On biclique coverings

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Abstract

It was proved by Fronček, Jerebic, Klavžar, and Kovář that if a complete bipartite graph $K_{n,n}$ with a perfect matching removed can be covered by k bicliques, then $n \le \binom{k}{\lfloor \frac{k}{2} \rfloor}$. We give a slightly simplified proof and we show that the result is tight. Moreover we use the result to prove analogous bounds for coverings of some other classes of graphs by bicliques.

1 Introduction

Let $G = (V, E)$ be a graph and $H_i = (V_i, E_i)$ for $i = 1, 2, \ldots, k$ be subgraphs of G. If $E = E_1 \cup E_2 \cup \ldots \cup E_k$, we say that G is *covered* by $H_1, H_2, \ldots, H_k$

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or that the subgraphs $H_1, H_2, \ldots, H_k$ form a *covering* of G. By a *biclique* we mean a complete bipartite graph.

There are several ways to define a minimum covering problem. For instance, Füredi and Kündgen [3] give general bounds for the total number of edges used in the covering of any graph G by bicliques, as well as sharp bounds for certain classes of graphs such as 4-colorable graphs and random graphs.

Chung [1] proved a conjecture of Bermond that $\lim_{n\to\infty} \rho(n)/n = 1$, where $\rho(n)$ denotes the smallest integer such that *any* graph with n vertices can be covered by $\rho(n)$ bicliques.

Froncek, Jerebic, Klavzar, and Kovar [2] proved that if $\tau(n)$ is the smallest number with the property that $K^-_{n,n}$ (the complete bipartite graph with a perfect matching removed) has a covering by $\tau(n)$ bicliques then $\lim_{n\to\infty} \frac{\tau(n)}{n} = 0$. They also proved that if there is a covering of $K^-_{n,n}$ by k bicliques, then $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$.

In this note we show that the result is tight and give a slightly simplified proof. We then use the result to prove analogous bounds for coverings of some other classes of graphs by bicliques.

2 Covering of $K^-_{n,n}$ revisited

The main tool used in the proof of Theorem 2.2 is Sperner's Theorem. An *antichain* $\{A_1, A_2, \ldots, A_n\}$ *on a set* A is a family of nonempty subsets of A such that $A_i \subseteq A_j$ implies that $i = j$. In other words, none of the subsets is fully contained in another one.

Theorem 2.1 **(Sperner)** *Let* $A = \{1, 2, \ldots, k\}$ *and let* $\{A_1, A_2, \ldots, A_n\}$ *be an antichain on* A. *Then* $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$. *Moreover, for each* $k \geq 1$, *there exists an antichain on* k *elements that contains* n *sets for every* $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$.

Let $K_{n,n} = (V \cup W, E)$ be the complete bipartite graph with the partite sets $V = \{v_1, v_2, \ldots, v_n\}, W = \{w_1, w_2, \ldots, w_n\}$ and the edge set $E = \{(v_i, w_j) \mid i, j = 1, 2, \ldots, n\}$. The graph $K_{n,n}$ with a perfect matching M removed will be denoted $K^-_{n,n}$. We assume without loss of generality that $M = \{(v_i, w_i) \mid i = 1, 2, \ldots, n\}$.

The following theorem was proved in [2]. We simplify the proof below.

Theorem 2.2 *Let* H_1, H_2, $\ldots$, H_k *be a covering of* $K^-_{n,n}$ *by* k *bicliques. Then* $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$.

Proof Suppose we have a covering of $K^-_{n,n}$ by k bicliques $H_1, H_2, \ldots, H_k$. For $i = 1, 2, \ldots, n$ we define $A_i = \{j \mid v_i \in H_j\}$. Obviously, every A_i is a subset of $A = \{1, 2, \ldots, k\}$. Because $H_1, H_2, \ldots, H_k$ form a covering, every edge is covered and every vertex v_i belongs to at least one biclique H_j. Hence, no A_i is empty. We only need to show that there is no pair of sets A_i and A_m such that $A_i \subseteq A_m$ while $i \neq m$.

We proceed by contradiction and suppose $A_i \subseteq A_m$ for some $i \neq m$. Let $j \in A_i$. Then also $j \in A_m$. This means that $v_i \in H_j$ implies $v_m \in H_j$. Because the edge $v_m w_m$ is not contained in $K^-_{n,n}$, w_m does not belong to any biclique H_p, where $p \in A_m$. Therefore, no biclique that contains v_i (and consequently v_m) can contain w_m and the edge $v_i w_m$ is not covered. This is a contradiction, as we assumed that $i \neq m$. Therefore, no set A_i is contained in another set A_m and $\{A_1, A_2, \ldots, A_n\}$ is an antichain on A.

By Sperner's Theorem an antichain on $A = \{1, 2, \ldots, k\}$ contains at most $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ sets. Thus $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$ and the proof is complete. □

The result can be stated in terms of a lower bound for the number k of bicliques that are needed to cover $K^-_{n,n}$. If $H_i = (V_i, E_i)$ is a biclique in the cover and $X = V_i \cap V$, without loss of generality we can assume $V_i \cap W = \overline{X}$. Now, in order to cover an edge $(v_i, w_j) \in E$, there should be a biclique H_t in the cover such that $v_i \in V_t$ and $w_j \notin V_t$. Equivalently, we are looking for a collection of subsets $\{X_1, X_2, \ldots, X_k\}$ of V such that for any ordered pair (v_i, v_j) with $i \neq j$ one has $v_i \in X_t$ and $v_j \notin X_t$ for some t, $t = 1, 2, \ldots, k$. Such a collection $\{X_1, X_2, \ldots, X_k\}$ is called a *completely separating system* of a set of n elements, and its minimum size is established in [5]. The result states that the minimum size is

$$\min \left\{ c \mid \binom{c}{\lfloor \frac{c}{2} \rfloor} \geq n \right\}.$$

Corollary 2.3 *Let c be the smallest integer such that $n \leq \binom{c}{\lfloor \frac{c}{2} \rfloor}$. Let k be the number of bicliques covering $K^-_{n,n}$. Then $k \geq c$.*

Now we show that the bound is sharp. The proof is in a sense a dual construction to that in Theorem 2.2.

Lemma 2.4 *Let $A = \{1, 2, \ldots, k\}$ and let $\{A_1,\ A_2,\ \ldots,\ A_n\}$ be an antichain on A. Then there exists a covering of $K^-_{n,n}$ by k bicliques.*

Proof We construct a biclique H_j for every $j = 1, 2, \ldots, k$. Each H_j will be uniquely determined by precisely the subsets of A that contain the element j.

Let $K^-_{n,n}$ have partite sets $V = \{A_1, A_2, \ldots, A_n\}$ and $W = \{\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n\}$, where $A_1, A_2, \ldots, A_n$ are the sets of an antichain on k elements and $\overline{A}_i$ is the complement of A_i. An edge $A_i \overline{A}_j$ belongs to the edge set of $K^-_{n,n}$ if and only if $i \neq j$. That means that the "missing matching" consists of the edges $(A_i, \overline{A}_i)$ for $i = 1, 2, \ldots, n$.

We define the covering bicliques $H_j = (V_j \cup W_j, E_j)$ for $j = 1, 2, \ldots, k$ in such a way that V_j consists of all sets A_i containing j, that is,

$$V_j = \{A_i \in V \mid j \in A_i\}$$

and W_j consists of all complements $\overline{A}_t$ containing j, that is,

$$W_j = \{\overline{A}_t \in W \mid j \in \overline{A}_t\} = \{\overline{A}_t \in W \mid A_t \notin V_j\}.$$

We observe that each biclique H_j is the maximal biclique containing the set V_j and that $|V_j \cup W_j| = n$ for every j. The edge sets are

$$E_j = \{(A_i, \overline{A}_t) \mid j \in A_i \text{ and } j \in \overline{A}_t\} = \{(A_i, \overline{A}_t) \mid A_i \in V_j \text{ and } \overline{A}_t \in W_j\}.$$

In other words, two sets A_i and $\overline{A}_t$ are joined by an edge if they have an element in common. Obviously, no edge $(A_i, \overline{A}_i)$ is covered by any H_j, since the intersection $A_i \cap \overline{A}_i$ is empty.

It remains to show that the bicliques H_j cover the graph $K^-_{n,n}$. Suppose to the contrary that there is an edge $(A_i, \overline{A}_m)$ for $i \neq m$, which is not covered by any biclique. Then the sets A_i and $\overline{A}_m$ have an empty intersection. But then all elements of A_i are contained in A_m and $A_i \subseteq A_m$, which contradicts our assumption that the family $\{A_1, A_2, \ldots, A_n\}$ is an antichain. $\square$

Summarizing Theorem 2.1 and Lemma 2.4 we get

Corollary 2.5 *Let k be the smallest integer such that $n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$. Then there exists a covering of $K^-_{n,n}$ by k bicliques.*

Proof Follows directly from Lemma 2.4 and the second part of Sperner's Theorem. $\square$

Hence, the bound $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ is sharp.

The results above can be now summarized as follows.

Theorem 2.6 *Let n and k be integers such that $\binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} < n \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$. Then k is the minimum number such that there exists a covering of $K^-_{n,n}$ by k bicliques.*

Proof According to Corollary 2.5 there exists a covering of $K^-_{n,n}$ by k bicliques. On the other hand from Corollary 2.3 it follows that a covering by less than k bicliques is not possible. $\square$

3 Related results

Using Theorem 2.2, we get upper bounds on the minimum number of bicliques in a biclique covering for some other classes of graphs.

Let $f(n)$ be the smallest k such that K_n can be covered by k bicliques $H_1, H_2, \ldots, H_k$. Then K_{nm} can be covered by $f(nm)$ bicliques. When we remove from K_{nm} edges of n disjoint copies of K_m to obtain the graph $K_{m,m,\ldots,m}$, we cannot use the same covering as for K_{nm}, since this would also cover the removed edges. One could then expect that we will need more than $f(nm)$ bicliques. It was proved by Füredi and Kündgen [3] that the minimum number of bipartite subgraphs needed to cover the edges of a graph G with chromatic number $\chi(G)$ is $\lceil \lg \chi(G) \rceil$. Therefore, their upper bounds on the minimum number of bicliques in a covering of K_n and $K_{m,m,\ldots,m}$ by bicliques is the same, namely $\lceil \lg n \rceil$.

Inspired by this result, we prove a slightly more general result for the covering of $K_{m,m,\dots,m}$. Using the covering $H_1, H_2, \dots, H_k$ of K_n we produce a covering of the complete n-partite graph $K_{m,m,\dots,m}$ also by k bicliques. The result is a special case of a more general result for lexicographic products of graphs.

The *lexicographic product* or *composition* $G[H]$ of graphs G and H is defined as follows: $V(G[H]) = V(G) \times V(H)$ and $(x_1, y_1)(x_2, y_2) \in E(G[H])$ if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $x_1 x_2 \in E(G)$.

Theorem 3.1 *If there exists a covering of G by k bicliques then there also exists a covering of the lexicographic product $G[\overline{K}_m]$ by k bicliques.*

Proof Let $H_1, H_2, \dots, H_k$ be a biclique covering of G with n vertices $x_1, x_2, \dots, x_n$. We will construct for each H_j a biclique I_j such that $I_1, I_2, \dots, I_k$ will be a covering of $G[\overline{K}_m]$. The vertex set of $G[\overline{K}_m]$ is the union of the partite sets $X_1, X_2, \dots, X_n$, where $X_i = \{x_i^1, x_i^2, \dots, x_i^m\}$. This means that every vertex of G is blown up into m independent vertices of $G[\overline{K}_m]$ and each edge of G is blown up into the complete bipartite subgraph $K_{m,m}$ of $G[\overline{K}_m]$. Using this observation, we can now construct the biclique I_j as the composition $H_j[\overline{K}_m]$. In other words, every vertex x_p in the original biclique H_j will be blown up into m independent vertices $x_p^1, x_p^2, \dots, x_p^m$ and every edge $x_p x_q$ of H_j will be blown up into the complete bipartite graph with partite sets X_p and X_q. Obviously, the graph $G[\overline{K}_m]$ is now covered by the bicliques $I_1, I_2, \dots, I_k$. For if $x_p^r x_q^s$ is an arbitrary edge in $G[\overline{K}_m]$, then it is covered by at least one biclique—in particular, by $I_j = H_j[\overline{K}_m]$, where H_j is one of the bicliques covering $x_p x_q$ in the original graph G. Since every $x_p x_q$ is covered in G, every $x_p^r x_q^s$ is covered in $G[\overline{K}_m]$. Obviously, no edge $x_i^r x_i^s$ is covered by any biclique I_j and the proof is complete. $\square$

By setting $G = K_n$ in the previous theorem, we get instantly the following.

Corollary 3.2 *If there exists a covering of K_n by k bicliques then there also exists a covering of the complete n-partite graph $K_{m,m,\dots,m}$ by k bicliques.*

We will denote by $CP(n)$ the *cocktail party graph*, i.e., the complete graph $K_{2n} = (V, E)$ with a perfect matching M removed. Since $CP(n)$ is isomorphic to the complete n-partite graph $K_{2,2,\dots,2}$, a result analogous to Theorem 2.2 now follows easily from Corollary 3.2.

Corollary 3.3 *If there exists a covering of K_n by k bicliques then there exists a covering of $CP(n)$ by k bicliques.*

We can also use the technique described in the proof of Theorem 3.1 to prove the following.

Theorem 3.4 *Let there exist a covering of a graph G by ℓ bicliques and a covering of K_n by k bicliques. Then the graph $K_n[G]$ can be covered by $k + \ell$ bicliques.*

Proof Suppose that K_n has vertices $x^1, x^2, \dots, x^n$ while G has m vertices $x_1, x_2, \dots, x_m$. First we observe that $K_n[G]$ can be covered by $K_n[\overline{K}_m] = K_{m,m,\dots,m}$ and $G[\overline{K}_n]$ in the following manner. Each vertex x^j of K_n is in

$K_n[G]$ blown up into the set $X^j = \{x_1^j, x_2^j, \ldots, x_m^j\}$. These sets correspond to the partite sets of the graph $K_n[\overline{K}_m] = K_{m,m,\ldots,m}$. Therefore, every edge $x_p^r x_q^s$ of $K_n[G]$ for $r \neq s$ is covered by the graph $K_n[\overline{K}_m] = K_{m,m,\ldots,m}$. In every set X^j in $K_n[G]$ "resides" a copy of G whose edges are not yet covered. However, all of these copies are covered by the graph $G[\overline{K}_n]$ in which every vertex x_i of G is blown up into the set $X_i = \{x_i^1, x_i^2, \ldots, x_i^n\}$ and every edge $x_i x_t$ is blown up into the complete bipartite graph with partite sets X_i and X_t.

Because G can be covered by ℓ bicliques, by Theorem 3.1 $G[\overline{K}_n]$ can be also covered by ℓ bicliques. Similarly, because K_n can be covered by k bicliques, by Corollary 3.2 $K_n[\overline{K}_m]$ can be also covered by k bicliques. Since $K_n[G]$ is covered by $G[\overline{K}_n]$ and $K_n[\overline{K}_m]$, the conclusion follows. □

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