Murphy element.

4.2.1. Moments and cumulants of partitions. Let $X$ be an ordered set. We define the moment $M_{\mathrm{X}}^{\mathrm{P}}$ (where the letter P stands for partition) by

$$
M_{\mathrm{X}}^{\mathrm{P}}=\sum_{\sigma \in \mathrm{P}(\mathrm{X})} \sigma
$$

where the sum denotes a formal linear combination and it runs over all partitions of $X$.

Let $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ be a non-crossing partition. Motivated by (4.5) we define

$$
M_{\rho}^{P}=\prod_{s} M_{\rho_{s}}^{P}=\sum_{\sigma^{1}, \ldots, \sigma^{r}} \prod_{s} \sigma^{s},
$$

where the sum runs over all tuples $\left(\sigma^{1}, \ldots, \sigma^{r}\right)$ such that $\sigma^{s}$ is a partition of the set $\rho_{s}$ and products denote $\rho$-ordered products of partitions.

Proposition 4.2. For every non-crossing partition $\rho$ of the set X we have

$$
\begin{equation*}
M_{\rho}^{\mathrm{P}}=\sum_{\sigma \geq \rho_{\mathrm{comp}^{-1}}} \sigma, \tag{4.4}
\end{equation*}
$$

where the sum runs over all partitions $\sigma$ of the set X such that $\sigma \geq \rho_{\text {comp }^{-1}}$.
Proof. Probably the best way to prove the proposition is to consider the graphical description of the partition multiplication from Section 3.2.2, We present an alternative proof below.

From the very definition of multiplication one can see that every partition $\sigma$ which appears in $M_{\rho}^{\mathrm{P}}$ must fulfill $\sigma \geq \rho_{\text {comp }^{-1}}$.

To any partition $\sigma$ of the set $X$ we can assign a tuple ( $\sigma^{1}, \ldots, \sigma^{r}$ ), where $\sigma^{s}$ (a partition of the set $\rho_{s}$ ) is obtained from $\sigma$ by restriction to the set $\rho_{s}$. Now it is easy to observe that every $\sigma \geq \rho_{\text {comp }^{-1}}$ appears exactly once in $\prod_{s} M_{\rho_{s}}^{\mathrm{P}}$, namely in the factor $\prod_{s} \sigma^{s}$.

The moment map $M^{P}$ gives rise to a sequence of free cumulants which will be denoted by $R_{n}^{P}$.
4.2.2. Moments and cumulants of Jucys-Murphy element. Inspired by (4.1) we shall denote by $M_{\rho}^{\mathrm{JM}}$ the partition-indexed moments of the JucysMurphy element, defined by

$$
\begin{equation*}
M_{\rho}^{\mathrm{JM}}=\prod_{1 \leq s \leq r} M_{\left|\rho_{s}\right|}^{\mathrm{JM}}=\prod_{1 \leq s \leq r} \mathbb{E}\left(J^{\left|\rho_{s}\right|}\right) \in \mathbb{C}\left(\widetilde{\mathrm{S}_{\infty}}\right) \tag{4.5}
\end{equation*}
$$

where $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ is a non-crossing partition. The moment map $M^{\mathrm{JM}}$ gives rise to a sequence of free cumulants which will be denoted by $R_{n}^{\mathrm{JM}}$. Please note that $M_{1}^{\mathrm{JM}}=\mathbb{E}(\mathrm{J})=0$ and therefore also $\mathrm{R}_{1}^{\mathrm{JM}}=0$.
4.2.3. The relation between the partitions and the Jucys-Murphy element. The following Claim provides a link between the moments of the JucysMurphy element and the moments of partitions.
Claim 4.3. Let X be a finite ordered set, let $\rho$ be a non-crossing partition of a finite ordered set and $n \geq 1$ be an integer. Then

$$
\begin{align*}
\Sigma\left(M_{X}^{\mathrm{P}}\right) & =M_{|X|}^{\mathrm{JM}},  \tag{4.6}\\
\Sigma\left(M_{\rho}^{\mathrm{P}}\right) & =M_{\rho}^{\mathrm{JM}},  \tag{4.7}\\
\Sigma\left(\mathrm{R}_{n}^{\mathrm{P}}\right) & =\mathrm{R}_{n}^{\mathrm{IM}}, \tag{4.8}
\end{align*}
$$

where $\Sigma$ is the map considered in Section 3.1.1
We postpone the proof of (4.6) to Theorem 5.12. Equation (4.6) implies (4.7) immediately because $\Sigma$ is a homomorphism. Equation (4.8) is an immediate consequence of 4.7).

This result implies that all questions about the moments of the JucysMurphy element can be easily translated into questions about moments of partitions and the machinery of Section 3 can be applied.

### 4.2.4. Free cumulants of partitions.

Proposition 4.4. For every $n \geq 1$

$$
\begin{align*}
\mathrm{R}_{\mathrm{n}}^{\mathrm{P}} & =\sum_{\pi \in \mathrm{P}(1,2, \ldots, n)} \mathrm{I}_{\pi} \pi,  \tag{4.9}\\
\mathrm{R}_{\mathrm{n}}^{\mathrm{IM}} & =\sum_{\pi \in \mathrm{P}(1,2, \ldots, \mathrm{n})} \mathrm{I}_{\pi} \Sigma_{\pi}, \tag{4.10}
\end{align*}
$$

where $I_{\pi} \in \mathbb{Z}$, called free index of $\pi$, is defined by

$$
\begin{equation*}
\mathrm{I}_{\pi}=\sum_{\substack{\rho \in \mathrm{NC}(1,2, \ldots, n) \\ \rho \leq \pi}} \text { Moeb }_{\rho} . \tag{4.11}
\end{equation*}
$$

Proof. From (4.2) and (4.4) it follows that

$$
\begin{aligned}
& R_{n}^{\mathrm{P}}=\sum_{\rho \in \mathrm{NC}(1,2, \ldots, n)} \operatorname{Moeb}_{\rho_{\text {comp }}} \sum_{\substack{\pi \in \mathrm{P}(1,2, \ldots, \mathrm{n}) \\
\pi \geq \rho_{\text {comp }}}} \pi= \\
& \sum_{\pi \in \mathrm{P}(1,2, \ldots, \mathrm{n})} \pi \sum_{\substack{\rho \in \mathrm{NC}(1,2, \ldots, n) \\
\rho_{\text {comp }} \leq \pi}} \operatorname{Moeb}_{\rho_{\text {comp }}}=\sum_{\pi \in \mathrm{P}(1,2, \ldots, \mathrm{n})} \pi \sum_{\substack{\rho \in \mathrm{NC}(1,2, \ldots, \mathrm{n}) \\
\rho \leq \pi}} \operatorname{Moeb}_{\rho},
\end{aligned}
$$

where in the last equality we used the fact that $\rho \mapsto \rho_{\text {comp }}$ is a permutation of the set of non-crossing partitions.

The second identity (4.10) follows now from Claim4.3,
4.2.5. Evercrossing partitions. We say that a partition $\pi$ of a finite ordered set $X$ is evercrossing if for any $a<c(a, c \in X)$ such that $a$ is connected with $c$ by $\pi$ there exist $b, d \in X$ such that
(1) elements $a, b, c, d$ are ordered up to a cyclic rotation, i.e. either $\mathrm{a}<\mathrm{b}<\mathrm{c}<\mathrm{d}$ or $\mathrm{d}<\mathrm{a}<\mathrm{b}<\mathrm{c}$,
(2) elements $b$ and $d$ are connected by $\pi$,
(3) elements $a, b, c, d$ are not elements of the same block of $\pi$.

Up to some technical details, this means that on the graphical representation of $\pi$ every line connecting a pair of elements must be crossed by some other line. For example, partition from Figure 4.1 is evercrossing, but partition from Figure 2.3 is not.

The following result shows that only evercrossing partitions contribute to sums (4.9) and (4.10).


Figure 4.1. Example of an evercrossing partition
Theorem 4.5. If a partition $\pi$ is not evercrossing then $\mathrm{I}_{\pi}=0$.
In order to prove this result we will need the following lemma.
Lemma 4.6. Let $\pi=\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ be a partition of $X$. Let $\sigma=\left\{\pi_{1}, \ldots, \pi_{s}\right\}$, $\tau=\left\{\pi_{s+1}, \ldots, \pi_{r}\right\}$ be a decomposition of $\pi$ into two partitions of disjoint sets $S=\pi_{1} \cup \cdots \cup \pi_{s}$ and $\mathrm{T}=\pi_{s+1} \cup \cdots \cup \pi_{r}$ such that $\sigma$ is non-crossing.

For $\rho \in \mathrm{NC}(\mathrm{T})$ we shall denote (with a small abuse of notation) by $\rho_{\text {comp }} \in \mathrm{NC}(\mathrm{S})$ the biggest non-crossing partition with a property that $\rho \cup$ $\rho_{\text {comp }}$ is non-crossing. We denote by $\sigma \wedge \rho_{\mathrm{comp}} \in \mathrm{NC}(\mathrm{S})$ the biggest noncrossing partition which is smaller both than $\sigma$ and $\rho_{\text {comp }}$.

Then the free index of $\pi$ is given by

$$
\begin{equation*}
\mathrm{I}_{\pi}=\sum_{\substack{\rho \in \mathrm{NC}(\mathrm{~T}) \\ \rho \leq \tau \\\left(\sigma \wedge \rho_{\text {comp }}\right) \text { is trivial }}} \text { Moeb }_{\rho} . \tag{4.12}
\end{equation*}
$$

Proof. Since every $\rho \in \operatorname{NC}(X), \rho \leq \pi$ can be decomposed as $\rho=\tilde{\rho} \cup \hat{\rho}$, where $\tilde{\rho} \in \operatorname{NC}(S), \tilde{\rho} \leq \sigma$ and $\hat{\rho} \in \operatorname{NC}(T), \hat{\rho} \leq \tau$ it follows from (4.11) that

$$
I_{\pi}=\sum_{\substack{\hat{\rho} \in N C(T) \\ \hat{\rho} \leq \tau}} \operatorname{Moeb}_{\hat{\rho}}\left(\sum_{\substack{\tilde{\rho} \in N C(S) \\ \tilde{\rho} \leq\left(\sigma \wedge \hat{\rho}_{\text {comp }}\right)}} \operatorname{Moeb}_{\tilde{\rho}}\right) .
$$

We apply Lemma 4.1 and observe that the second sum is equal to zero unless $\sigma \wedge \hat{\rho}_{\text {comp }}$ is trivial; otherwise it is equal to 1 .
Proof of Theorem 4.5 Let $a, c$ be the vertices which have the property required for $\pi=\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ not to be evercrossing. By a change of numbering of blocks we can always assume that $\mathrm{a}, \mathrm{c} \in \pi_{1}$. Following the notation of Lemma4.6 we set $S=\pi_{1}, T=\pi_{2} \cup \cdots \cup \pi_{r}, \sigma=\left\{\pi_{1}\right\}, \tau=\left\{\pi_{2}, \ldots, \pi_{r}\right\}$.


Figure 4.2. Example of an evercrossing partition with the free index equal to 0 .

Observe that for every $\rho \in \mathrm{NC}(\mathrm{T})$ such that $\rho \leq \tau$, we have that the partition $\rho \cup\{\{a, c\}\}$ is non-crossing, therefore $\rho_{\text {comp }} \geq\{\{a, c\}\}$ and $\rho_{\text {comp }} \wedge \sigma \geq\{\{a, c\}\}$ must contain a non-trivial block and the sum in (4.12) contains no summands.

Remark. The converse implication is not true, as one can see on an example from Figure 4.2

### 4.3. First-order asymptotics of free cumulants.

Theorem 4.7. Let integers $k_{1}, \ldots, k_{t} \geq 2$ be given and let us express $R_{k_{1}}^{\mathrm{JM}} \cdots \mathrm{R}_{\mathrm{k}_{\mathrm{t}}}^{\mathrm{JM}} \in \mathfrak{P}$ as a linear combination of normalized conjugacy class indicators $\Sigma$. Then

$$
\begin{align*}
& \mathrm{R}_{\mathrm{k}_{1}}^{\mathrm{JM}} \cdots \mathrm{R}_{\mathrm{k}_{\mathrm{t}}}^{\mathrm{JM}}= \Sigma_{\mathrm{k}_{1}-1, \ldots, \mathrm{k}_{\mathrm{t}}-1}+  \tag{4.13}\\
&\left.\quad \text { (terms of degree at most } \mathrm{k}_{1}+\cdots+\mathrm{k}_{\mathrm{t}}-2\right) \text {, }
\end{align*}
$$

where the degree is taken with respect to the filtration (2.8).
Proof. Let us consider the case $\mathrm{t}=1$; we set $\mathrm{n}=\mathrm{k}_{1}$. Propositions 3.4 and 3.5 imply that the highest-order terms in the expansion (4.10) correspond to non-crossing partitions and that the degree of all remaining terms will be at most $n-2$; by Theorem 4.5 a partition contributes in the sum (4.10) only if it is evercrossing. In order to find the leading term we need to find all non-crossing partitions of the set $\{1,2, \ldots, n\}$ which are at the same time evercrossing. This combination of adjectives sounds oxymoronic which suggests that there should not be too many of such partitions and indeed it
is easy to check that the only such partition is the trivial one. Hence

$$
\begin{aligned}
& R_{n}^{\mathrm{JM}}=\Sigma_{\{\{1\},\{2\}, \ldots,\{n\}\}}+(\text { terms of degree at most } n-2)= \\
& \quad \Sigma_{n-1}+(\text { terms of degree at most } n-2)
\end{aligned}
$$

which finishes the proof of the case $t=1$.
The general case follows easily from Corollary 3.8 ,
Corollary 4.8. Elements $\left(\mathrm{R}_{\mathrm{k}}^{\mathrm{JM}}\right)_{\mathrm{k} \geq 2}$ are algebraically free; also elements $\left(M_{k}^{\mathrm{JM}}\right)_{\mathrm{k} \geq 2}$ are algebraically free and therefore the gradation (2.6) is welldefined.

Proof. Let $\mathrm{P}\left(\mathrm{x}_{2}, \mathrm{x}_{3}, \ldots\right) \neq 0$ be a polynomial; our goal is to show that $P\left(R_{2}^{\mathrm{IM}}, R_{3}^{\mathrm{JM}}, \ldots\right) \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ is non-zero. In order to show this we express $P\left(R_{2}^{\mathrm{JM}}, R_{3}^{\mathrm{JM}}, \ldots\right)$ as a linear combination of conjugacy class indicators $\Sigma$ and Theorem 4.7 shows that the highest-order terms do not cancel.

In order to show that $\left(M_{\mathrm{k}}^{\mathrm{JM}}\right)_{\mathrm{k} \geq 2}$ are algebraically free we consider a polynomial $\mathrm{Q}\left(x_{2}, x_{3}, \ldots\right) \neq 0$. We apply (4.3) and obtain a polynomial $P\left(x_{2}, x_{3}, \ldots\right)$ such that $Q\left(M_{2}^{\mathrm{JM}}, M_{3}^{\mathrm{JM}}, \ldots\right)=P\left(R_{2}^{\mathrm{JM}}, R_{3}^{\mathrm{JM}}, \ldots\right)$. Equation (4.2) implies that $P\left(x_{2}, x_{3}, \ldots\right) \neq 0$. It follows that $Q\left(M_{2}^{\mathrm{JM}}, M_{3}^{\mathrm{JM}}, \ldots\right)=$ $P\left(R_{2}^{\mathrm{JM}}, R_{3}^{\mathrm{JM}}, \ldots\right) \neq 0$.

The following result was proved in [Bia98, [O02].
Theorem 4.9. For every tuple $k_{1}, \ldots, k_{t}$ of positive integers there exists a polynomial $\mathrm{K}_{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{t}}}\left(\mathrm{R}_{2}^{\mathrm{JM}}, \mathrm{R}_{3}^{\mathrm{JM}}, \ldots\right)$, called Kerov polynomial, such that

$$
\Sigma_{k_{1}, \ldots, k_{t}}=\mathrm{K}_{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{t}}}\left(\mathrm{R}_{2}^{\mathrm{JM}}, \mathrm{R}_{3}^{\mathrm{JM}}, \ldots\right) .
$$

In other words, each of the families $\left(R_{n}^{\mathrm{JM}}\right)_{n \geq 2},\left(M_{n}^{\mathrm{JM}}\right)_{n \geq 2}$ and $\left(\Sigma_{k_{1}, \ldots, k_{m}}\right)_{k_{1}, \ldots, k_{m} \geq 1}$ generates the same commutative algebra $\mathfrak{P}$.

With respect to the gradation (2.6) the leading term of the Kerov polynomial is given by

$$
\begin{align*}
\Sigma_{k_{1}, \ldots, k_{t}} & =\mathrm{R}_{k_{1}+1}^{\mathrm{JM}} \cdots \mathrm{R}_{\mathrm{k}_{\mathrm{t}}+1}^{\mathrm{JM}}+  \tag{4.14}\\
& \left(\text { terms of degree at most }\left(\mathrm{k}_{1}+1\right)+\cdots+\left(\mathrm{k}_{\mathrm{t}}+1\right)-2\right)
\end{align*}
$$

and

$$
\operatorname{deg} \Sigma_{k_{1}, \ldots, k_{t}}=\left(k_{1}+1\right)+\cdots+\left(k_{t}+1\right)
$$

In other words, the filtration (2.8) is induced by gradation (2.6).
Proof. We shall prove this theorem by induction over $\left(k_{1}+1\right)+\cdots+\left(k_{t}+1\right)$.
Let us express $\Sigma_{k_{1}, \ldots, k_{t}}-R_{k_{1}+1}^{J M} \cdots R_{k_{t}+1}^{\mathrm{JM}}$ as a linear combination of normalized conjugacy class indicators $\Sigma$. Theorem 4.7 implies that with respect to the filtration (2.8) all these terms have degree at most $\left(k_{1}+1\right)+\cdots+$ $\left(k_{t}+1\right)-2$ and hence the inductive hypothesis can be applied: every of


Figure 4.3. Partition from Figure 4.1 after the first step of the simplification algorithm.
these terms can be expressed by the corresponding Kerov polynomial. Observe that from the very definition of free cumulants (4.2) it follows that $\operatorname{deg} R_{k_{1}+1}^{J M} \cdots R_{k_{t}+1}^{J M}=\left(k_{1}+1\right)+\cdots+\left(k_{t}+1\right)$ with respect to the gradation (2.6) and hence the theorem follows.
4.4. Higher order expansion for free cumulants. In this section we present a method of enumerating all evercrossing partitions with a given genus. The main concept is to simplify a given evercrossing partition by a number of steps. After these simplifications we obtain an evercrossing pair partition (with some extra properties) which can be enumerated easily. By reversing the process of simplifications we shall therefore obtain all evercrossing partitions with a given genus.
4.4.1. Simplification, step 1. Let an evercrossing partition be given. In the first step we remove all its trivial blocks. For example, Figure 4.3 depicts the outcome of this step performed on an evercrossing partition from Figure 4.1 In fact we do not really care what are the labels of vertices and it would be a good idea not to write them down at all; in this example we do write them down for the reason of clarity.
4.4.2. Simplification, step 2. In the second step of simplification, we consider the fat partition corresponding to the outcome of the first step (cf example on Figure 4.4). In other words, we perform the operation depicted on Figure 4.5 on each of the blocks.
4.4.3. Simplification, step 3. The outcome of the second step is a pair partition. In the third step of the simplification algorithm we perform the operation depicted on Figure 4.6. To be precise: for $n \geq 2$ let $\left(p_{1}, \ldots, p_{n}\right)$


Figure 4.4. Partition from Figure 4.1 after the second step of the simplification algorithm.


Figure 4.5. Elementary operation of step 2 of the simplification algorithm: one of the blocks is replaced by its fat version.
and $\left(q_{n}, q_{n-1}, \ldots, q_{1}\right)$ be two sequences of consecutive vertices of this pair partition such that $p_{k}$ is connected with $q_{k}$. We remove all vertices $p_{2}, p_{3}, \ldots, p_{n}, q_{2}, q_{3}, \ldots, q_{n}$ and leave vertices $p_{1}$ and $q_{1}$ unchanged. We iterate this removal of vertices as long as it is possible (cf example on Figure 4.7).

A careful reader might observe that after rotating the Figure 4.6 upsidedown the roles played by $p_{1}, q_{1}$ are interchanged with $q_{n}$ and $p_{n}$ therefore the above rule is not very precise which labels should be carried by the vertices of the surviving block; but in fact we do not really care about these labels. In particular, we do not care which of the labels is the smallest, therefore we identify pair partitions which are cyclic rotations of each other.

### 4.4.4. Outcome of the simplification algorithm.



Figure 4.6. The elementary operation of step 3 of the simplification algorithm: a number of parallel lines is replaced by a single one.


Figure 4.7. The outcome of the step 3 of the simplifying algorithm for partition from Figure 4.1

Theorem 4.10. The outcome of the simplification algorithm from Sections 4.4.1 4.4.3 is an evercrossing pair-partition which has the same genus and free index as the original partition.

Let n denote the number of vertices of the outcome pair partition $\pi$. If genus $_{\pi} \neq 0$ then

$$
\begin{equation*}
\mathrm{n} \leq 12 \operatorname{genus}_{\pi}-6 \tag{4.15}
\end{equation*}
$$

Proof. In order to show that all three steps of the simplification algorithm preserve the genus of a partition, it is enough to notice that the surface without the boundary we constructed in Section 3.1.3 changes in each of the steps into a homeomorphic one.

We shall prove now that all three steps preserve the free index of a partition. Let $\pi$ be a partition of some set $X$, let $\chi \notin X$, then $\pi^{\prime}=\pi \cup\{x\}$ is
a partition of the set $X \cup\{x\}$ which has an additional trivial block. Observe that every partition $\rho^{\prime} \leq \pi^{\prime}$ must have a form $\rho^{\prime}=\rho \cup\{\{x\}\}$ where $\rho \leq \pi$. Therefore

$$
I_{\pi^{\prime}}=\sum_{\substack{\left.\rho^{\prime} \in \mathbb{N C}(X \cup\{x\}\}\right) \\ \rho^{\prime} \leq \pi^{\prime}}} \operatorname{Moeb}_{\rho^{\prime}}=\sum_{\substack{\rho \in \mathbb{N C}(X) \\ \rho \leq \pi}} \operatorname{Moeb}_{\rho}=I_{\pi} .
$$

It follows that two partitions which differ with one trivial block have the same free index and, by iterating, the first step preserves the free index.

The second and the third step can be divided into a number of elementary operations (for the second step the elementary operation is the replacement of only one of the blocks of the original partition by the corresponding 'fat block', cf Figure 4.5, for the third step the elementary operation is the performance of only one operation from Figure 4.6. Our goal is to prove that every elementary operation preserves the free index. Let $\pi$ be a partition and let $\pi^{\prime}$ be an outcome of one of the elementary operations performed on $\pi$. We can decompose $\pi=\sigma \cup \tau$, $\pi^{\prime}=\sigma^{\prime} \cup \tau$ (we use the notations of Lemma 4.6, where $\tau \in \mathrm{P}(\mathrm{T})$ is the set of blocks of $\pi$ not changed by the elementary operation. One can easily check that for both elementary operations $\sigma \in \mathrm{NC}(S)$ and $\sigma^{\prime} \in \mathrm{NC}\left(\mathrm{S}^{\prime}\right)$ are non-crossing partitions, furthermore for every $\rho \in \mathrm{P}(\mathrm{T})$ we have that ( $\sigma \wedge \rho_{\text {comp }}$ ) $\in \mathrm{NC}(\mathrm{S})$ is trivial if and only if $\left(\sigma^{\prime} \wedge \rho_{\text {comp }}\right) \in \operatorname{NC}\left(S^{\prime}\right)$ is trivial. From Lemma 4.6 applied to $\pi$ and $\pi^{\prime}$ it follows that

$$
\mathrm{I}_{\pi}=\sum_{\substack{\rho \in \mathrm{NC}(\mathrm{~T}), \rho \leq \tau \\\left(\sigma \wedge \rho_{\text {comp }}\right) \text { is trivial }}} \operatorname{Moeb}_{\rho}=\sum_{\substack{\rho \in \mathrm{NC}(\mathrm{~T}), \rho \leq \tau,\left(\sigma^{\prime} \wedge \rho_{\text {comp }}\right) \text { is trivial }}} \operatorname{Moeb}_{\rho}=I_{\pi^{\prime}}
$$

and hence both elementary operations preserve the free index.
Observe also, that the first and the third step of the algorithm preserve clearly the partition property of being evercrossing. It should be clear from the graphical interpretation that also the second step preserves this property; nevertheless we provide a more detailed proof below. Let a partition $\pi$ be an outcome of the first step of the algorithm. Any pair connected by $\pi_{\text {fat }}$ must be of the form ( $\pi_{s, t}^{\prime}, \pi_{s, t+1}$ ) (we use the notation from Section 2.2.2). As an outcome of the first step of the simplification algorithm, $\pi$ contains no trivial blocks hence $\pi_{s, t} \neq \pi_{s, t+1}$. Since $\pi$ is evercrossing, there exist $\mathrm{b}, \mathrm{d} \in \pi_{\mathrm{u}}$, $u \neq s$, such that $\pi_{s, t}, b, \pi_{s, t+1}, d$ are ordered up to a cyclic rotation. In other words: on the graphical representation on a circle, elements of the block $\pi_{u}$ are not all on the same side of the line which passes through $\pi_{s, t}$ and $\pi_{s, t+1}$. Therefore there must be some consequent elements $\pi_{u, v}, \pi_{u, v+1}$ of the block $\pi_{u}$ such that $\pi_{s, t}, \pi_{u, v}, \pi_{s, t+1}, \pi_{u, v+1}$ are ordered up to a cyclic
rotation. It is easy to see that $b=\pi_{u, v}^{\prime}$ and $d=\pi_{s, t+1}$ are the vertices required by the property of $\pi_{\mathrm{fat}}$ to be evercrossing.

Let $\pi$ be the outcome of the simplification algorithm and such that genus $_{\pi} \neq 0$. We keep notation from Section 3.1.3, i.e. $r=\frac{n}{2}$ denotes the number of blocks of $\pi$ and $t$ denotes the number of cycles of $\pi_{\text {fat }} \circ \mathrm{c}$ (or, alternatively, the number of holes after gluing the first collection of discs). Observe that on the boundary of every hole there must be at least three intervals which touch some discs from the first collection (if some hole does not touch any discs then $\pi=\emptyset$ and genus ${ }_{\pi}=0$; if some hole touches only one disc we denote by $a, c$ the only two vertices which are on the boundary of the hole and clearly $(a, c)$ is a pair of vertices required for $\pi$ to be nonevercrossing; if some hole touches exactly two discs then it is possible to perform some simplifications of step 3; a careful reader might observe that it is possible and perfectly legal for a hole to touch some of the discs twice, cf the left-hand side partition on Figure 4.10). Since $\pi$ is a pair partition, every disc of the first collection touches the holes on exactly two intervals. It follows that $n=2 r \geq 3 t$. Equation (3.4) implies that

$$
n \leq 12 \text { genus }_{\pi}-6
$$

which finishes the proof.
4.4.5. Interpretation of the free index? Free index of a partition is a quite strange combinatorial object. We can think that it measures how far a given partition is away from non-crossing partitions or how much it is evercrossing. Its particularly interesting feature appears in Theorem 4.10 namely that it is being preserved by a number of operations one can perform on a partition. The list of such operations is far from being complete and we leave it as an exercise to the reader to find at least one such operation which does not appear in the simplification algorithm. Suppose we have completed such a list of natural operations preserving free index; now we can treat them as analogues of Reidemeister moves in the knot theory and say that two partitions are isotopic when one can be obtained from another by a sequence of such elementary operations. Of course free index will be an invariant of the isotopy class but we should not expect that it will always be able to distinguish different isotopy classes. A question arises: is there some natural (geometric?) interpretation of such isotopy?

### 4.5. The second main result: the second-order expansion of free cumulants and characters.

4.5.1. Evercrossing partitions with genus 1. Theorem 4.10 allows us to enumerate (in finite time) all possible outcomes of the simplifying algorithm for a fixed genus of the original evercrossing partition. By reversing


Figure 4.8. General pattern of evercrossing partitions with genus 1 and free index equal to -1 (only non-trivial blocks were shown).


Figure 4.9. General pattern of evercrossing partitions with genus 1 and free index equal to -2 (only non-trivial blocks were shown).
all three steps of the simplifying algorithm we are able to write a number of "templates" which describe all evercrossing partitions with a prescribed genus. Let us have a look on an example.

Theorem 4.11. Every evercrossing partition with genus 1 must be either of the form depicted on Figure 4.8 (in this case it has the free index equal to -1) or Figure 4.9 (in this case it has the free index equal to -2 ). On both figures only non-trivial blocks were shown.


Figure 4.10. The only two possible outcomes of the simplification algorithm for an evercrossing partition with genus 1. The pair partition on the left-hand side has free index equal to -1 and the pair partition on the right-hand side has free index equal to -2 .

Proof. Theorem 4.10 implies that an outcome of the simplification algorithm obtained for an evercrossing partition with genus 1 must have at most 6 vertices. By direct inspection of all such pair partitions one can find that there are only two possible outcomes of the simplifying algorithm and they are depicted on Figure 4.10

By reverting the third step of the simplifying algorithm we see that the outcome of the first two steps of the simplifying algorithm must be a pair partition either of the form from Figure 4.8 or Figure 4.9. For Figure 4.9 we have to study $2^{4}$ cases now: each of the numbers $p, q, r$ can be either even or odd, we also have to guess which of the vertices carry primed and which carry non-primed labels (we have two possibilities); the case of Figure 4.8 is slightly simpler. For each of the $2^{4}$ cases we try to reverse the second step of the algorithm (and for most of the cases it is not possible; a less patient reader will find easily arguments which would allow to find only the few cases when it is possible). It turns out that after the first step of the simplification algorithm must be a pair partition either of the form depicted on Figure 4.8 or on Figure 4.9. Reverting the first step means that we are allowed to add any number of trivial blocks, which finishes the proof.
4.5.2. Second-order asymptotics of free cumulants. We leave the proof of the following simple lemma to the reader.

Lemma 4.12. For any $y, n \in \mathbb{N}$ there are $\binom{y-1}{n-1}$ solutions of the equation $x_{1}+\cdots+x_{n}=y$ if we require $x_{1}, \ldots, x_{n}$ to be positive integers.

Theorem 4.13. For every $\mathfrak{n} \in \mathbb{N}$ we have

$$
\begin{gather*}
R_{n}^{\mathrm{JM}}=\Sigma_{n-1}-\sum_{\substack{m_{2}, m_{3}, \ldots \geq 0 \\
2 m_{2}+3 m_{3}+4 m_{4}+\cdots=n-2}} \frac{n(n-1)(n-2)}{24}\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots} \times  \tag{4.16}\\
\prod_{s \geq 2}((s-1))^{m_{s}} \underbrace{\sum_{m_{3}}, \ldots, 1}_{m_{2} \text { tines }} \underbrace{2, \ldots, 2, \ldots, 3}_{m_{3} \text { tines }} \underbrace{3, \ldots}_{m_{4} \text { tines }}+
\end{gather*}
$$

(terms of degree at most $\mathrm{n}-4$ )
Proof. The highest-order term in expansion (4.10) was already found in Theorem4.7 The next highest-order terms correspond to evercrossing partitions with genus 1 . We shall calculate first the contribution of the partitions of the form depicted on Figure 4.8 .

For the partition from Figure 4.8 the tuple $\left(k_{1}, \ldots, k_{p+q-1}\right)$ given by the algorithm from Section 3.1.1 is equal to

$$
\begin{align*}
& \left(\left(\left(a_{s+1}-a_{s}\right)+\left(c_{s}-c_{s+1}\right)-1\right)_{s=1,2, \ldots, p-1}\right.  \tag{4.17}\\
& \left(\left(b_{1}-a_{p}\right)+\left(c_{p}-b_{q}\right)+\left(d_{q}-c_{1}\right)+\left(a_{1}-d_{1}\right)\right)-3 \\
& \left.\quad\left(\left(b_{s+1}-b_{s}\right)+\left(d_{s}-d_{s+1}\right)-1\right)_{s=1,2, \ldots, q-1}\right)
\end{align*}
$$

Here and in the following, the subtraction is taken modulo $n$, i.e.

$$
y-x= \begin{cases}y-x & \text { if } y>x \\ (y+n)-x & \text { if } y<x\end{cases}
$$

This subtraction has a natural interpretation as the number of elements between $x$ and $y$ going counterclockwise from $x$ to $y$.

Let $m_{2}, m_{3}, \ldots$ be a sequence of nonnegative integers such that $2 m_{2}+$ $3 m_{3}+\cdots=n-2$. We shall count now for how many different partitions of the form depicted on Figure 4.8 the tuple 4.17) is equal (up to a permutation) to

$$
\begin{equation*}
\left(k_{1}^{\prime}, k_{2}^{\prime}, \ldots, k_{p+q-1}^{\prime}\right)=(\underbrace{1, \ldots, 1}_{m_{2} \text { times }}, \underbrace{2, \ldots, 2}_{\mathfrak{m}_{3} \text { times }}, \underbrace{3, \ldots, 3}_{\mathfrak{m}_{4} \text { times }} \ldots) . \tag{4.18}
\end{equation*}
$$

Firstly, observe that all the numbers $\left(a_{s}\right),\left(b_{s}\right),\left(c_{s}\right),\left(d_{s}\right)$ are uniquely determined by $a_{1}$ and the collection of increments
(4.19) $\left(a_{s+1}-a_{s}\right)_{s=1, \ldots, p-1},\left(b_{1}-a_{p}\right),\left(b_{s+1}-b_{s}\right)_{s=1, \ldots, q-1}$,
$\left(c_{p}-b_{q}\right),\left(c_{s}-c_{s+1}\right)_{s=1, \ldots, p-1},\left(d_{q}-c_{1}\right),\left(d_{s}-d_{s+1}\right)_{s=1, \ldots, q-1},\left(a_{1}-d_{1}\right)$,
where numbers (4.19) are positive integers the sum of which is equal to $n$.

Every partition of the form depicted on Figure 4.8 has free index equal to $(-1)$. There are $n$ choices of $a_{1}$. There are $\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots}$ different permutations of the tuple 4.18). We need to specify $1 \leq p \leq m_{2}+m_{3}+\cdots$. Now we count in how many different ways every of the numbers 4.17) can be written as a sum of two specified elements of 4.19 minus 1 except for one of the numbers (4.17) which should be written as a sum of four specified elements of (4.19) minus 3 ; the position of this exceptional number is specified by the index $p$. Finally, we have to take into account the symmetry factor $\frac{1}{4}$ of Figure 4.8 since for every partition of this form we can choose sequences $\left(a_{s}\right),\left(b_{s}\right),\left(c_{s}\right),\left(d_{s}\right)$ in four different ways corresponding to four cyclic rotations of the template. From Lemma 4.12 it follows that the contribution of partitions from Figure 4.8 to the summand $\underbrace{1_{1} \ldots, 1}_{m_{2} \text { times }} \underbrace{2, \ldots, 2}_{m_{3} \text { times }} \ldots$ is equal to

$$
\begin{equation*}
(-1) \frac{1}{4} n\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots} \sum_{1 \leq p \leq m_{2}+m_{3}+\cdots}\binom{k_{p}^{\prime}+2}{3} \prod_{p^{\prime} \neq p}\binom{k_{p^{\prime}}^{\prime}}{1} . \tag{4.20}
\end{equation*}
$$

By very similar considerations one can easily show that the contribution of partitions from Figure 4.9 to the summand $\sum_{m_{2} \text { times }}^{1_{2}, \ldots, 1} \underbrace{2, \ldots, 2}_{m_{3} \text { times }}, \ldots$ is equal to

$$
\begin{align*}
& (-2) \frac{1}{6} n\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots} \times  \tag{4.21}\\
& \quad \sum_{1 \leq p_{1}<p_{2} \leq m_{2}+m_{3}+\cdots}\binom{k_{p_{1}}^{\prime}+1}{2}\binom{k_{p_{2}}^{\prime}+1}{2} \prod_{p^{\prime} \neq p_{1}, p_{2}}\binom{k_{p^{\prime}}^{\prime}}{1} .
\end{align*}
$$

By an application of Leibnitz rule we see that the sum of (4.20) and (4.21) can be written in a compact form as

$$
\begin{aligned}
& {\left[x^{-\left(2 m_{2}+3 m_{3}+\cdots\right)-2} z_{2}^{m_{2}} z_{3}^{m_{3}} \cdots\right] \frac{(-1) n}{24} \times} \\
& \frac{d^{2}}{d x^{2}}\left(\sum_{s}(s-1) z_{s} x^{-s}\right)^{m_{2}+m_{3}+\cdots}= \\
& {\left[z_{2}^{m_{2}} z_{3}^{m_{3}} \cdots\right] \frac{(-1) n(n-1)(n-2)}{24}\left(\sum_{s}(s-1) z_{s}\right)^{m_{2}+m_{3}+\cdots}=} \\
& \quad \frac{(-1) n(n-1)(n-2)}{24}\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \cdots} 1^{m_{2}} 2^{m_{3}} 3^{m_{4}} \cdots
\end{aligned}
$$

which finishes the proof.
4.5.3. Second-order expansion of characters. The following result was conjectured by Biane [Bia03].

Theorem 4.14. For every $n \geq 2$ we have

$$
\begin{aligned}
& \Sigma_{n-1}=R_{n}^{\mathrm{JM}}+ \\
& \quad \sum_{\substack{m_{2}, m_{3}, \ldots \geq 0 \\
2 m_{2}+3 m_{3}+\cdots=n-2}} \frac{1}{4}\binom{n}{3}\binom{m_{2}+m_{3}+\cdots}{m_{2}, m_{3}, \ldots} \prod_{s \geq 2}\left((s-1) R_{s}^{J M}\right)^{m_{s}}+
\end{aligned}
$$

(terms of degree at most $\mathrm{n}-4$ ).
Proof. In (4.16) we use (4.14) to express $\underbrace{\sum_{m_{3} \text { times }}, \ldots, 1}_{m_{2} \text { times }} \underbrace{2, \ldots, 2}_{2}$, in terms of $R_{2}^{\mathrm{JM}}, \mathrm{R}_{3}^{\mathrm{JM}}, \ldots$
4.6. Higher order expansion of characters. In principle, there are no obstacles to repeat the reasoning from Section 4.5 for partitions with genus 2 and in this way obtain higher order asymptotic expansions.

The first step is to enumerate all possible outcomes of the simplification algorithm. Theorem 4.10 shows that such outcome is a pair partition with at most 9 lines. The patience of the most human beings is not sufficient enumerate all such partitions, nevertheless the use of computer allows us to find them all [Śni04]. After removing cyclic rotations and mirror images there are 61 such partitions-which is still an accessible number. By reverting all three steps of the simplification algorithm we could therefore find general patterns of evercrossing partitions with genus 2 and therefore express $R_{n}^{J M}$ in terms of elements $\Sigma$.

However, the above method seems to be far more complicated than the result of Goulden and Rattan [GR05] who found an explicit formula for the general coefficients of Kerov polynomials.

## 5. PRoofs of technical results

5.1. Admissible and pushing sequences. In this article we manipulate with moments of the Jucys-Murphy element J (or with products of such moments) and we need an efficient way to enumerate the summands in sums similar to (2.5]. Conceptually the simplest way is to enumerate the summands by sequences $a_{1}, \ldots, a_{n} \in A$ and which will be called admissible. However, it turns out that manipulating admissible sequences is quite complicated. It turns out that the suitable objects to enumerate summands in (2.5) are pushing sequences. We will describe pushing sequences in terms of admissible sequences.
5.1.1. Admissible sequences. Map $\Phi_{\bar{S}_{\AA}}^{\text {as }}$. We recall that $* \notin A$. We say that a sequence $a_{1}, \ldots, a_{n}$ is admissible if it contributes in the sum (2.5) or equivalently if $a_{1}, \ldots, a_{n} \in A$ and

$$
\begin{equation*}
\left[\left(a_{1} *\right) \cdots\left(a_{n} *\right)\right](*)=* \tag{5.1}
\end{equation*}
$$

The set of admissible sequences is a semigroup if for multiplication we take the concatenation of sequences:

$$
\mathbf{a b}=\left(a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{o}\right)
$$

for admissible sequences $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ and $\mathbf{b}=\left(b_{1}, \ldots, b_{o}\right)$. The unit is the empty sequence.

We consider a map $\Phi_{S_{A}}^{\text {as }}$ from the set of admissible sequences to the symmetric group $S_{\text {A }}$ by setting

$$
\Phi_{S_{A}}^{\text {as }}:\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(a_{1} *\right) \cdots\left(a_{n} *\right) .
$$

We can treat the right-hand side as a partial permutation of $A$ with the support $\left\{a_{1}, \ldots, a_{n}\right\}$; in this way we obtain a map $\Phi_{\widetilde{S_{A}}}^{\text {as }}$ from the set of admissible sequences to the semigroup $\widetilde{S_{A}}$ of partial permutations. It is easy to check that both maps are homomorphisms of semigroups.
5.1.2. Pushing sequences. We say that no neighbor elements of a sequence $\left(p_{1}, \ldots, p_{n}\right)$ are equal if $p_{l} \neq p_{l+1}$ holds for any $1 \leq l \leq n$. We should think that the elements of the sequence are arranged in a circle and the successor of $p_{n}$ is $p_{1}$, therefore in the above definition the case $l=n$ means that we require $p_{n} \neq p_{1}$. In particular, it cannot happen that $n=1$.

We will say that a sequence $\left(p_{1}, \ldots, p_{n}\right)$ is a pushing sequence if $p_{1}, \ldots, p_{n} \in A \cup\{*\}$ and if $p_{1}=*$ and if no neighbor elements of $\left(p_{1}, \ldots, p_{n}\right)$ are equal.
5.1.3. Commutative diagram. In the following we will construct functions such that the following diagram commutes. Later on we shall equip pushing sequences with a multiplicative structure in such a way that all maps will become homomorphisms of semigroups.

5.1.4. $\operatorname{Map} \Phi_{\mathrm{ps}} \mathrm{as}^{\text {as }}$ Let $\mathbf{a}=\left(\mathrm{a}_{1}, \ldots, \mathrm{a}_{\mathrm{n}}\right), \mathrm{a}_{1}, \ldots, \mathrm{a}_{\mathrm{n}} \in \mathcal{A}$ be an admissible sequence. We assign to it a sequence of permutations $\sigma_{1}, \ldots, \sigma_{n+1} \in S_{A \cup\{*\}}$ defined by

$$
\begin{equation*}
\sigma_{l}=\left(a_{l} *\right)\left(a_{l+1} *\right) \cdots\left(a_{n} *\right) \tag{5.3}
\end{equation*}
$$

and a sequence $\mathbf{p}=\Phi_{\mathrm{ps}}^{\text {as }}(\mathbf{a})=\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{n}}\right), \mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{n}} \in A \cup\{*\}$ defined by

$$
\begin{equation*}
p_{\mathrm{l}}=\sigma_{\mathrm{l}}^{-1}(*) . \tag{5.4}
\end{equation*}
$$

Notice that since $\mathbf{a}$ is admissible hence $p_{1}=*$ and $p_{1}=* \neq a_{n}=p_{n}$. Furthermore since $\sigma_{l-1}=\left(a_{l-1} *\right) \sigma_{l}$ hence $p_{l-1}=\sigma_{l}^{-1}\left(a_{l-1}\right) \neq \sigma_{l}^{-1}(*)=$ $p_{l}$ and $\mathbf{p}$ is a pushing sequence.

Equation (5.4) should explain the name of pushing sequences: imagine that $\mathcal{A} \cup\{*\}$ is a set of some items and initially every item $x$ is located within a box which carries a label ' $x$ '. In the process of computing the product $\left(a_{1} *\right) \cdots\left(a_{n} *\right)$ (from the right to the left) we obtain a sequence of partial products $\sigma_{n}, \sigma_{n-1}, \ldots, \sigma_{1}$ : permutation $\sigma_{l-1}$ is obtained from $\sigma_{l}$ by transposition of the item which is in the box labeled by ' $*$ ' and the item (which turns out to be the item $p_{l-1}$ ) from the box labeled by ' $a_{l-1}$ '. We can think therefore that the item $p_{l-1}$ is pushing out the contents of the box labeled by ' $*$ '.
5.1.5. Map $\Phi_{\mathrm{as}}^{\mathrm{ps}}$. Let a pushing sequence $\mathbf{p}=\left(p_{1}, \ldots, p_{\mathrm{n}}\right)$ be given. By backward induction we will assign to it a unique pair of sequences $\left(\sigma_{1}, \ldots, \sigma_{\mathfrak{n}+1}\right), \Phi_{\mathrm{as}}^{\mathrm{ps}}(\mathbf{p})=\left(a_{1}, \ldots, a_{n}\right)$, where $\sigma_{1}, \ldots, \sigma_{n+1} \in S_{A \cup\{ \}\}}$ and $\left(a_{1}, \ldots, a_{n}\right)$ is an admissible sequence such that (5.3) and (5.4) are fulfilled.

We start the construction by setting $\sigma_{\mathfrak{n}+1}=e$ since it is the only choice if we want (5.3) to be fulfilled for $l=n+1$. If $\sigma_{l}$ is already defined we set $a_{l-1}=\sigma_{l}\left(p_{l-1}\right)$ and $\sigma_{l-1}=\left(a_{l-1} *\right) \sigma_{l}$. Again, it is the only choice that we have since equations (5.3) and (5.4) imply

$$
\begin{equation*}
a_{l-1}=\left(a_{l-1} *\right)(*)=\sigma_{l} \sigma_{l-1}^{-1}(*)=\sigma_{l}\left(p_{l-1}\right) . \tag{5.5}
\end{equation*}
$$

Since $\mathbf{p}$ is pushing hence $\sigma_{l}^{-1}\left(a_{l-1}\right)=p_{l-1} \neq p_{l}=\sigma_{l}^{-1}(*)$ therefore $a_{1}, \ldots, a_{n} \neq *$; furthermore $\left[\left(a_{1} *\right) \cdots\left(a_{n} *\right)\right]^{-1}(*)=\sigma_{1}^{-1}(*)=p_{1}=*$ hence $\mathbf{a}$ is admissible.

Equations (5.3) and (5.4) are indeed fulfilled by construction, therefore $\Phi_{\mathrm{ps}}^{\mathrm{as}}\left(\Phi_{\mathrm{as}}^{\mathrm{ps}}(\mathbf{p})\right)=\mathbf{p}$ for every pushing sequence $\mathbf{p}$. Furthermore the uniqueness of the pair $\left(a_{1}, \ldots, a_{n}\right),\left(\sigma_{1}, \ldots, \sigma_{n+1}\right)$ implies that there exists at most one preimage of $\mathbf{p}$ with respect to $\Phi_{\mathrm{ps}}^{\text {as }}$. Therefore we proved the following result:

Proposition 5.1. Maps $\Phi_{\mathrm{as}}^{\mathrm{ps}}$ and $\Phi_{\mathrm{ps}}^{\mathrm{as}}$ are inverses of each other.
5.1.6. Map $\Phi_{\widetilde{\mathcal{S}_{A}}}^{\mathrm{ps}}$. The bijections between pushing and admissible sequences allow us to define the map $\Phi_{\widehat{\mathrm{S}_{\mathrm{A}}}}^{\mathrm{ps}}$ in a unique way which makes the diagram (5.2) commute.

Let $\mathbf{p}$ be a pushing sequence and $\mathbf{a}$ be the corresponding admissible sequence. Observe that if $\sigma_{1}, \ldots, \sigma_{n+1}$ is the sequence of permutations which was constructed in Sections 5.1.4 and 5.1.5 then

$$
\begin{equation*}
\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}(\mathbf{p})=\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{as}}(\mathbf{a})=\sigma_{1} \tag{5.6}
\end{equation*}
$$

In order to compute $\Phi_{\widetilde{S_{A}}}^{\mathrm{ps}}(\mathbf{p})$ we need to specify the support of this permutation. It is the support of $\Phi_{S_{A}}^{\text {as }}(\mathbf{a})$, namely $\left\{a_{1}, \ldots, a_{n}\right\}=\left\{p_{1}, \ldots, p_{n}\right\} \backslash\{*\}$; the inclusions which imply the latter equality follow directly from the construction of the maps $\Phi_{\mathrm{ps}}^{\mathrm{as}}$ and $\Phi_{\mathrm{as}}^{\mathrm{ps}}$. In Section 5.2.1 we shall compute $\Phi_{{\widetilde{S_{A}}}_{\mathrm{ps}}^{\mathrm{ps}}}$ and $\Phi_{\mathrm{as}}^{\mathrm{ps}}$ more explicitly.
5.1.7. Multiplication of pushing sequences. The bijections between pushing and admissible sequences allow us to define multiplication of pushing sequences in the unique way which makes arrows of (5.2) to be homomorphisms of semigroups in such a way that the diagram commutes.

The exact form of this multiplication is given by the following proposition

Proposition 5.2. Let $\mathbf{p}=\left(p_{1}, \ldots, p_{k}\right)$ and $\mathbf{q}=\left(q_{1}, \ldots, q_{\imath}\right)$ be pushing sequences and let $\pi=\Phi_{S_{A}}^{\mathrm{ps}}\left(\mathrm{q}_{1}, \ldots, \mathrm{q}_{1}\right)$. Then

$$
\mathbf{p q}=\left(\pi^{-1}\left(p_{1}\right), \ldots, \pi^{-1}\left(p_{k}\right), q_{1}, \ldots, q_{l}\right) .
$$

Proof. Let $\mathbf{p}=\Phi_{\mathrm{ps}}^{\mathrm{as}}(\mathbf{a})$ and let $\sigma_{1}^{\mathrm{a}}, \ldots, \sigma_{\mathrm{k}}^{\mathrm{a}}$ be the sequence of permutations constructed in Section 5.1 .4 for a, i.e. $\sigma_{m}^{a}=\left(a_{m} *\right) \cdots\left(a_{k} *\right)$ for $1 \leq m \leq k$. Let $\mathbf{b}=\Phi_{\mathrm{ps}}^{\text {as }}(\mathbf{q})$ and let $\sigma_{1}^{\mathrm{b}}, \ldots, \sigma_{\mathrm{l}}^{\mathrm{b}}$ be the analogous sequence of permutations obtained for $\mathbf{b}$.

Let $\sigma_{1}, \ldots, \sigma_{k+l}$ be the sequence of permutations obtained for the concatenated sequence ab. Observe that

$$
\begin{aligned}
& \sigma_{\mathfrak{m}}= \begin{cases}\left(b_{\mathfrak{m}-k^{*}}\right) \cdots\left(b_{l} *\right) & \text { for } \mathfrak{m} \geq k+1 \\
\left(a_{\mathfrak{m}} *\right) \cdots\left(a_{k} *\right)\left(b_{1 *}\right) \cdots\left(b_{l} *\right) & \text { for } m \leq k\end{cases} \\
&= \begin{cases}\sigma_{\mathfrak{m}-k}^{b} & \text { for } m \geq k+1 \\
\sigma_{\mathfrak{m}}^{a} \pi & \text { for } m \leq k\end{cases}
\end{aligned}
$$

where $\pi=\left(\mathrm{b}_{1^{*}}\right) \cdots\left(\mathrm{b}_{l^{*}}\right)=\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{as}}(\mathbf{b})=\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}(\mathbf{q})$.

It follows that $\mathbf{r}=\left(\mathrm{r}_{1}, \ldots, \mathrm{r}_{\mathrm{n}+\mathrm{l}}\right)=\mathbf{p q}=\Phi_{\mathrm{ps}}^{\text {as }}(\mathbf{a b})$ is given by

$$
r_{m}=\sigma_{m}^{-1}(*)= \begin{cases}q_{m-n} & \text { for } m \geq n+1 \\ \pi^{-1}\left(p_{m}\right) & \text { for } m \leq n\end{cases}
$$

which finishes the proof.
5.1.8. Invariance of the conjugacy class assigned to a pushing sequence. For the purpose of this section let us forget about the set $A$ and the distinguished element $*$. Let a sequence $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right)$ be given such that no neighbor elements are equal: $p_{l} \neq p_{l+1}$ for all $1 \leq l \leq n$. Now we can define the distinguished element $*$ by setting $*=p_{1}$ and set $\mathcal{A}$ by $A=\left\{p_{1}, \ldots, p_{n}\right\} \backslash\{*\}$.

Let $f$ be a one to one function defined on the set $\left\{p_{1}, \ldots, p_{n}\right\}$. We set $\mathbf{p}^{\prime}=\left(f\left(p_{1}\right), \ldots, f\left(p_{n}\right)\right), *^{\prime}=f(*), A^{\prime}=f(A)$.

Pushing sequence $\mathbf{p}$ defines a partial permutation $\Phi_{\widetilde{S_{A}}}^{\mathrm{ps}}(\mathbf{p}) \in \widetilde{\mathrm{S}_{\mathcal{A}}}$ and the pushing sequence $\mathbf{p}^{\prime}$ defines a partial permutation $\widetilde{\Phi^{\mathrm{ps}}}\left(\mathbf{p}^{\prime}\right) \in \widetilde{\mathbf{S}_{A^{\prime}}}$.

Proposition 5.3. We use the above notations. Then

$$
\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}\left(\mathbf{p}^{\prime}\right)=\mathrm{f} \circ \Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}(\mathbf{p}) \circ \mathrm{f}^{-1}
$$

and the support of $\Phi_{\stackrel{\mathrm{Ss}}{\mathcal{A}^{\prime}}}^{\mathrm{ps}}\left(\mathbf{p}^{\prime}\right)$ is equal to the image of the support of $\Phi_{\stackrel{\bar{S}_{A}}{\mathrm{ps}}}(\mathbf{p})$ under the map f .

Proof. Let $a_{1}, \ldots, a_{n}$ and $\sigma_{1}, \ldots, \sigma_{n+1}$ be the sequences we considered in Sections 5.1.4 and 5.1.5 and let $a_{1}^{\prime}, \ldots, a_{n}^{\prime}$ and $\sigma_{1}^{\prime}, \ldots, \sigma_{n+1}^{\prime}$ be their analogues obtained for the pushing sequence $\mathbf{p}^{\prime}$.

We leave to the reader to show by backward induction that $a_{l}^{\prime}=f\left(a_{l}\right)$ and $\sigma_{l}^{\prime}=f \circ \sigma_{\imath} \circ f^{-1}$ for every $1 \leq l \leq n$. From (5.6) (and its analogue for $\mathbf{p}^{\prime}$ ) it follows that

$$
\Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}\left(\mathbf{p}^{\prime}\right)=\mathrm{f} \circ \Phi_{\mathrm{S}_{\mathrm{A}}}^{\mathrm{ps}}(\mathbf{p}) \circ \mathrm{f}^{-1}
$$

Now it is enough to observe that the support of $\Phi_{\frac{S_{A^{\prime}}}{\mathrm{ps}}}\left(\mathbf{p}^{\prime}\right)$ is equal to $A^{\prime}$ and the support of $\Phi_{\widetilde{S_{A}}}^{\mathrm{ps}}(\mathbf{p})$ is equal to $A$.
5.2. Pushing partitions. In this article we are concerned mostly with conjugacy classes of permutations and not permutations themselves. For this reason we need a method to enumerate in the sums similar to (2.5) not the individual summands, but whole classes of summands (summands within each class will be conjugate). Pushing partitions will turn out to be the suitable objects to enumerate such classes.

We say that a partition $\pi$ of a finite ordered set $X$ (by changing the labels we can assume that $X=\{1, \ldots, n\}$ ) is a pushing partition if no neighbors $l$
and $l+1$ are connected by $\pi$ for $1 \leq l \leq n$. It is useful to arrange numbers $1, \ldots, \mathrm{n}$ in a circle, therefore the case $\mathrm{l}=\mathrm{n}$ should be understood that neighbors $n$ and 1 are not connected by $\pi$. In particular, it cannot happen that $n=1$.

We say that $\mathbf{p} \sim \pi$ if $\pi$ is a partition of $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right)$ is a sequence of length $n$ such that for every $1 \leq l, m \leq$ $n$ numbers $l$ and $m$ are connected by $\pi$ iff $p_{l}=p_{m}$. For every pushing sequence $\mathbf{p}$ there exists a unique pushing partition $\pi$ such that $\mathbf{p} \sim \pi$.
5.2.1. Explicit form of the maps $\Phi_{\widetilde{\mathrm{S}}_{\mathrm{A}}}^{\mathrm{ps}}$ and $\Phi_{\mathrm{as}}^{\mathrm{ps}}$. Let a pushing partition $\pi=\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ of the set $\{1,2, \ldots, n\}$ be given. To every block $\pi_{s}$ of the partition we assign a label $\min \pi_{s}$, i.e. the smallest element of $\pi_{s}$. To every element of $\{1,2, \ldots, n\}$ we assign the label of the block it belongs to. In this way we constructed a sequence $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right)$ such that $\mathbf{p} \sim \pi$. We set $*=p_{1}=1$ and $\mathcal{A}$ to be the set of the other labels, i.e. $A=\left\{p_{1}, \ldots, p_{n}\right\} \backslash\{*\}=\left\{\min \pi_{s}: 1 \leq s \leq r\right\} \backslash\{1\}$. We decorate on the graph of the fat partition $\pi_{\text {fat }}$ all elements of the set $\mathcal{A}$ (cf Figure 3.1).

In the following we shall compute $\Phi_{\stackrel{\mathrm{S}}{\mathrm{A}}}^{\mathrm{ps}}(\mathbf{p})$ for $\mathbf{p}$ defined as above; please note that Proposition 5.3 allows us to rename the labels and to find $\Phi_{\overline{\mathrm{S}_{A}}}^{\mathrm{ps}}(\mathbf{p})$ for a general $\mathbf{p}$.

As we already mentioned in Section 3.1.1 we can view the fat partition $\pi_{\text {fat }}$ as a bijection $\pi_{\text {fat }}:\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\} \rightarrow\{1,2, \ldots, n\}$. We also consider a bijection $\mathrm{c}:\{1,2, \ldots, n\} \rightarrow\left\{1^{\prime}, 2^{\prime}, \ldots, n^{\prime}\right\}$ given by $\ldots, 3 \mapsto 2^{\prime}, 2 \mapsto$ $1^{\prime}, 1 \mapsto n^{\prime}, n \mapsto(n-1)^{\prime}, \ldots$. We consider the cycle decomposition of the permutation $\pi_{\text {fat }} \circ \mathrm{c} \in \mathrm{S}_{\{1,2, \ldots, \mathrm{n}\}}$ and remove from this decomposition all elements except the elements of $A$. In this way we constructed a permutation $\left.\left(\pi_{\text {fat }} \circ c\right)\right|_{A} \in S_{A}$, called restriction of $\pi_{\text {fat }} \circ c$ to the set $A$. This operation has a nice interpretation: on the graphical representation of $\pi_{\mathrm{fat}}$ and c we travel along the cycles by following the arrows and we write down only decorated vertices. We equip $\left.\left(\pi_{\text {fat }} \circ \mathrm{c}\right)\right|_{\mathrm{A}}$ with a structure of a partial permutation by setting its support to be $A$.

For example, for $\pi=\{\{1,3\},\{2,5,7\},\{4\},\{6\}\}$ depicted on Figure 2.3 we have $\mathbf{p}=(1,2,1,4,2,6,2), A=\{2,4,6\}$ and the decorations of $\pi_{\text {fat }}$ are depicted on Figures 3.1 3.2 and 3.3. The composition $\pi_{\text {fat }} \circ \mathrm{c}$ has a cycle decomposition $(1,2,3,5,4)(6,7)$. It follows that $\left.\left(\pi_{\text {fat }} \circ \mathrm{c}\right)\right|_{A}=(\mathbf{2}, \mathbf{4})(\mathbf{6})$.

Theorem 5.4. Let $\pi$ and $\mathbf{p}$ be as above. Then

$$
\Phi_{\widehat{S_{\mathrm{A}}}}^{\mathrm{ps}}(\mathbf{p})=\left.\left(\pi_{\mathrm{fat}} \circ \mathrm{c}\right)\right|_{\mathrm{A}}
$$

Proof. The equality of permutations follows easily from Lemma[5.5] below by setting $l=1$ and the equality of supports is trivial.

Lemma 5.5. Let $\pi$ and $\mathbf{p}$ be as above, we set $p_{n+1}:=p_{1}=1=*$ and let $\sigma_{1}, \ldots, \sigma_{n+1}$ be as in Sections 5.1.4 and 5.1.5 Let $x \in A \cup\{*\}$ and let $1 \leq l \leq n+1$.

If $x=p_{\mathrm{l}}$ then $\sigma_{\mathrm{l}}(\mathrm{x})=*$.
Suppose that $x \neq p_{1}$; let m be the smallest index such that $\mathrm{l} \leq \mathrm{m} \leq \mathrm{n}+1$ and $x=p_{\mathfrak{m}}$. If no such index exist then $\sigma_{\mathfrak{l}}(x)=x$. Otherwise we start in the vertex m a walk on the graphical representation of $\pi_{\text {fat }}$ and c by following the arrows until we enter some decorated vertex $y$ (after having made at least one step, i.e. starting in a decorated vertex do not count as entering $i t$ ). The above walk always stops after a finite number of steps and $\sigma_{\mathrm{l}}(\mathrm{x})=\mathrm{y}$.

Proof. The shall prove the lemma by backward induction. In the case $l=$ $n+1$ it holds trivially.

Suppose that the statement of lemma is true for all $l^{\prime}>l$. The first part, $\sigma_{l}\left(p_{l}\right)=*$ follows from the very construction of $\sigma_{l}$ in (5.4).

Observe that for every $r$ we have $\sigma_{r}=\left(a_{r} *\right) \sigma_{r+1}$ hence $\sigma_{r}(x)=\sigma_{r+1}(x)$ if $x \notin\left\{\sigma_{r}^{-1}(*), \sigma_{r+1}^{-1}(*)\right\}=\left\{p_{r}, p_{r+1}\right\}$. This has twofold implications.

Firstly, if $x \notin\left\{p_{l}, p_{l+1}, \ldots, p_{n}, p_{n+1}\right\}$ then $\sigma_{l}(x)=\sigma_{l+1}(x)=\cdots=$ $\sigma_{n+1}(x)=x$, which finishes the proof.

Secondly, when the last case in the lemma statement holds then

$$
\sigma_{l}(x)=\sigma_{l+1}(x)=\cdots=\sigma_{m-1}(x)=\left(\left(a_{m-1} *\right) \circ \sigma_{m}\right)(x)=a_{m-1},
$$

where we used the fact that $p_{m}=x$ is equivalent to $\sigma_{m}(x)=*$. Equation (5.5) implies that $\sigma_{l}(x)=\sigma_{m}\left(p_{m-1}\right)$. We denote by $\pi_{h}$ the block of $\pi=$ $\left\{\pi_{1}, \ldots, \pi_{r}\right\}$ such that the element $m-1$ belongs to it. Let $r$ be the smallest index such that $p_{m-1}=p_{r}$ and $m \leq r \leq n+1$. If no such index exists then $\sigma_{\mathfrak{l}}(x)=\sigma_{\mathfrak{m}}\left(p_{m-1}\right)=p_{\mathfrak{m}-1}$. Otherwise, if such an index exists then

$$
\sigma_{l}(x)=\sigma_{m}\left(p_{m-1}\right)=\sigma_{m+1}\left(p_{m-1}\right)=\cdots=\sigma_{r}\left(p_{m-1}\right)=\sigma_{r}\left(p_{r}\right) .
$$

We compare the value of $\sigma_{l}(x)$ computed above with the answer given by the graphical algorithm: we start our trip in vertex $m$ and then go to vertex $(m-1)^{\prime}$. If the index $r$ considered above does not exist then $m-1$ is the biggest element of $\pi_{\mathrm{h}}$ and therefore $(m-1)^{\prime}$ is connected by $\pi_{\text {fat }}$ with the smallest element of $\pi_{h}$ which is therefore decorated (the exceptional case $1 \in \pi_{h}$ is not possible since this would mean that $p_{m-1}=p_{1}=*$ and in this case $p_{m-1}=p_{n+1}$ and the index $r$ would exist) and the algorithm terminates giving the answer $\min \pi_{h}=p_{m-1}$ which coincides with the correct value. Otherwise, if the index $r$ exists then $(m-1)^{\prime}$ is connected by $\pi_{\text {fat }}$ with the vertex $r, l<m+1 \leq r \leq n+1$ (the vertex $n+1$ should be understood as the vertex 1). From the inductive hypothesis it follows that the algorithm will return the answer $\sigma_{r}\left(p_{r}\right)$ which is again correct.

Corollary 5.6. Let $\pi$ and $\mathbf{p}$ be as above, let $\mathbf{a}=\Phi_{\mathrm{as}}^{\mathrm{ps}}(\mathbf{p})$ and let $1 \leq l \leq n$. We start in the vertex $l$ a walk on the on the graphical representation of $\pi_{\mathrm{fat}}$ and c by following the arrows until we enter some decorated vertex y (after having made at least one step).

Then

$$
a_{l}=y
$$

Proof. It is a simple application of (5.5) and Lemma 5.5.
5.2.2. Maps $\Phi_{\mathrm{ps}}^{\mathrm{P}}$ and $\Sigma=\Phi_{\mathbb{C}\left(\widetilde{\left.\mathcal{S}_{\mathrm{A}}\right)}\right.}^{\mathrm{P}}$. Let us fix the finite set $A$ and the extra element $*$. We consider an algebra of pushing sequences in which the multiplication is as in Section 5.1.7 and the addition is understood as an addition of formal sums. The map $\Phi_{\stackrel{\mathrm{S}}{\mathrm{s}}}^{\mathrm{p}}$ extends naturally to an algebra homomorphism from the algebra of pushing sequences to the algebra $\mathbb{C}\left(\widetilde{S_{A}}\right)$ of partial permutations.

We define a map from pushing partitions to the algebra of pushing sequences by

$$
\begin{equation*}
\Phi_{\mathrm{ps}}^{\mathrm{P}}(\pi)=\sum_{\mathbf{p} \sim \pi} \mathbf{p} . \tag{5.7}
\end{equation*}
$$

We define the map $\Phi_{\mathbb{C}\left(\widetilde{\left.S_{A}\right)}\right.}^{P}$ in the unique way which makes the diagram below commute. This map is explicitly given by Claim 3.1 which will be proved below.


Theorem 5.7. Claim 3.1 is true. In other words: let $\pi$ be a partition of the set $\{1,2, \ldots, n\}$ and let numbers $k_{1}, \ldots, k_{\mathrm{t}}$ be given by the above construction. Then

$$
\Sigma_{\pi}:=\Phi_{\mathbb{C}\left(\widetilde{\left.S_{\infty}\right)}\right.}^{\mathrm{P}}(\pi)=\Sigma_{\mathrm{k}_{1}, \ldots, k_{\mathrm{t}}}
$$

where $\Sigma_{k_{1}, \ldots, k_{t}} \in \mathbb{C}\left(\widetilde{S_{\infty}}\right)$ on the right-hand side should be understood as in Section 2.1.10

Proof. Equation (5.7) and requirement that (5.8) commutes imply that

$$
\Sigma(\pi)=\sum_{\mathbf{p} \sim \pi} \Phi_{\widetilde{S_{\mathrm{A}}}}^{\mathrm{ps}}(\mathbf{p}) .
$$

By an appropriate renaming of labels Theorem 5.4 allows us to compute each summand $\Phi_{\widetilde{\mathrm{S}_{\mathcal{A}}}}^{\mathrm{ps}}(\mathbf{p})$. We decompose $\Phi_{\widetilde{\mathrm{S}_{\mathcal{A}}}}^{\mathrm{ps}}(\mathbf{p})=\left.(\pi \circ \mathrm{c})\right|_{\mathrm{A}} \in$
$S_{\left\{p_{1}, \mathfrak{p}_{2}, \ldots, \mathfrak{p}_{n}\right\} \backslash\left\{p_{1}\right\}}$ into a product of disjoint cycles and we denote by $k_{1}, \ldots, k_{t}$ the lengths of these cycles. It follows that

$$
\Sigma_{\pi}=\Sigma_{k_{1}, \ldots, k_{t}}
$$

where the right-hand side was defined in (2.7).
From the proof of Theorem 5.4 it follows that for each $1 \leq \mathrm{s} \leq \mathrm{t}$ the number $k_{s}$ is equal to the number of decorated elements in the corresponding cycle of the permutation $\pi_{\mathrm{fat}} \circ \mathrm{c}$. One can easily see that the original definition of the decorated elements is equivalent to the following one: element $m \in\{1,2, \ldots, n\}$ is decorated if and only if going counterclockwise from m to $\left(\pi_{\mathrm{fat}} \circ \mathrm{c}\right)^{-1}(\mathrm{~m})$ one does not cross the line between the marked starting point between 1 and $1^{\prime}$ and the central disc (see Figure 3.1). Therefore $k_{s}$ is equal to the number of elements in the corresponding cycle of $\pi_{\mathrm{fat}} \circ \mathrm{c}$ minus the number of times this cycle going clockwise crosses the line between the starting point and the central disc. It follows that the numbers $k_{1}, \ldots, k_{t}$ defined above coincide with (3.1) which finishes the proof.
5.2.3. Multiplication of pushing partitions. In Section 3.2 we defined the multiplication of partitions. Diagram (5.8) commutes and we shall prove that $\Sigma=\Phi_{\widetilde{C}\left(\widetilde{S_{\mathcal{A}}}\right)}^{\mathrm{P}}$ is an algebra homomorphism (we already know that $\Phi_{\widetilde{S_{A}}}^{\mathrm{ps}}$ is an algebra homomorphism; we will not need it but one can show that in a certain sense $\Phi_{\mathrm{ps}}^{\mathrm{P}}$ is also a homomorphism).

Lemma 5.8. Let $1 \leq k<l \leq m$, let $\pi^{1}$ be a pushing partition of the set $\rho_{1}=\{l+1, l+2, \ldots, m, 1,2, \ldots, k\}$, let $\pi^{2}$ be a pushing partition of the set $\rho_{2}=\{k+1, k+2, \ldots, l\}$. We define a non-crossing partition $\rho=\left\{\rho_{1}, \rho_{2}\right\}$.

Then

$$
\Phi_{\widetilde{C}\left(\widetilde{S_{\mathcal{A}}}\right)}^{\mathrm{P}}\left(\pi^{1} \cdot \pi^{2}\right)=\Phi_{\mathbb{C}\left(\widetilde{S_{A}}\right)}^{\mathrm{P}}\left(\pi^{1}\right) \Phi_{\widetilde{C}\left(\widetilde{S_{A}}\right)}^{\mathrm{P}}\left(\pi^{2}\right),
$$

where $\pi^{1} \cdot \pi^{2}$ denotes the $\rho$-ordered product of partitions $\pi^{1}$ and $\pi^{2}$ as defined in Section 3.2 and the product on the right-hand side is the usual product of the elements of $\mathbb{C}\left(\widetilde{\mathrm{S}_{A}}\right)$.

Proof. Let us consider a simpler case $\rho_{1}^{\prime}=\{1,2, \ldots, k+m-l\}, \rho_{2}^{\prime}=$ $\{k+m-l+1, k+m-l+2, \ldots, m\}$. From Proposition 5.2 it follows that $\Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\pi^{1}\right) \Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\pi^{2}\right)$ consists of all pushing sequences $\left(\mathrm{p}_{1}, \ldots, \mathrm{p}_{\mathrm{m}}\right)$ such that $p_{1}=p_{k+m-l+1}=*$ and $\left(p_{1}, \ldots, p_{k+m-l}\right) \sim \pi^{1},\left(p_{k+m-l+1}, \ldots, p_{m}\right) \sim \pi^{2}$. On the other hand the only non-trivial block of $\rho_{\text {comp }^{-1}}^{\prime}$ is $\{1, k+m-l+1\}$ and therefore

$$
\Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\pi^{1}\right) \Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\pi^{2}\right)=\Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\pi^{1} \cdot \pi^{2}\right) .
$$

To both sides of the equation we apply the map $\Phi_{\widehat{s_{\mathrm{A}}}}^{\mathrm{ps}}$ and use the fact that it is a homomorphism and the diagram (5.2) commutes.

The general case follows from the observation that the partition $\rho$ can be obtained from $\rho^{\prime}$ by a cyclic rotation. One can easily see that a product of rotated partitions is a rotation of the product of the original partitions. Finally, we apply Proposition 3.2
Lemma 5.9. Multiplication of partitions, as defined in Section 3.2 is associative. To be precise: let $\tau=\left\{\tau_{1}, \ldots, \tau_{r}\right\}$ be a non-crossing partition of the set $\{1,2, \ldots, n\}$. For each slet $\tilde{\tau}^{s}=\left\{\tau_{s, 1}, \ldots, \tau_{s, l_{s}}\right\}$ be a non-crossing partition of the set $\tau_{s}$. We denote by $\tau=\tilde{\tau}^{1} \cup \cdots \cup \tilde{\tau}^{r}=\left\{\tau_{s, k}: 1 \leq\right.$ $\left.s \leq r, 1 \leq k \leq l_{s}\right\}$ the non-crossing partition of the set $\{1,2, \ldots, n\}$. Let furthermore $\pi^{\mathrm{s}, \mathrm{k}}$ be a pushing partition of the set $\tau_{\mathrm{s}, \mathrm{k}}$ for each choice of $1 \leq \mathrm{s} \leq \mathrm{r}$ and $1 \leq \mathrm{k} \leq \mathrm{l}_{\mathrm{s}}$.

Then

$$
\begin{equation*}
\prod_{s}\left(\prod_{k} \pi^{s, k}\right)=\prod_{s, k} \pi^{s, k} \tag{5.9}
\end{equation*}
$$

where the left-hand side is a $\tau$-ordered product of $\tilde{\tau}^{s}$-ordered products and the right-hand side is a $\tau$-ordered product.

Proof. Probably the best way to prove this lemma is to use the graphical description of the partition multiplication from Section 3.2.2 since-speaking informally-cutting holes, gluing discs and merging discs are all associative operations. We provide a more detailed proof below.

Observe that every partition $\sigma$ which contributes to the left-hand side has a property that for every $a, b \in \tau_{s, k}$ elements $a$ and $b$ are connected by $\sigma$ if and only if they are connected by $\pi^{s, k}$. Furthermore since every partition $\sigma^{s}$ which contributes in $\prod_{k} \pi^{s, k}$ must fulfill $\sigma^{s} \geq\left(\tilde{\tau}^{s}\right)_{\text {comp }^{-1}}$ therefore $\sigma \geq$ $\tau_{\text {comp }^{-1}} \vee\left(\bigcup_{s}\left(\tilde{\tau}^{s}\right)_{\text {comp }^{-1}}\right)=\hat{\tau}_{\text {comp }^{-1}}$ (for the last equality see Lemma 5.10 below). It follows that every partition which appears on the left-hand side of (5.9) appears also on the right-hand side.

To show the opposite inclusion, for a summand $\sigma$ which appears on the right-hand side of (5.9) we set $\sigma^{s} \in \mathrm{P}\left(\tau_{s}\right)$ to be the partition which connects $a, b \in \tau_{s}$ if and only if $a, b$ are connected by $\sigma$. Lemma 5.10 implies that $\sigma^{s} \geq\left(\tilde{\tau}^{s}\right)_{\text {comp }^{-1}}$ hence $\sigma^{s}$ is one of the summands which appear in the product $\prod_{k} \pi^{s, k}$. By using a similar argument we see that $\sigma$ is one of the summands which contribute to the product $\prod_{s} \sigma^{s}$ and therefore contributes to the left-hand side of (5.9).

Lemma 5.10. We keep the notation from Lemma 5.9 Then

$$
\begin{equation*}
\tau_{\text {comp }^{-1}} \vee\left(\bigcup_{\mathrm{s}}\left(\tilde{\tau}^{s}\right)_{\text {comp }^{-1}}\right)=\hat{\tau}_{\text {comp }^{-1}} \tag{5.10}
\end{equation*}
$$

Proof. Similarly as in Section 3.2.2 we consider a sphere with a collection of holes, each corresponding to one of the blocks of $\tau$. In the second step
we replace each initial hole $\tau_{s}$ by a collection of holes $\tilde{\tau}^{s}$ : in this way we obtained a large sphere with a collection of holes $\tau_{s, k}$. Alternatively we can treat it as one big hole to which was glued a collection of discs; every of these discs corresponds to one of the blocks of $\hat{\tau}_{\text {comp }^{-1}}$ which corresponds to the right-hand side of (5.10).

On the other hand, we can treat the sphere with the first collection of holes as a sphere with a single hole glued with a collection of discs corresponding to $\tau_{\text {comp }}{ }^{-1}$. When we replace the hole $\tau_{s}$ by $\tilde{\tau}^{s}$ it corresponds to gluing another collection of discs $\left(\tilde{\tau}^{s}\right)_{\text {comp }^{-1}}$. Therefore the whole collection of discs corresponds to the left-hand side of (5.10).

Theorem 5.11. Claim 3.6 is true. In other words: let $\rho=\left\{\rho_{1}, \ldots, \rho_{r}\right\}$ be a non-crossing partition of the set $\{1,2, \ldots, n\}$ and for every $1 \leq s \leq r$ let $\pi^{s}$ be a partition of the set $\rho_{s}$. Then

$$
\Sigma\left(\prod_{s} \pi^{s}\right)=\prod_{1 \leq s \leq r} \Sigma_{\pi^{s}}
$$

where the multiplication on the left hand side should be understood as the $\rho$-ordered product of partitions and on the right hand side it should be understood as the usual product of commuting elements in $\mathbb{C}\left(\widetilde{S_{\infty}}\right)$.

Proof. We shall use the induction with respect to $r$, the number of blocks of the partition $\rho$. The case $r=1$ is of course trivial.

Observe that since $\rho$ is non-crossing therefore at least one block $\rho_{h}$ has a form $\rho_{h}=\{a, a+1, a+2, \ldots, b\}$ (for example, it is one of the blocks for which the expression $\left(\max \rho_{h}-\min \rho_{h}\right)$ takes the minimal value). Lemma 5.9 implies that

$$
\prod_{s} \pi^{s}=\pi^{h} \cdot\left(\prod_{s \neq h} \pi^{s}\right)
$$

We apply to both sides the map $\Phi_{\mathbb{C}\left(\widetilde{\mathcal{S}_{A}}\right)}^{P}=\Sigma$ and apply Lemma 5.8 to the right hand side. The inductive hypothesis can be applied to the second factor on the right hand side which finishes the proof.

### 5.2.4. Jucys-Murphy element.

Theorem 5.12. Claim 4.3 is true. In other words, let X be a finite ordered set, let $\rho$ be a non-crossing partition of a finite ordered set and $n \geq 1$ be an integer. Then

$$
\begin{aligned}
\Sigma\left(M_{X}^{P}\right) & =M_{|X|}^{\mathrm{JM}}, \\
\Sigma\left(M_{\rho}^{\mathrm{P}}\right) & =M_{\rho}^{\mathrm{IM}}, \\
\Sigma\left(\mathrm{R}_{n}^{\mathrm{P}}\right) & =\mathrm{R}_{\mathrm{n}}^{\mathrm{JM}},
\end{aligned}
$$

where $\sum$ is the map considered in Section 3.1.1]

Proof. Observe that every pushing sequence of length n appears in $\Phi_{\mathrm{ps}}^{\mathrm{P}}\left(\mathcal{M}_{\{1,2, \ldots, n\}}^{\mathrm{P}}\right)$ exactly once. The bijection between pushing and admissible sequences together with the definition (2.5) finish the proof.

## 6. Final remarks

6.1. Connection with the work of Biane. The following connection with article [Bia98] was pointed out to me by Philippe Biane.

Biane shows [Bia98] that the matrix $\Gamma$ defined in (2.1) fulfills

$$
\begin{aligned}
& M_{n}^{\mathrm{JM}}=\frac{1}{q+1} \operatorname{Tr} \Gamma^{n}= \\
& \frac{1}{q+1} \sum_{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{\mathrm{n}} \in\{1, \ldots, q+1\}} \Gamma_{\mathfrak{p}_{1} p_{2}} \Gamma_{\mathfrak{p}_{2} p_{3}} \cdots \Gamma_{\mathfrak{p}_{n-1} p_{n}} \Gamma_{\mathfrak{p}_{n} p_{1}},
\end{aligned}
$$

where the sum runs only over tuples $\left(p_{1}, \ldots, p_{n}\right)$ such that no neighbor elements are equal. Under identification $*=\mathrm{q}+1$ summands in Biane's formula coincide with pushing sequences; the only modification is that one does not assume that $p_{1}=*$.

This connection goes much deeper: in Section 4.3 of [Bia98] one considers pairs $(J, \pi)$ where $J=\left\{r_{1}, \ldots, r_{k}\right\} \subseteq\{1,2, \ldots, n\}$ is the set of indices such that $p_{r_{1}}=\cdots=p_{r_{k}}=*$ and $\pi$ is a partition of the set $\{1,2, \ldots, n\} \backslash J$ which tells us which of the elements in the tuple $\left(p_{1}, \ldots, p_{n}\right)$ are equal. One can easily see that the union $\{J\} \cup \pi$ is a pushing partition. We leave it as an exercise to the reader to verify that the conjugacy class assigned to a pair $(J, \pi)$ by Biane coincides with the conjugacy class which appears in $\Phi_{\mathrm{S}_{\mathcal{A}}}^{\mathrm{P}}(\{J\} \cup \pi)$. Methods presented in this article can be used therefore to simplify some of the arguments in the papers [Bia98, Bia01].

### 6.2. Ramified coverings and pushing partitions.

6.2.1. Ramified covering and an admissible sequence. Suppose that an admissible sequence $a_{1}, \ldots, a_{n} \in\{1,2, \ldots, q\}$ is given. Following the idea of Okounkov Oko00] we consider a two-dimensional sphere $\mathbb{S}^{2}$ with distinguished points $0,1,2, \ldots, n, \infty$. Figure 6.1 depicts a small region on this sphere (unfortunately, some conventions used in the pictures in this article do not coincide with the ones of Okounkov).

Let $S$ be an oriented surface. We shall consider a covering $S \rightarrow \mathbb{S}^{2}$ with simple ramifications over points $1,2, \ldots, n$ and an unspecified ramification over 0 . We consider $n+1$ cuts from points $0,1,2, \ldots, n$ to $\infty$. We choose $S$ in such a way that in the fiber over any point not lying on a cut there are $q+1$ sheets marked $1,2, \ldots, q, *$. Sheet $*$ will be called special sheet. We also require that the monodromy around any point $k \in\{1, \ldots, n\}$ be a transposition of the sheet $a_{k}$ and the sheet $*$. Since a loop around 0 is homotopic to


Figure 6.1. Two homotopic loops on a sphere.
a loop around points $1,2, \ldots, n$ hence the monodromy around 0 is a product of monodromies around $1, \ldots, n$ and hence is equal to $\left(a_{1} *\right) \cdots\left(a_{n} *\right)$ (cf Figure 6.1). The difference with the situation from the paper [Oko00] is that we do not assume that $\left(a_{1} *\right) \cdots\left(a_{n} *\right)=e$ and for this reason we might have a non-trivial ramification in 0 and therefore we need the extra cut between 0 and $\infty$, not present in [Oko00].
6.2.2. Shape of the special sheet. Since the sequence $a_{1}, \ldots, a_{n}$ is admissible, hence the monodromy around 0 preserves the special sheet. It follows that when we consider only cuts from points $1,2, \ldots, n$ to $\infty$ then $S$ splits into the special sheet and the union of non-special sheets (which are, possibly, glued together along the cut between 0 and $\infty$ ). Let us have a look on the shape of the special sheet. Along the cuts from $1,2, \ldots, n$ to $\infty$ it is glued with non-special sheets and we are not interested how does this gluing look like. Much more interesting is the vertex $\infty$ since some points on the boundary of the special sheet might be glued there together.

To check if two vertices: the one between $k-1$ and $k$ and the one between $l-1$ and $l$ are glued together or not, we consider a loop as on Figure 6.2 To indicate better the shape of the special sheet we inflated slightly the cuts. Our question is equivalent to the following one: is it true that the monodromy along this loop preserves the special sheet. This holds if and only if

$$
\left(\left(a_{k} *\right)\left(a_{k+1} *\right) \cdots\left(a_{l-1} *\right)\right)^{-1}(*)=*
$$

which is equivalent to

$$
p_{k}=\left(\left(a_{k} *\right)\left(a_{k+1} *\right) \cdots\left(a_{n} *\right)\right)^{-1}(*)=\left(\left(a_{l} *\right) \cdots\left(a_{n} *\right)\right)^{-1}(*)=p_{l}
$$

where $p_{1}, \ldots, p_{n}$ is the corresponding pushing sequence.


Figure 6.2. A loop on a sphere.


Figure 6.3. Shape of the special sheet (the white region).

It follows that the special sheet looks as it is depicted on Figure 6.3 where points $p_{1}, p_{2}, \ldots$ all cover $\infty$ and those of them which carry the same labels should be glued together. The pushing partition $\pi$ corresponding to the pushing sequence $p_{1}, \ldots, p_{n}$ can be therefore identified as a receipt for gluing vertices $p_{1}, \ldots, p_{n}$ of the special sheet covering $\infty$. The meaning of the additional labeling of the edges will be explained below.
6.2.3. Collapsing non-special sheets. Let us inflate the point $\infty$ a little bit (or-speaking more precisely-while performing cuts from points $1,2, \ldots, n$ to $\infty$ let us leave some neighborhood of $\infty$ untouched). The special sheet becomes now an oriented surface with a boundary which coincides with a construction from Section 3.1.3 where we glued a hole in a sphere with the first collection of discs (see also Figure 22 in [Oko00]).

Let us collapse the union of non-special sheets (for details of this construction we refer to [Oko00]). After this operation the 2 n edges of the special sheet (denoted by $1,1^{\prime}, 2,2^{\prime}, \ldots, n, n^{\prime}$ ) become glued together in pairs. We know that the vertices $p_{1}, \ldots, p_{n}$ are glued together according to the pushing partition $\pi$; this implies that the edges of the special sheet are glued according to the pair partition $\pi_{\mathrm{fat}}$. This provides an alternative description of the Okounkov's mapping $\Psi$ from coverings to maps on surfaces.

A careful reader will notice that the above statement is not completely true: the reason is that Okounkov collapses non-special sheets with two edges in a different way than the non-special sheets with three or more edges. This means that in fact in the process of computing $\pi_{\text {fat }}$ we should treat two-element blocks of $\pi$ differently. We leave the details as an exercise to the reader.
6.2.4. Many Jucys-Murphy elements. In this article we deal with only one Jucys-Murphy element J while in the work [Oko00] one considers products of many such elements $J_{1}, J_{2}, \ldots$ and therefore a careful reader might wonder if our methods are general enough. However, one can easily translate Okounkov's statements concerning products of many Jucys-Murphy elements into our favorite language of products of moments of a single JucysMurphy element which translates easily into properties of corresponding pushing partitions.
6.2.5. The case $\left(a_{1} *\right) \cdots\left(a_{n} *\right)=e$. Suppose now that $\left(a_{1} *\right) \cdots\left(a_{n} *\right)=$ $e$. It follows that the construction from Section 3.1.1 assigns to every cycle $b_{s}$ of the permutation $\pi_{\text {fat }} \circ c$ the number $k_{s}=1$ (otherwise ( $\left.a_{1} *\right) \cdots\left(a_{n} *\right)$ would have a non-trivial cycle). Now we see from (3.2) that it is equivalent to the statement that every cycle $b_{s}$ of the product $\pi_{\text {fat }} \circ \mathrm{c}$ makes exactly one counterclockwise wind around the circle.

As in Section 6.2.3 let us collapse the non-special sheets; the 2 n vertices of the polygon representing the special sheet fall into two classes: those which were covering $\infty$ (and which correspond to glued vertices $p_{1}, \ldots, p_{n}$ ) and those corresponding to the ramification points $1,2, \ldots, n$. Suppose we are going clockwise around the boundary of the polygon constituting the special sheet and thus we visit vertices in the order $n, p_{n}, n-$ $1, p_{n-1}, \ldots, 1, p_{1}$; let us have a look in which order we visit corners which
meet in one of the vertices of the second class. One can see that the edges which meet in such a vertex correspond to one of the polygons which constitute the boundary of the special sheet after gluing in $\infty$ or-in other words-to one of the cycles of $\pi_{\text {fat }} \circ \mathrm{c}$. From our previous discussion it follows that the corners will be visited in the counterclockwise order; in Okounkov's notation this means that the vertex is left.

Every group of the first class of vertices (covering $\infty$ ) which is glued together corresponds to one of the blocks of the partition $\pi$. It follows that they will be visited in a clockwise order; in Okounkov's notation this means that the vertex is right.

The above classification of vertices into the class of left and the class of right ones is one of the key points of the paper [Oko00] and it is interesting that we were able to reconstruct this result by the techniques of pushing partitions.

### 6.3. Connection with orthogonal polynomials and Wick product. Sup-

 pose that a probability measure $\mu$ on the real line is given. We consider a sequence $P_{0}, P_{1}, \ldots$ of orthogonal polynomials with respect to the measure $\mu$. Some questions concerning these polynomials are extremely simple, for example the mean value is given by a trivial equation $\int_{\mathbb{R}} P_{k}(x) d \mu(x)=\delta_{0 k}$, but the price for working with orthogonal polynomials is that we need a formula for expressing a product $P_{k} P_{l}$ as a linear combination of polynomials P (in the non-commutative probability theory such formulas are called Wick products). Also, a question arises how to write monomials $x^{k}$ in this basis.From this viewpoint one can easily see an analogy between orthogonal polynomials and indicator functions $\Sigma$ considered in this article. Some questions concerning $\Sigma$ are very simple: how to evaluate its value on a given permutation or how to write a central function as a linear combination of $\Sigma$. The price for this is that we need to find formulas for computation of the products $\Sigma_{k_{1}, \ldots, k_{m}} \cdot \Sigma_{k_{1}^{\prime}, \ldots, k_{n}^{\prime}}$ and need a formula for expressing the moments of J as a linear combination of $\Sigma$. The 'Wick product' (3.6) is very similar to combinatorial objects which appear in this context in the study of (generalized) Gaussian random variables [Gut02]. It would be very interesting to investigate the connections between these two objects.

## 7. Acknowledgments

I thank Philippe Biane for introducing me into the subject and many discussions. I also thank Roland Speicher, Ilona Królak and Akihito Hora for many discussions and encouragement.

Research supported by State Committee for Scientific Research (Komitet Badań Naukowych) grant No. 2P03A00723; by EU Network "QPapplications", contract HPRN-CT-2002-00729; by KBN-DAAD project $36 / 2003 / 2004$. The research was conducted in École Normale Supérieure du Paris, Institute des Hautes Etudes Scientifiques (Bures-sur-Yvette, France), Syddansk Universitet (Odense, Denmark) and Banach Center (Warszawa, Poland) on a grant funded by European Post-Doctoral Institute for Mathematical Sciences.

## References

[Bia97] Philippe Biane. Some properties of crossings and partitions. Discrete Math., 175(1-3):41-53, 1997.
[Bia98] Philippe Biane. Representations of symmetric groups and free probability. Adv. Math., 138(1):126-181, 1998.
[Bia01] Philippe Biane. Approximate factorization and concentration for characters of symmetric groups. Internat. Math. Res. Notices, (4):179-192, 2001.
[Bia03] Philippe Biane. Characters of symmetric groups and free cumulants. In Anatoly M. Vershik, editor, Asymptotic Combinatorics with Applications to Mathematical Physics, volume 1815 of Lecture Notes in Mathematics, pages 185-200. Springer, 2003.
[Ful97] William Fulton. Young tableaux, volume 35 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
[GJ96] I. P. Goulden and D. M. Jackson. Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. Trans. Amer. Math. Soc., 348(3):873-892, 1996.
[GJL01] I. P. Goulden, D. M. Jackson, and F. G. Latour. Inequivalent transitive factorizations into transpositions. Canad. J. Math., 53(4):758-779, 2001.
[Gou94] Alain Goupil. Decomposition of certain products of conjugacy classes of $S_{n}$. J. Combin. Theory Ser. A, 66(1):102-117, 1994.
[Gou90] Alain Goupil. On products of conjugacy classes of the symmetric group. Discrete Math., 79(1):49-57, 1989/90.
[GR05] I.P. Goulden and A. Rattan. An explicit form for Kerov's character polynomials. Preprint arXiv:math.CO/0505317, 2005.
[GS98] Alain Goupil and Gilles Schaeffer. Factoring n-cycles and counting maps of given genus. European J. Combin., 19(7):819-834, 1998.
[Gut02] M.I. Guta. Gaussian Processes in Non-commutatie Probability Theory. PhD thesis, University of Nijmegen, The Netherlands, 2002.
[HP00] Fumio Hiai and Dénes Petz. The semicircle law, free random variables and entropy. American Mathematical Society, Providence, RI, 2000.
[IK99] V. Ivanov and S. Kerov. The algebra of conjugacy classes in symmetric groups, and partial permutations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 256(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 3):95120, 265, 1999.
[IO02] Vladimir Ivanov and Grigori Olshanski. Kerov's central limit theorem for the Plancherel measure on Young diagrams. In S. Fomin, editor, Symmetric Functions 2001: Surveys of Developments and Perspectives, volume 74 of NATO

Science Series II. Mathematics, Physics and Chemistry, pages 93-151. Kluwer, 2002.
[Ker93] S. V. Kerov. Transition probabilities of continual Young diagrams and the Markov moment problem. Funktsional. Anal. i Prilozhen., 27(2):32-49, 96, 1993.
[Ker98] Sergei Kerov. Interlacing measures. In Kirillov's seminar on representation theory, volume 181 of Amer. Math. Soc. Transl. Ser. 2, pages 35-83. Amer. Math. Soc., Providence, RI, 1998.
[Ker99] S. Kerov. A differential model for the growth of Young diagrams. In Proceedings of the St. Petersburg Mathematical Society, Vol. IV, volume 188 of Amer. Math. Soc. Transl. Ser. 2, pages 111-130, Providence, RI, 1999. Amer. Math. Soc.
[KO94] Serguei Kerov and Grigori Olshanski. Polynomial functions on the set of Young diagrams. C. R. Acad. Sci. Paris Sér. I Math., 319(2):121-126, 1994.
[Kre72] G. Kreweras. Sur les partitions non croisées d'un cycle. Discrete Math., 1(4):333-350, 1972.
[LT01] A. Lascoux and J.-Y. Thibon. Vertex operators and the class algebras of symmetric groups. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 283(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6):156-177, 261, 2001.
[Mey93] Paul-André Meyer. Quantum probability for probabilists, volume 1538 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1993.
[Oko00] Andrei Okounkov. Random matrices and random permutations. Internat. Math. Res. Notices, (20):1043-1095, 2000.
[OV96] A. Okounkov and A.M. Vershik. A new approach to the representation theory of symmetric groups. Selecta Math. (N.S.), 4:581-605, 1996.
[Śni03a] Piotr Śniady. Free probability and representations of large symmetric groups. Preprint arXiv:math.CO/0304275, 2003.
[Śni03b] Piotr Śniady. Symmetric groups and random matrices. Preprint arXiv:math.CO/0301299, 2003.
[Śni04] Piotr Śniady. Table of reduced pair-partitions with genus 2. Preprint, 2004.
[Śni05] Piotr Śniady. Gaussian fluctuations of characters of symmetric groups and of Young diagrams. To appear in Probab. Theory Related Fields. Preprint arXiv:math.CO/0501112, 2005.
[Spe94] Roland Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution. Math. Ann., 298(4):611-628, 1994.
[Spe97] Roland Speicher. Free probability theory and non-crossing partitions. Sém. Lothar. Combin., 39:Art. B39c, 38 pp. (electronic), 1997.
[Spe98] Roland Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. Mem. Amer. Math. Soc., 132(627): $x+88,1998$.
[Sta02a] Richard Stanley. Irreducible symmetric group characters of rectangular shape. Preprint arXiv:math.CO/0109091, 2002.
[Sta02b] Richard Stanley. Kerov's character polynomial and irreducible symmetric group characters of rectangular shape. Transparencies from a talk in Québec City, available at http://www-math.mit.edu/~rstan/trans.html, 2002.
[SU91] Rodica Simion and Daniel Ullman. On the structure of the lattice of noncrossing partitions. Discrete Math., 98(3):193-206, 1991.
[VDN92] D. V. Voiculescu, K. J. Dykema, and A. Nica. Free random variables. American Mathematical Society, Providence, RI, 1992.
[VK85] A. M. Vershik and S. V. Kerov. Asymptotic behavior of the maximum and generic dimensions of irreducible representations of the symmetric group. Funktsional. Anal. i Prilozhen., 19(1):25-36, 96, 1985.
[Voi95] Dan Voiculescu. Free probability theory: random matrices and von Neumann algebras. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), pages 227-241, Basel, 1995. Birkhäuser.
[Voi00] Dan Voiculescu. Lectures on free probability theory. In Lectures on probability theory and statistics (Saint-Flour, 1998), pages 279-349. Springer, Berlin, 2000.
[Zvo97] A. Zvonkin. Matrix integrals and map enumeration: an accessible introduction. Math. Comput. Modelling, 26(8-10):281-304, 1997. Combinatorics and physics (Marseilles, 1995).

Institute of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland

E-mail address: Piotr.Sniady@math.uni.wroc.pl

