PARAMETRIZING THE ABSTRACT ELLENTUCK THEOREM

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ABSTRACT. We give a parametrization with perfect subsets of 2^{∞} of the abstract Ellentuck theorem (see [1], [16] or [17]). The main tool for achieving this goal is a sort of parametrization of an abstract version of the Nash-Williams theorem. As corollaries, we obtain some known classical results like the parametrized version of the Galvin-Prikry theorem due to Miller and Todorcevic [9], and the parametrized version of Ellentuck's theorem due to Paw-likowski [13]. Also, we obtain parametized versions of nonclassical results such as Milliken's theorem [10], and we prove that the family of perfectly Ramsey subsets of $2^{\infty} \times FIN_k^{[\infty]}$ is closed under the Souslin operation.

1. INTRODUCTION.

In [3], Ellentuck considers the space $\mathbb{N}^{[\infty]}$, of all the infinite sets of natural numbers, with the *exponential* (or *Ellentuck's*) topology to obtain his famous topological generalization of Ramsey's theorem [14]. The basic open sets of this topology are the neighborhoods $[a, \alpha] =$ $\{\beta \in \mathbb{N}^{[\infty]} : a \subset \beta \subseteq \alpha\}$, where a (resp. α) is a finite (resp. infinite) subset of N. We recall that a set $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is *Ramsey* (completely *Ramsey* in [3]) if for every Ellentuck neighborhood $[a, \alpha]$ there exists an infinite $\beta \subseteq \alpha$ such that either $[a, \beta] \subseteq \mathcal{X}$ or $[a, \beta] \cap \mathcal{X} = \emptyset$. In [3], Ellentuck proved that a set is Ramsey if and only if it has the Baire property relative to Ellentuck's topology, proving in this way that the family of Ramsey sets is closed under the Souslin operation and hence obtaining a simpler topological proof of Silver's theorem [15] that classical analytic subsets of $\mathbb{N}^{[\infty]}$ are Ramsey, and generalizing the Galvin-Prikry theorem [6] that classical Borel subsets of $\mathbb{N}^{[\infty]}$ are Ramsey.

Theorems analog to that of Ellentuck have also been proven on other spaces and different contexts, provided an analog "exponential" topology is given in each case (see for instance [2], [4] and [10]). These are the so called *Ellentuck type theorems*. Spaces where an Ellectuck type theorem can be proven are called (topological) *Ramsey spaces*. The

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main tool for proving such theorems (including Ellentuck's) is deviced for each particular case as a sort of "combinatorial forcing", inspired by the works of Galvin-Prikry [6] and Nash-Williams [12]. Recently, based on a previous work of Carlson and Simpson (see [1]), S. Todorcevic has isolated abstract combinatorial features which are conditions of sufficiency for topological spaces provided of a suitable "exponential" topology, to guarantee that Ellentuck type theorems can be obtained in such spaces. These combinatorial features enable to device a sort of "abstract Galvin-Prikry machinery" which constitutes the main means for obtaining an abstract Ellectuck theorem. It turns out that spaces satisfying these conditions are Ramsey spaces. We present a summary of Todorcevic's recent presentation of the abstract topological Ramsey theory in section 2.

In this work we will show that the conditions proposed in [16], and summarized in the next section, are also sufficient to obtain a parametrized version of the abstract Ellentuck theorem. In this way, we obtain as corollaries some known classical results like the parametrized version of the Galvin-Prikry theorem due to Miller and Todorcevic [9], and the parametrized version of the Ellentuck theorem due to Pawlikowski [13] which makes use of the notion of abstract Baire property of Morgan [11]. Also, we obtain parametized vesions of nonclassical results such as Milliken's theorem [10], and we prove that the family of perfectly Ramsey subsets of $2^{\infty} \times FIN_k^{[\infty]}$ is closed under the Souslin operation, and hence that analytic subsets of $2^{\infty} \times FIN_k^{[\infty]}$ are perfectly Ramsey. Here 2^{∞} is the space of all the infinite sequences of 0's and 1's, with the product topology regarding $2 = \{0, 1\}$ as a disdrete space. The space $FIN_k^{[\infty]}$, of infinite block basic sequences of functions $p: \mathbb{N} \to \{0, 1, \cdots, k\}$ of finite domain and with p(n) = k for some n, will be described in section 4.

2. Abstract Topological Ramsey Theory. Summary of Main facts.

All the definitions and results throughout this section were taken from [17] and are expected to appear in [16]. A previous presentation of the following notions can also be found in [1].

We will consider triplets of the form $(\mathcal{R}, \leq, (p_n)_{n \in \mathbb{N}})$, where \mathcal{R} is a set, \leq is a quasi order on \mathcal{R} , \mathbb{N} is the set of natural numbers, and for every $n \in \mathbb{N}$, $p_n : \mathcal{R} \to \mathcal{P}_n$ is a function with range \mathcal{P}_n . For each $A \in \mathcal{R}$, we say that $p_n(A)$ is the *n*th approximation of A. In order to capture the combinational structure required to ensure the provability of an Ellentuck type theorem, we will impose some assumptions on $(\mathcal{R}, \leq, (p_n)_n)$. The first three of them are the following:

- (A1) For any $A \in \mathcal{R}$, $p_0(A) = \emptyset$.
- (A2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n)p_n(A) \neq p_n(B)$.
- (A3) If $p_n(A) = p_m(B)$ then n = m and $(\forall i < n)p_i(A) = p_i(B)$.

These three assumptions allow us to identify each $A \in \mathcal{R}$ with the sequence $(p_n(A))_n$ of its approximations. In this way, if we consider the space $\mathcal{P} = \bigcup_n \mathcal{P}_n$ with the discrete topology, we can identify \mathcal{R} with a subspace of the (metric) space $\mathcal{P}^{[\infty]}$ (with the product topology) of all the sequences of elements of \mathcal{P} . Via this identification, we will regard \mathcal{R} as a subspace of $\mathcal{P}^{[\infty]}$, and we will say that \mathcal{R} is *metrically closed* if it is a closed subspace of $\mathcal{P}^{[\infty]}$.

Also, for $a \in \mathcal{P}$ we define the *lenth* of a, |a|, as the unique n such that $a = p_n(A)$ for some $A \in \mathcal{R}$.

We also consider on \mathcal{R} the *Ellentuck type neighborhoods*

$$[a, A] = \{ B \in \mathcal{R} : (\exists n)(a = p_n(B)) \text{ and } B \le A \}$$

where $a \in \mathcal{P}$ and $A \in \mathcal{R}$. If $[a, A] \neq \emptyset$ we will say that a is *compatible* with A (or A is compatible with a). Let $\mathcal{P}[A] = \{a \in \mathcal{P} : a \text{ is compatible with } A\}$.

We write [n, A] for $[p_n(A), A]$, and $Exp(\mathcal{R})$ for the family of all the neighborhoods [n, A]. This family generates the natural "exponential" topology on \mathcal{R} which is finer than the product topology.

A sequence $([n_k, A_k])_k$ of elements of $Exp(\mathcal{R})$ is called a *fusion sequence* if it is infinite and if

- (i) $(n_k)_k \subseteq \mathbb{N}$ is nondecreasing and $\lim_{k\to\infty} n_k = \infty$,
- (ii) $A_{k+1} \in [n_k, A_k]$ for all k.

Since \mathcal{R} is closed, building fusion sequences constitute a very useful procedure for defining desired elements of \mathcal{R} : for every fusion sequence $([n_k, A_k])_k$ we have that $\bigcap_k [n_k, A_k] \neq \emptyset$ (and a singleton). The *limit* of the fusion sequence is the unique element A_{∞} of $\bigcap_k [n_k, A_k]$. Note that $p_{n_k}(A_{\infty}) = p_{n_k}(A_k)$, for all k.

With this notation, we can define an analog notion for subsets of \mathcal{R} , to that of *Ramseyness* for subsets of $\mathbb{N}^{[\infty]}$:

Definition 1.0.1. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every neighborhood [a, A] there exists $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$. **Definition 1.0.2.** We say that $(\mathcal{R}, \leq, (p_n)_n)$ is a *Ramsey space* if subsets of \mathcal{R} with the Baire property are Ramsey and meager subsets of \mathcal{R} are Ramsey null.

In [16] it is shown that A1, A2 and A3, together with the following three assumptions are conditions of suficiency for a triplet $(\mathcal{R}, \leq, (p_n)_n)$, with \mathcal{R} metrically closed, to be a Ramsey space (see also [1]):

- (A4)(*Finitization*) There is a quasi order \leq_{fin} on \mathcal{P} such that:
- (i) $A \leq B$ iff $\forall n \exists m \ p_n(A) \leq_{fin} p_m(B)$.
- (ii) $\{b \in \mathcal{P} : b \leq_{fin} a\}$ is finite, for every $a \in \mathcal{P}$.

Given a and A, we define the depth of a in A, $depth_A(a)$, as the minimal n such that $a \leq_{fin} p_n(A)$.

- (A5) (Amalgamation) Given compatible a and A with $depth_A(a) = n$, the following holds:
 - (i) $\forall B \in [n, A] \ ([a, B] \neq \emptyset).$
 - (ii) $\forall B \in [a, A] \quad \exists A' \in [n, A] \quad ([a, A'] \subseteq [a, B]).$
- (A6) (*Pigeon Hole Principle*) Given compatible a and A with $depth_A(a) = n$, for each partition $\phi : \mathcal{P}_{|a|+1} \to \{0,1\}$ there is $B \in [n, A]$ such that ϕ is constant in $p_{|a|+1}[a, B]$.

Abstract Ellentuck Theorem:

Theorem 2.0.1. [Carlson] Any $(\mathcal{R}, \leq, \mathcal{P}_n, (p_n)_n)$ with \mathcal{R} metrically closed and satisfying A1-A6 is a Ramsey space.

For instance, take $\mathcal{R} = \mathbb{N}^{[\infty]}$, the set of infinite subsets of $\mathbb{N}, \leq = \subseteq$ and $p_n(A) =$ the first n elements of A, for each $A \in \mathbb{N}^{[\infty]}$. So, the set of approximations is $\mathcal{P} = \mathbb{N}^{[<\infty]}$, the set of finite subsets of \mathbb{N} . The family of neighborhoods [a, A], with $a \in \mathbb{N}^{[<\infty]}$ and $A \in \mathbb{N}^{[\infty]}$, is the family of Ellentuck neighborghoods defined in the introduction. Define \leq_{fin} as $a \leq_{fin} b$ iff $(a = b = \emptyset$ or $a \subseteq b$ and max(a) = max(b)), for $a, b \in \mathbb{N}^{[<\infty]}$. With these definitions, A1-A6 are easily verified. In this case A6 reduces to a natural variation of the classical pigeon hole principle for finite partitions of an infinite set of natural numbers. Note also that $\mathbb{N}^{[\infty]}$ is easily identified with a closed subspace of $\mathcal{P}^{[\infty]}$, namely, the set of all the sequences $(x_n)_n$ of finite sets such that $x_n =$ $x_{n+1} \setminus \{max(x_{n+1})\}$, for each $n \in \mathbb{N}$. Then $(\mathbb{N}^{[\infty]}, \subseteq, (p_n)_n)$) is a Ramsey space in virtue of the abstract Ellentuck theorem. Hence, Ellentuck's theorem is obtained as corollary: **Corollary 2.0.2.** [Ellentuck] Given $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$, the following hold:

- (a) \mathcal{X} is Ramsey iff \mathcal{X} has the Baire Property, relative to Ellentuck's topology.
- (b) X is Ramsey null iff X is meager, relative to Ellentuck's topology.

3. PARAMETRIZING WITH PERFECT SETS.

In this section we will present our main result. We recall that 2^{∞} denotes the space of infinite sequences of 0's and 1's, with the product topology regarding $2 = \{0, 1\}$ as a discrete space. Also, $2^{<\infty}$ denotes the set of finite sequences of 0's and 1's. Let us consider some features of the perfect subsets of 2^{∞} , following [13]:

Some notation is needed. For $x = (x_n)_n \in 2^{\infty}$, $x|_k$ denotes the finite sequence $(x_0, x_1, \dots, x_{k-1})$. For $u \in 2^{<\infty}$, let $[u] = \{x \in 2^{\infty} : (\exists k)(u = x|_k)\}$ and let |u| be the lenth of u. Given a perfect set $Q \subseteq 2^{\infty}$, let T_Q be its associated perfect tree. Also, for $u, v = (v_0, v_1, \dots, v_{|v|-1}) \in 2^{<\infty}$ we write $u \sqsubseteq v$ to mean $(\exists k \le |v|)(u = (v_0, v_1, \dots, v_{k-1}))$. For each $u \in 2^{<\infty}$, let $Q(u) = Q \cap [u(Q)]$, where $u(Q) \in T_Q$ is define inductively, as follows: $\emptyset(Q) = \emptyset$. Suppose u(Q) defined. Find $\sigma \in T_Q$ such that σ is the \sqsubseteq -extension of u(Q) where the first ramification occurs. Then, set $(u * i)(Q) = \sigma * i, i = 0, 1$. Here "*" denotes "concatenation". Note that for each $n, Q = \bigcup \{Q(u) : u \in 2^n\}$.

Given $n \in \mathbb{N}$, and perfect sets S and Q we say that $S \subseteq_n Q$ if $S(u) \subseteq Q(u)$, for every $u \in 2^n$. The relation " \subseteq_n " is a partial order. If for every $u \in 2^n$ we have chosen a $S_u \subseteq Q(u)$, then $S = \bigcup_u S_u$ is perfect and we have $S(u) = S_u$ and $S \subseteq_n Q$. As pointed out in [13], the most important feature of this partial order is the *property of fusion*: if $Q_{n+1} \subseteq_{n+1} Q_n$, $n \in \mathbb{N}$, then the fusion $Q = \bigcap_n Q_n$ is a perfect set and $Q \subseteq_n Q_n$, for each n.

Our goal in this section is proving a parametrized version of the abstract Ellentuck theorem of the previous section. This constitutes the main result contained in this work and is stated as thereom 3.0.3 below.

We introduce now the abstract version of the notion of "perfectly Ramsey" (see [13]). We will use the same name: given a triplet $(\mathcal{R}, \leq , (p_n)_n)$ as defined at the previous section, we say that a set $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ is *perfectly Ramsey* if for every perfect set $Q \subseteq 2^{\infty}$ and every neighborhood $[a, A] \neq \emptyset$ there exist a perfect set $S \subseteq Q$ and $B \in [a, A]$ such that $S \times [a, B] \subseteq \mathcal{X}$ or $S \times [a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ is *perfectly Ramsey null* if for every perfect set $Q \subseteq 2^{\infty}$ and every neighborhood $[a, A] \neq \emptyset$ there exist a perfect set $S \subseteq Q$ and $B \in [a, A]$ such that $S \times [a, B] \cap \mathcal{X} = \emptyset$.

Also, we will need to generalize the notion of abstract Baire property (see [11]) to this context:

Let \mathbb{P} be the family of perfect subsets of 2^{∞} . We will say that a set $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ has the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire property if for every perfect set $Q \subseteq 2^{\infty}$ and every neighborhood [a, A] there exist a perfect set $S \subseteq Q$ and a neighborhood $[b, B] \subseteq [a, A]$ such that $S \times [b, B] \subseteq \mathcal{X}$ or $S \times [b, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ is $\mathbb{P} \times Exp(\mathcal{R})$ -meager if for every perfect set $Q \subseteq 2^{\infty}$ and every neighborhood [a, A] there exist a perfect set $S \subseteq Q$ and a neighborhood $[b, B] \subseteq [a, A]$ such that $S \times [b, B] \cap \mathcal{X} = \emptyset$.

Now we are ready to state our main result:

Theorem 3.0.3. Given $(\mathcal{R}, \leq, (p_n)_n)$, with \mathcal{R} metrically closed and satisfying A1-A6, the following are true:

- (a) $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ is perfectly Ramsey iff \mathcal{X} has the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire Property.
- (b) $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ is perfectly Ramsey null iff \mathcal{X} is $\mathbb{P} \times Exp(\mathcal{R})$ -meager.

The main tool for proving theorem 3.0.3 is the following fact which is a sort of parametrization of an abstract version of Nash-Williams' theorem:

Theorem 3.0.4. Given $(\mathcal{R}, \leq, (p_n)_n)$, with \mathcal{R} metrically closed and satisfying A1-A6, the following is true:

For every $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{P}$, perfect $P \subseteq 2^{\infty}$ and $A \in \mathcal{R}$ there exist a perfect $S \subseteq P$ and $D \leq A$ such that one of the following holds:

- (a) for every $x \in S$ and every $C \leq D$ there exist integers k and m > 0 such that $(x|_k, p_m(C)) \in \mathcal{F}$.
- (b) $(T_S \times \mathcal{P}[D]) \cap \mathcal{F} = \emptyset.$

Theorem 3.0.4 is inspired on a parametrized version of the semiselective Nash-Williams theorem proved by I. Farah (see theorems 2.2 and 2.3 of [4]). Before proving it, we need to start up our parametrized combinatorial machinery, based on the tecniques used in [4], [13][13] and [17].

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Combinatorial Forcing 1. Fix $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{P}$. For a perfect $Q \subseteq 2^{\infty}$, $A \in \mathcal{R}$ and a pair $(u, a) \in 2^{<\infty} \times \mathcal{P}$, we say that (Q, A) accepts (u, a) if for every $x \in Q(u)$ and for every $B \in [a, A]$ there exist integers k and m such that $(x|_k, p_m(B)) \in \mathcal{F}$. We say that (Q, A) rejects (u, a) if for every perfect $S \subseteq Q(u)$ and every $B \leq A$, compatible with a, (S, B) does not accepts (u, a). Also, we say that (Q, A) decides (u, a) if it accepts or rejects it.

Combinatorial Forcing 2. Fix $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$. For a perfect $Q \subseteq 2^{\infty}$, $A \in \mathcal{R}$ and a pair $(u, a) \in 2^{<\infty} \times \mathcal{P}$, we say that (Q, A) accepts (u, a) if $Q(u) \times [a, A] \subseteq \mathcal{X}$. We say that (Q, A) rejects (u, a) if for every perfect $S \subseteq Q(u)$ and every $B \leq A$, compatible with a, (S, B) does not accepts (u, a). And as before, we say that (Q, A) decides (u, a) if it accepts or rejects it.

Note: Lemmas 3.0.5, 3.0.6, and 3.0.7 below hold *for both* combinatorial forcings defined above.

Lemma 3.0.5. The following are true:

- (a) If (Q, A) accepts (rejects) (u, a) then (S, B) also accepts (rejects) (u, a), for every perfect $S \subseteq Q(u)$ and every $B \leq A$ compatible with a.
- (b) If (Q, A) accepts (rejects) (u, a) then (Q, B) also accepts (rejects) (u, a), for every $B \leq A$ compatible with a.
- (c) For all (u, a) and (Q, A) such that A is compatible with a, there exist a perfect $S \subseteq Q$ and $B \leq A$, compatible with a, such that (S, B) decides (u, a).
- (d) If (Q, A) accepts (u, a) then (Q, A) accepts (u, b) for every $b \in p_{|a|+1}[a, A]$.
- (e) If (Q, A) rejects (u, a) then there exist $B \in [depth_A(a), A]$ such that (Q, A) does not accept (u, b) for every $b \in p_{|a|+1}[a, B]$.
- (f) (Q, A) accepts (rejects) (u, a) iff (Q, A) accepts (rejects) (v, a), for every $v \in 2^{<\infty}$ with $u \sqsubseteq v$.

Proof. (a), (b), (c), (d) and (f) follow from the definitions. Now to proof (e), take (u, a) with |a| = m and suppose (Q, A) rejects it. Define $\phi : \mathcal{P}_{m+1} \to 2$ such that $\phi(b) = 1$ iff (Q, A) accepts (u, b). Let $n = depth_A(a)$. By A6, there exists $B \in [n, A]$ such that ϕ is constant on $p_{m+1}[a, B]$.

If ϕ takes value 1 on $p_{m+1}[a, B]$ then (Q, B) accepts (u, a). So, in virtue of part (b), ϕ must take value 0 on $p_{m+1}[a, B]$ since (Q, A) rejects (u, a). Then B is as required.

Lemma 3.0.6. For every perfect $P \subseteq 2^{\infty}$ and $A \in \mathcal{R}$ there exist a perfect $Q \subseteq P$ and $B \leq A$ such that (Q, B) decides (u, a), for every $(u, a) \in 2^{<\infty} \times \mathcal{P}[B]$ with $depth_B(a) \leq |u|$.

Proof. Let $\langle \rangle$ be the empty sequence of 0's and 1's, and recall from section 2 that $\emptyset \in \mathcal{P}$. Using lemma 3.0.5(c), find a perfect $Q_0 \subseteq P$ and $B_0 \leq A$ such that (Q_0, B_0) decides $(\langle \rangle, \emptyset)$.

Suppose we have defined Q_n and B_n such that (Q_n, B_n) decides every $(u, a) \in 2^n \times \mathcal{P}[B_n]$ with $depth_{B_n}(a) = n$. Let $u_0, u_1, \ldots, u_{2^{n+1}-1}$ be a list of the elements of 2^{n+1} ; and let b_0, b_1, \ldots, b_r be a list of the $b \in \mathcal{P}[B_n]$ such that $depth_{B_n}(b) = n + 1$.

Using lemma 3.0.5(c), find a perfect $Q_n^{0,0} \subseteq Q_n(u_0)$ and $B_n^{0,0} \leq B_n$ compatible with b_0 such that $(Q_n^{0,0}, B_n^{0,0})$ decides (u_0, b_0) . We can suppose $B_n^{0,0} \in [b_0, B_n]$. Hence, we can assume $B_n^{0,0} \in [n+1, B_n]$ in virtue of A5(ii) and lemma 3.0.5(b).

In a similar way, for every $(i, j) \in \{0, 1, \dots, 2^{n+1}-1\} \times \{0, 1, \dots, r\}$, we can find $Q_n^{i,j}$ and $B_n^{i,j}$ with: $Q_n^{i,j+1} \subseteq Q_n^{i,j}(u_i), B_n^{i,j+1} \in [b_{j+1}, B_n^{i,j}]$, $Q_n^{i+1,0} \subseteq Q_n(u_{i+1}), B_n^{i+1,0} \in [b_0, B_n^{i,r}]$; and such that $(Q_n^{i,j}, B_n^{i,j})$ decides (u_i, b_j) . (Notice that this construction is possible in virtue of A5(i). Again, we can assume $B_n^{i,j} \in [n+1, B_n]$ in virtue of A5(ii) and lemma 3.0.5(b)).

Let $Q_{n+1} = \bigcup_{i=0}^{2^{n+1}-1} Q_n^{i,r}$ and $B_{n+1} = B_n^{2^{n+1}-1,r}$. Then, (Q_{n+1}, B_{n+1}) decides (u, b), for every $(u, b) \in 2^{n+1} \times \mathcal{P}[B_{n+1}]$ with $depth_{B_{n+1}}(b) = n + 1$: for such (u, b), there exist $(i, j) \in \{0, 1, \dots, 2^{n+1}-1\} \times \{0, 1, \dots, r\}$ such that $u = u_i$ and $b = b_j$. Then $(Q_n^{i,j}, B_n^{i,j})$ decides (u, b). Notice that

$$Q_{n+1}(u_i) = Q_n^{i,r} \subseteq Q_n^{i,r-1}(u_i) \subseteq \dots \subseteq Q_n^{i,j}(u_i)$$

and

$$B_{n+1} = B_n^{2^{n+1}-1,r} \le B_n^{i,j}.$$

Hence (Q_{n+1}, B_{n+1}) decides (u_i, b_j) .

Besides, $Q_{n+1} \subseteq_{n+1} Q_n$ and $B_{n+1} \in [n+1, B_n]$

Now let $Q = \bigcap_n Q_n$ and take $B \in \bigcap_n [n+1, B_n]$. A similar argument shows that (Q, B) decides (u, b), for every $(u, b) \in 2^{<\infty} \times \mathcal{P}[B]$ with $depth_B(b) = |u|$. Then in virtue of lemma 3.0.5(f), Q and B are as required.

Lemma 3.0.7. Let Q and B be as in lemma 3.0.6. Suppose (Q, B) rejects $(<>, \emptyset)$. Then there exists $D \leq B$ such that (Q, D) rejects (u, b), for every $(u, b) \in 2^{<\infty} \times \mathcal{P}[D]$ with $depth_D(b) \leq |u|$.

Proof. Let us build a fusion sequence $([n, D_n])_n$. Let $D_0 = B$. Then by hipothesis (Q, D_0) rejects $(\langle \rangle, \emptyset)$. Suppose D_n is given such that (Q, D_n) rejects (u, b), for every $(u, b) \in 2^n \times \mathcal{P}[D_n]$ with $depth_{D_n}(b) = n$.

Now, let $u_0, u_1, \ldots, u_{2^{n+1}-1}$ be a list of the elements of 2^{n+1} ; and let b_0, b_1, \ldots, b_r be a list of the $b \in \mathcal{P}[D_n]$ such that $depth_{D_n}(b) = n$. By lemma $3.0.5(f), (Q, D_n)$ rejects (u_i, b_j) for every $(i, j) \in \{0, 1, \cdots, 2^{n+1} - 1\} \times \{0, 1, \cdots, r\}$. Now, by lemma 3.0.5(e) there exists $D_n^{0,0} \in [n, D_n]$ such that $(Q, D_n^{0,0})$ rejects (u_0, b) for every $b \in p_{|b_0|+1}[b_0, D_n^{0,0}]$.

In the same way, for every $(i, j) \in \{0, 1, \dots, 2^{n+1} - 1\} \times \{0, 1, \dots, r\}$, we can find a $D_n^{i,j}$ with: $D_n^{i,j+1} \in [n, D_n^{i,j}], D_n^{i+1,0} \in [n, D_n^{i,r}]$; and such that $(Q, D_n^{i,j})$ rejects (u_i, b) for every $b \in p_{|b_j|+1}[b_j, D_n^{i,j}]$. Let $D_{n+1} = D_n^{2^{n+1}-1,r}$. Notice that if $(u, b) \in 2^{n+1} \times \mathcal{P}[D_{n+1}]$ and $depth_{D_{n+1}}(b) = n+1$ then $u = u_i$ and $b \in p_{|b_j|+1}[b_j, D_n^{i,j}]$ for some $(i, j) \in \{0, 1, \dots, 2^{n+1} - 1\} \times \{0, 1, \dots, r\}$. Hence (Q, D_{n+1}) rejects (u, b). Besides, $D_{n+1} \in [n, D_n]$.

Now take $D \in \bigcap_n [n, D_n]$. Then D is as required.

Proof of theorem 3.0.4. Given $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{P}$, perfect $P \subseteq 2^{\infty}$ and $A \in \mathcal{R}$, consider the combinatorial forcing 1. Let $Q \subseteq P$ and $B \leq A$ be as in lemma 3.0.6. If (Q, B) accepts $(<>, \emptyset)$ then part (a) of theorem 3.0.4 holds by the definition of "accepts". So suppose (Q, B) does not accept (and hence, rejects) $(<>, \emptyset)$. By lemma 3.0.7, find $D \leq B$ such that (Q, D) rejects (u, b), for every $(u, b) \in 2^{<\infty} \times \mathcal{P}[D]$ with $depth_D(b) \leq |u|$. Suppose towards a contradiction that there exist (t, b) in $(T_Q \times \mathcal{P}[D]) \cap \mathcal{F}$. Find $u_t \in 2^{<\infty}$ such that $Q(u_t) \subseteq Q \cap [t]$. Then (Q, D) accepts (u_t, b) : for $x \in Q(u_t)$ and $C \in [b, D]$, let k = |t| and m be such that $p_m(C) = b$. Then $(x|_k, p_m(C)) = (t, b) \in \mathcal{F}$.

But then, by lemma 3.0.5 (f), (Q, D) accepts (v, b), for every $v \in 2^{<\infty}$ such that $u_t \sqsubseteq v$ and $|v| \ge depth_D(b)$. This is a contradiction with the choice of D. Therefore, for S = Q and D part (b) of theorem 3.0.4 holds.

Now we are ready to prove our main result:

Proof of theorem 3.0.3. (a) The implication from left to right is obvious. So suppose $\mathcal{X} \subseteq 2^{\infty} \times \mathcal{R}$ has the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire Property, and let $P \times [a, A]$ be given. In order to make the proof notationally simpler, we will assume $a = \emptyset$ without a loss of generality.

Claim 3.0.8. Given $\hat{\mathcal{X}} \subseteq 2^{\infty} \times \mathcal{R}$, perfect $\hat{P} \subseteq 2^{\infty}$ and $\hat{A} \in \mathcal{R}$, there exist $Q \subseteq \hat{P}$ and $B \leq \hat{A}$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{P}[B]$ with $|u| \geq depth_B(b)$ one of the following holds:

- i.) $Q(u) \times [b, B] \subseteq \mathcal{X}$
- ii.) $R \times [b, C] \not\subseteq \hat{\mathcal{X}}$, for every $R \subseteq Q(u)$ and every $C \leq B$ compatible with b.

Proof of theorem Claim. Consider the Combinatorial Forcing 2 and apply lemma 3.0.6.

Apply the claim to \mathcal{X} , P and A to find $Q_1 \subseteq P$ and $B_1 \leq A$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{P}[B_1]$ with $|u| \geq depth_{B_1}(b)$ one of the following holds:

- (1) $Q_1(u) \times [b, B_1] \subseteq \mathcal{X}$ or
- (2) $R \times [b, C] \not\subseteq \mathcal{X}$, for every $R \subseteq Q_1(u)$ and every $C \leq B_1$ compatible with b.

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\infty}$ such that $u_1^t(Q_1) \sqsubseteq t$.

Let

$$\mathcal{F}_1 = \{(t,b) \in T_{Q_1} \times \mathcal{P}[B_1] : Q_1(u_1^t) \times [b,B_1] \subseteq \mathcal{X}\}$$

Now, pick $S_1 \subseteq Q_1$ and $D_1 \leq B_1$ satisfying theorem 3.0.4. If (a) of 3.0.4 holds then $S_1 \times [0, D_1] \subseteq \mathcal{X}$ and we are done. So suppose (b) holds.

Apply the claim to \mathcal{X}^c , S_1 and D_1 to find $Q_2 \subseteq S_1$ and $B_2 \leq D_1$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{P}[B_2]$ with $|u| \geq depth_{D_2}(b)$ one of the following holds:

- (3) $Q_2(u) \times [b, B_2] \subseteq \mathcal{X}^c$ or
- (4) $R \times [b, C] \not\subseteq \mathcal{X}^c$, for every $R \subseteq Q_2(u)$ and every $C \leq B_2$ compatible with b.

As before, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\infty}$ such that $u_2^t(Q_2) \sqsubseteq t$. Let

$$\mathcal{F}_2 = \{(t,b) \in T_{Q_2} \times \mathcal{P}[B_2] : Q_2(u_2^t) \times [b,B_2] \subseteq \mathcal{X}^c\}$$

Again, pick $S_2 \subseteq Q_2$ and $D_2 \leq B_2$ satisfying theorem 3.0.4. If (a) of 3.0.4 holds then $S_2 \times [0, D_2] \cap \mathcal{X} = \emptyset$ and we are done. So suppose (b) holds again. Let us see that this contradicts the fact that \mathcal{X} has the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire Property:

Notice that for every $(t, b) \in T_{S_2} \times \mathcal{P}[D_2]$ the following holds:

(i)
$$Q_1(u_1^t) \times [b, B_1] \not\subseteq \mathcal{X}$$
, and
(ii) $Q_1(u_1^t) \times [b, B_1] \not\subset \mathcal{X}_c$

(ii) $Q_2(u_2^t) \times [b, B_2] \not\subseteq \mathcal{X}^c$.

So, suppose there is a nonempty $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathcal{X}$, and pick $t \in T_R$ with $|u_1^t| \geq depth_{B_1}(b)$. Notice that $R \cap [t] \subseteq Q_1(u_1^t)$. On the one hand we have that $R \cap [t] \times [b, C] \subseteq R \times [b, C] \subseteq \mathcal{X}$. But in virtue of (i), $Q_1(u_1^t) \times [b, B_1] \not\subseteq \mathcal{X}$ and hence by (2) above we have that $R \cap [t] \times [b, C] \not\subseteq \mathcal{X}$. If we suppose that there is a nonempty $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathcal{X}^c$ we reach to a similar contradiction in virtue of (ii) and (4) above. So there is neither $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathcal{X}$ nor $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathcal{X}^c$. But this is impossible because \mathcal{X} has the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire Property.

(b) Again, the implication from left to right is obvious. Conversely, the result follows easily from (a) and the fact that \mathcal{X} is $\mathbb{P} \times Exp(\mathcal{R})$ -meager.

This completes the proof of theorem 3.0.3.

4. Some Particular Cases.

Several interesting consequences can be derived from the facts obtained in the previous section. Some of them are known classical results and the others are parametrized versions of known Ellentuck type theorems in nonclassical spaces.

Let k be a positive integer. For $p : \mathbb{N} \to \{0, 1, \dots, k\}$, let supp(p) denote the set $\{n : p(n) \neq 0\}$ and let rang(p) denote the range of p.

Let us consider the set

 $FIN_k := \{p : \mathbb{N} \to \{0, 1, \cdots, k\} : supp(p) \text{ is finite and } k \in rang(p)\}.$

A block basic sequence is any finite or infinite sequence $X = (x_n)_{n \in I \subseteq \mathbb{N}}$ of elements of FIN_k such that

$$max(supp(x_n)) < min(supp(x_m))$$
 whenever $n < m$.

We shall use a, b, c, ... for finite block basic sequences, and A, B, C, ... for infinite block basic sequences. In this latter case we will assume that the set of indexes is $I = \mathbb{N}$.

Define $T: FIN_k \to FIN_{k-1}$ by

$$T(p)(n) = max\{p(n) - 1, 0\}.$$

In [16] this is called the *tetris operation*. For every $j \in \mathbb{N}$, $T^{(j)}$ is the j-th iteration of T, where $T^{(0)}(p) = p$.

For a given block basic sequence $X = (x_n)_{n \in I \subseteq \mathbb{N}}$, the subspace of FIN_k generated by X, denoted by [X], is the set of elements of FIN_k of the form:

$$T^{(j_0)}(x_{n_0}) + T^{(j_1)}(x_{n_1}) + \dots + T^{(j_r)}(x_{n_r})$$

where $n_0 < n_1 < \cdots < n_r$ is a finite sequence of elements of I and $j_0 < j_1 < \cdots < j_r$ is a sequence of elements of $\{0, 1, \cdots, k\}$ such that $j_i = 0$ for some $i \in \{0, 1, \cdots, r\}$.

The following result shows us an important feature of FIN_k , which provides us of a pigeon hole principle within this context:

Theorem 4.0.9. [Gowers [5]] For every integer r > 0 and every partition $\phi : FIN_k \to \{0, 1, \dots, r-1\}$ there exists an infinite block basic sequence A such that ϕ is constant in [A].

In the case k = 1, FIN_k is the set FIN of nonempty finite subsets of N, and theorem 4.0.9 is Hindman's theorem [7].

Let $FIN_k^{[\infty]}$ be the set of infinite block basic sequences and define,

$$A \leq B$$
 iff $A \subseteq [B]$

for $A,B\in FIN_k^{[\infty]}.$ Also, for every $A\in FIN_k^{[\infty]},$ let the n-th approximation of A be

$$p_n(A) =$$
 the first *n* elements of *A*.

Then the set \mathcal{P} of approximations is $FIN_k^{[<\infty]}$, the set of finite block basic sequences. Now, for $a, b \in FIN_k^{[<\infty]}$ define $a \leq_{fin} b$ if and only if

$$a = b = \emptyset$$
 or $a \subset [b]$ and $max(supp \bigcup a) = max(supp \bigcup b)$.

With this terminology and the obvious definition of the neighborhoods [a, A] (and the family $Exp(FIN_k^{[\infty]})$), the triplet $(FIN_k^{[\infty]}, \leq, (p_n)_{n \in \mathbb{N}})$ satisfies A1-A6. Also, $FIN_k^{[\infty]}$ is readily identified to a closed subset of $\mathcal{P}^{[\infty]}$ and hence it is a Ramsey space, in virtue of the Abstract Ellentuck Theorem (2.0.1 above). Here A6 reduces to a natural variation of Gowers' theorem (4.0.9 above). This yields the following corollary of theorem 3.0.3:

Corollary 4.0.10. Let $\mathcal{X} \subseteq 2^{\infty} \times FIN_k^{[\infty]}$ be given. \mathcal{X} is perfectly Ramsey iff \mathcal{X} has the $\mathbb{P} \times Exp(FIN_k^{[\infty]})$ -Baire Property. \mathcal{X} is perfectly Ramsey null iff \mathcal{X} is $\mathbb{P} \times Exp(FIN_k^{[\infty]})$ -meager.

In the same way, in virtue of corollary 2.0.2, we have the following:

Corollary 4.0.11. Let $\mathcal{X} \subseteq 2^{\infty} \times \mathbb{N}^{[\infty]}$ be given. \mathcal{X} is perfectly Ramsey iff \mathcal{X} has the $\mathbb{P} \times Exp(\mathbb{N}^{[\infty]})$ -Baire Property. \mathcal{X} is perfectly Ramsey null iff \mathcal{X} is $\mathbb{P} \times Exp(\mathbb{N}^{[\infty]})$ -meager.

Corollary 4.0.11 is the parametrization of Ellentuck's theorem [3] obtained by Pawlikowski in [13]. And corollary 4.0.10 gives a parametrized version of Milliken's theorem [10], when k = 1, and a parametrized version of the corresponding Ellentuck type theorem in the Ramsey space $(FIN_k^{[\infty]}, \leq, (p_n)_{n \in \mathbb{N}})$ defined in this section, when k > 1.

5. CLOSEDNESS UNDER THE SOUSLIN OPERATION.

In this section we go back to our parametrization of (abstract) Ramsey spaces and study the family of perfectly Ramsey sets in relation to the Souslin operation. First, we proof the following fact which turns out to be crucial for this study.

Lemma 5.0.12. Given $(\mathcal{R}, \leq, (p_n)_{n \in \mathbb{N}})$ satisfying A1-A6 and with \mathcal{R} metrically closed, the perfectly Ramsey null subsets of $2^{\infty} \times \mathcal{R}$ form a σ -ideal.

Proof. Let $(\mathcal{X}_n)_n$ be a sequence of perfectly Ramsey null subsets of $2^{\infty} \times \mathcal{R}$ and fix $P \times [a, A]$. We can assume $a = \emptyset$. Also notice that the finite union of perfectly Ramsey null sets yields a perfectly Ramsey null set; so we will assume $(\forall n) \mathcal{X}_n \subseteq \mathcal{X}_{n+1}$. Proceeding as in the proof of lemma 3.0.6 we build fusion sequences Q_n , $[n + 1, B_n]$ as follows: take $Q_0 \subseteq P$, $B_0 \leq A$ such that $Q_0 \times [0, B_0] \cap \mathcal{X}_0 = \emptyset$. Suppose Q_n , $[n + 1, B_n]$ have been defined such that

$$Q_n \times [b, B_n] \cap \mathcal{X}_n = \emptyset$$

for every $b \in \mathcal{P}[B_n]$ with $depth_{B_n}(b) = n$. Since \mathcal{X}_{n+1} is perfectly Ramsey null, applying this fact successively we find $Q_{n+1} \subseteq_{n+1} Q_n$,

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and $B_{n+1} \in [n+1, B_n]$ such that

 $Q_{n+1} \times [b, B_{n+1}] \cap \mathcal{X}_{n+1} = \emptyset$

for every $b \in \mathcal{P}[B_{n+1}]$ with $depth_{B_{n+1}}(b) = n+1$. Let

$$Q = \bigcap_{n} Q_n$$
 and $B = \bigcap_{n} [n+1, B_n].$

Then $Q \times [0, B] \cap \bigcup_n \mathcal{X}_n = \emptyset$: take $(x, C) \in Q \times [0, B]$ and fix arbitrary n. To show that $(x, C) \notin \mathcal{X}_n$ let k be large enough so that $depth_B(p_k(C)) = m \ge n$. Then by construction $Q \times [p_k(C), B] \cap \mathcal{X}_m = \emptyset$ and hence, since $\mathcal{X}_n \subseteq \mathcal{X}_m$, we have $(x, C) \notin \mathcal{X}_n$. This completes the proof.

Now, we borrow some terminology from [13]:

Let \mathcal{A} be a family of subsets of a set \mathcal{Z} . We say that $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{Z}$ are *compatible* (with respect to \mathcal{A}) if there exists $\mathcal{W} \in \mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{X} \cap \mathcal{Y}$. Also, we say that \mathcal{A} is *M*-like if for any $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| < |\mathcal{A}|$, every member of \mathcal{A} which is not compatible with any member of \mathcal{B} is compatible with $\mathcal{Z} \setminus \bigcup \mathcal{B}$.

Notice that the family \mathbb{P} of perfect subsets of 2^{∞} is *M*-like, as well as the family $Exp(\mathcal{R})$ (this is true of any topological basis). Therefore, according to lemma 2.7 in [13], if we require that $|Exp(\mathcal{R})| = |\mathbb{P}|$ (= 2^{\aleph_0}), then the family $\mathbb{P} \times Exp(\mathcal{R}) = \{P \times [n, A] : P \in \mathbb{P} \text{ and } A \in \mathcal{R}\}$ is also *M*-like. This lead us to the following:

Corollary 5.0.13. Let $(\mathcal{R}, \leq, (p_n)_{n \in \mathbb{N}})$ satisfying A1-A6, with \mathcal{R} metrically closed be such that $|\mathcal{R}| = 2^{\aleph_0}$. Then, the family of perfectly Ramsey subsets of $2^{\infty} \times \mathcal{R}$ is closed under the Souslin operation.

Proof. In virtue of theorem 3.0.3, the family of perfectly Ramsey subsets of $2^{\infty} \times \mathcal{R}$ coincides with the family of subsets of $2^{\infty} \times \mathcal{R}$ which have the $\mathbb{P} \times Exp(\mathcal{R})$ -Baire property. And as we pointed out in the previous paragraph, $\mathbb{P} \times Exp(\mathcal{R})$ is *M*-like. So the proof follows from lemma 5.0.12on top of this section and lemmas 2.5 and 2.6 of [13] (which refer to a well-known result of Marczewski [8]).

Corollary 5.0.14. (a)[Pawlikowski] The family of perfectly Ramsey subsets of $2^{\infty} \times \mathbb{N}^{[\infty]}$ is closed under the Souslin operation. (b)The

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family of perfectly Ramsey subsets of $2^{\infty} \times FIN_k^{[\infty]}$ is closed under the Souslin operation.

Corollary 5.0.15. (a)[Miller-Todorcevic] Analytic subsets of $2^{\infty} \times \mathbb{N}^{[\infty]}$ are perfectly Ramsey. (b)Analytic subsets of $2^{\infty} \times FIN_k^{[\infty]}$ are perfectly Ramsey.

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