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Abstract

It is known that if (A, A^*) is a Leonard pair, then the linear transformations A, A^* satisfy the Askey-Wilson relations

$$\begin{aligned} A^2A^* - \beta AA^*A + A^*A^2 - \gamma\,(AA^*+A^*A) - \varrho\, A^* &= \gamma^*A^2 + \omega A + \eta\, I, \\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A+AA^*) - \varrho^* A &= \gamma A^{*2} + \omega A^* + \eta^* I, \end{aligned}$$

for some scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$. The problem of this paper is the following: given a pair of Askey-Wilson relations as above, how many Leonard pairs are there that satisfy those relations? It turns out that the answer is 5 in general. We give the generic number of Leonard pairs for each Askey-Wilson type of Askey-Wilson relations.

1 Introduction

Throughout the paper, $\mathbb{K}$ denotes an algebraically closed field. We assume the characteristic of $\mathbb{K}$ is not equal to 2. Recall that a tridiagonal matrix is a square matrix which has non-zero entries only on the main diagonal, on the superdiagonal and the subdiagonal. A tridiagonal matrix is called irreducible whenever all entries on the superdiagonal and superdiagonal are non-zero.

Definition 1.1 Let V be a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations which satisfy the following two conditions:

1. There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tridiagonal.
2. There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal.

Remark 1.2 In this paper we do not use the conventional notation A^* for the conjugate-transpose of A. In a Leonard pair (A, A^*), the linear transformations A and A^* are arbitrary subject to the conditions *(i)* and *(ii)* above.

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Definition 1.3 Let V, W be vector spaces over $\mathbb{K}$ with finite positive dimensions. Let (A, A^*) denote a Leonard pair on V, and let (B, B^*) denote a Leonard pair on W. By an *isomorphism of Leonard pairs* we mean an isomorphism of vector spaces $\sigma : V \mapsto W$ such that $\sigma A \sigma^{-1} = B$ and $\sigma A^* \sigma^{-1} = B^*$. We say that (A, A^*) and (B, B^*) are *isomorphic* if there is an isomorphism of Leonard pairs from (A, A^*) to (B, B^*).

Leonard pairs occur in the theory of orthogonal polynomials, combinatorics, the representation theory of the Lie algebra sl_2 or the quantum group $U_q(sl_2)$. We refer to [Ter04] as a survey on Leonard pairs, and as a source of further references.

We have the following result [TV04, Theorem 1.5].

Theorem 1.4 *Let V denote a vector space over $\mathbb{K}$ of finite positive dimension. Let (A, A^*) be a Leonard pair on V. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$, ω, η, η^* taken from $\mathbb{K}$ such that*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma\,(AA^*+A^*A) - \varrho\,A^* \;=\; \gamma^*A^2 + \omega A + \eta\,I, \qquad (1)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A+AA^*) - \varrho^*A \;=\; \gamma A^{*2} + \omega A^* + \eta^* I. \qquad (2)$$

The sequence is uniquely determined by the pair (A, A^*) provided the dimension of V is at least 4.

The equations (1)–(2) are called the *Askey-Wilson relations*. They first appeared in the work [Zhe91] of Zhedanov, where he showed that the Askey-Wilson polynomials give pairs of infinite-dimensional matrices which satisfy the Askey-Wilson relations. We denote this pair of equations by $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$. We refer to the 8 scalar parameters as the *Askey-Wilson coefficients*.

A natural question is the following: does a particular pair of Askey-Wilson relations determines a Leonard pair uniquely? An example in the next Section shows that the answer is negative in general. One may ask then: if we fix the dimension of V and the 8 scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*,\ \omega, \eta, \eta^*$, how many Leonard pairs are there which satisfy $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$? This is the question that we consider in this paper.

It turns out that there may be up to 5 different Leonard pairs satisfying the same Askey-Wilson relations. As a preliminary check, one may consider the case $\dim V = 1$. Then we have 2 equations in 2 commuting unknowns A and A^*. After computing a resultant or a Gröbner basis one concludes that there are 5 solutions in general.

Table 1 represents our main results: the number of Leonard pairs, up to isomorphism, with the same Askey-Wilson relations for various sequences of the Askey-Wilson coefficients. We distinguish cases according to the classification of Askey-Wilson relations in [Vid05, Section 8], which mimics Terwilliger's classification of parameter arrays representing Leonard pairs; see [Ter02] or [Ter04, Section 35] and Section 3 here. Our results are valid if $\dim V \geq 4$.

The first column of Table 1 characterizes the distinguished cases in terms of the Askey-Wilson coefficients. The underlined expressions are not defining conditions; they mean that the Askey-Wilson relations can be normalized by affine transformations

$$(A, A^*) \mapsto (tA + c, t^*A^* + c^*), \qquad \text{with } c, c^*, t, t^* \in \mathbb{K};\ \ t, t^* \neq 0, \qquad (3)$$

Askey-Wilson coefficients	Leonard pairs	Askey-Wilson type
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}\,\widehat{\varrho}^* \neq 0$	5	q-Racah
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = 0$, $\widehat{\varrho}^*\widehat{\eta} \neq 0$	4	q-Hahn
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}^* = 0$, $\widehat{\varrho}\,\widehat{\eta}^* \neq 0$	4	Dual q-Hahn
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = \widehat{\eta} = 0$, $\widehat{\varrho}^*\widehat{\eta}^* \neq 0$	1	q-Krawtchouk
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}^* = \widehat{\eta}^* = 0$, $\widehat{\varrho}\,\widehat{\eta} \neq 0$	1	Dual q-Krawtchouk
$\beta \neq \pm 2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho} = \widehat{\varrho}^* = 0$, $\widehat{\eta}\,\widehat{\eta}^* \neq 0$	3	Quantum/affine q-Krawtchouk
$\beta = 2$, $\gamma\,\gamma^* \neq 0$, $\underline{\varrho = \varrho^* = 0}$	4	Racah
$\beta = 2$, $\gamma = 0$, $\gamma^*\varrho \neq 0$, $\underline{\varrho^* = \omega = 0}$	3	Hahn
$\beta = 2$, $\gamma^* = 0$, $\gamma\,\varrho^* \neq 0$, $\underline{\varrho = \omega = 0}$	3	Dual Hahn
$\beta = 2$, $\gamma = \gamma^* = 0$, $\varrho\varrho^* \neq 0$, $\underline{\eta = \eta^* = 0}$	1	Krawtchouk
$\beta = -2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}\,\widehat{\varrho}^* \neq 0$; $\dim V$ odd	5	Bannai-Ito
$\beta = -2$, $\underline{\gamma = \gamma^* = 0}$, $\widehat{\varrho}\,\widehat{\varrho}^* \neq 0$; $\dim V$ even	4	Bannai-Ito

Table 1: Leonard pairs with fixed Askey-Wilson relations, if $\dim V \geq 4$

into a form where the underlined expressions hold (provided that the preceding conditions are satisfied). Normalization of Askey-Wilson relations is adequately explained in [Vid05, Section 4]. Particularly, if $\beta \neq 2$ then the Askey-Wilson relations can be normalized so that $\gamma = 0$ and $\gamma^* = 0$. By $\widehat{\varrho}$, $\widehat{\varrho}^*$, $\widehat{\omega}$, $\widehat{\eta}$, $\widehat{\eta}^*$ we denote other Askey-Wilson coefficients in such a normalization.

The second column indicates the generic number of Leonard pairs satisfying Askey-Wilson relations restricted by the conditions in the first column. The results are generic, so for some special values of the Askey-Wilson coefficients the number of distinct Leonard pairs may be smaller. In these special cases, one may interpret missing Leonard pairs as degenerate, or one may argue that generically different Leonard pairs are isomorphic in the special case. This is explained in Remark 3.2 and demonstrated in Example 5.3 here below. If a sequence of Askey-Wilson coefficients satisfies neither condition set of the first column, there are no Leonard pairs satisfying those Askey-Wilson relations.

The third column gives the Askey-Wilson type of Askey-Wilson relations as defined in [Vid05, Section 8]. Leonard pairs have the same Askey-Wilson type as the Askey-Wilson relations that the satisfy, according to [Vid05, Theorem 8.1].

We use Terwilliger's classification of parameters arrays representing Leonard pairs. Therefore in Section 3 we recall definition of parameter arrays and classification terminology. In Section 4 we present normalized general parameter arrays and the Askey-Wilson relations for Leonard pairs represented by them. The results of Table 1 are proved in Section 5.

2 An example

Here we give an example of Askey-Wilson relations satisfied by different Leonard pairs. This example was observed by Curtin [Cur04] as well.

Let d be a non-negative integer, and let V be a vector space with dimension $d+1$ over $\mathbb{K}$. Let q denote a scalar which is not zero and not a root of unity. Set $\beta = q + q^{-1}$. We look for Leonard pairs on V which satisfy

$$AW\left(\beta,\, 0,\, 0,\, \beta^2 - 4,\, \beta^2 - 4,\, 0,\, 0,\, 0\right). \tag{4}$$

Existence of a Leonard pair satisfying these relations follows from [Cur01], where Terwilliger algebras for 2-homogenous bipartite distance regular graphs are computed. The Terwilliger algebra is defined by two non-commuting generators and two relations. The relations differ from (4) by a scaling of the generators. The two generators can be represented as a Leonard pair (A, A^*). It has the property that tridiagonal forms for A and A^* of Definition 1.1 have only zero entries on the main diagonal. A rescaled version of (A, A^*) must satisfy (4). Additionally, [Cur04] computed "almost 2-homogenous almost bipartite" Leonard pairs satisfying the same defining relations of the Terwilliger algebra. For these Leonard pairs, the tridiagonal forms of Definition 1.1 have precisely one non-zero entry on the main diagonal.

Here we present Leonard pairs of both kinds explicitly. They are scaled so that they satisfy (4). Let us denote some square roots:

$$C_1 = \sqrt{q^{-d}}, \qquad C_2 = \sqrt{-q^{-1}}. \tag{5}$$

Let A_1, A_1^*, A_2, A_2^* be the following matrices:

- A_1 is tridiagonal, with zero entries on the main diagonal, the entries
$$C_1\,\frac{q^{2d-2j}-1}{q^{d-2j}+1}, \qquad \text{for } j = 0, \ldots, d-1,$$
on the superdiagonal, and the entries
$$C_1\,\frac{q^{2j}-1}{q^{2j-d}+1}, \qquad \text{for } j = 1, \ldots, d,$$
on the subdiagonal.

- A_1^* is diagonal, with the entries
$$C_1\left(q^{d-j} - q^j\right), \qquad \text{for } j = 0, \ldots, d,$$
on the main diagonal.

- A_2 is tridiagonal, with the upper-left entry equal to $C_2\,\dfrac{q^{2d+2}-1}{q^d\,(q-1)}$, all other diagonal entries equal to zero, with the entries
$$-C_2\,\frac{q^{2d-2j}-1}{q^{d-2j-1}\,(q^{2j+1}-1)}, \qquad \text{for } j = 0, \ldots, d-1,$$

on the superdiagonal, and the entries

$$C_2\,\frac{q^{2d+2j+2}-1}{q^d\left(q^{2j+1}-1\right)}, \qquad \text{for } j=1,\ldots,d.$$

on the subdiagonal.

- A_2^* is diagonal, with the diagonal entries $C_2\left(q^{-j}+q^{j+1}\right)$, for $j=0,1,\ldots,d$.

One can routinely check that the pairs (A_1,A_1^*) and (A_2,A_2^*) satisfy Askey-Wilson relations (4). Compared with the intersection numbers for 2-homogenous bipartite distance regular graphs in [Cur01], the non-zero entries of A_1 differ by the factor $C_1\left(q^2-1\right)/\left(q^d+q^2\right)$.

One can see that both (A_1,A_1^*) and (A_2,A_2^*) are Leonard pairs in one of the following ways:

- Using Theorem [TV04, Teorem 6.2]. For $i=1,2$, the sufficient conditions for (A_i,A_i^*) to be a Leonard pair are the following:
  - There exists a sequence of scalars $\beta,\gamma,\gamma^*,\varrho,\varrho^*,\omega,\eta,\eta^*$ taken from $\mathbb{K}$ such that the Askey-Wilson relations as in (1)–(2) hold.
  - q is not a root of unity, where $q+q^{-1}=\beta$.
  - Both A_i and A_i^* are multiplicity free.
  - V is irreducible as an A_i, A_i^* module.
- By using classification of Leonard pairs [Ter04, Section 35]. Consider the most general q-Racah type:

$$\theta_i \;=\; \theta_0+h\left(1-q^i\right)\left(1-s\,q^{i+1}\right)q^{-i}, \tag{6}$$
$$\theta_i^* \;=\; \theta_0^*+h^*\left(1-q^i\right)\left(1-s^*q^{i+1}\right)q^{-i}, \tag{7}$$
$$\varphi_i \;=\; hh^*q^{1-2i}\left(1-q^i\right)\left(1-q^{i-d-1}\right)\left(1-r_1\,q^i\right)\left(1-r_2\,q^i\right), \tag{8}$$
$$\phi_i \;=\; hh^*q^{1-2i}\left(1-q^i\right)\left(1-q^{i-d-1}\right)\left(r_1-s^*q^i\right)\left(r_2-s^*q^i\right)/s^*. \tag{9}$$

  Here $q\neq 0,\pm1$, the constants h,h^*,s,s^*,r_1,r_2 are non-zero and satisfy $r_1r_2=s\,s^*q^{d+1}$, none of $q^i,r_1q^i,r_2q^i,s^*q^i/r_1,s^*q^i/r_2$ is equal to 1 for $i=1,\ldots,d$, and neither of $s\,q^i,s^*q^i$ is equal to 1 for $i=2,\ldots,2d$. To get the pair (A_1,A_1^*) we must take

$$\theta_0=\theta_0^*=C_1\big(q^d-1\big)\,, \quad h=h^*=\frac{1}{C_1},$$
$$s=s^*=-q^{-d-1}, \qquad r_1=-r_2=\sqrt{-q^{-d-1}},$$

  and use explicit expressions in [Ter04, Section 27]. To get the pair (A_2,A_2^*) we must take

$$\theta_0=\theta_0^*=C_2(q+1), \quad h=h^*=\frac{1}{C_2}, \quad s=s^*=1, \quad r_1=-1, \quad r_2=-q^{d+1}.$$

- By exhibiting explicit transition matrices to a base mentioned in part *(ii)* of Definition 1.1. Entries of the transition matrices are q-Racah polynomials [Ter04, Section 24]. Let P_1 denote the matrix with the (i,j)-th entry equal to

$$(-1)^j\, q^{dj} \big(1+q^{2j-d}\big) \frac{(q^{-2d};q^2)_j}{(q^2;q^2)_j} \times \\ R_i\Big(\, q^{-j}-q^{j-d};\; \sqrt{-q^{-d-1}},\; \sqrt{-q^{-d-1}},\; q^{-d-1}, -1 \,\Big|\; q\Big)\,. \tag{10}$$

  and let P_2 denote the matrix with the (i,j)-th entry equal to

$$\frac{(1-q^{2j+1})}{q^{di+i-dj}} \frac{(-q^{d+2};q)_i\,(q^{-d};q)_j}{(-q^{-d};q)_i\,(q^{d+2};q_j)}\, R_i\big(\, q^{-j}+q^{j+1}; -1,-1,q^{-d-1},q^{d+1} \big|\; q\big)\,. \tag{11}$$

  Recall [KS94, Section 3.2] that in the q-hypergeometric notation,

$$R_n(z(x);\,\alpha,\beta,\gamma,\delta\,|\,q) = {}_4\phi_3 \left(\begin{matrix} q^{-n},\ \alpha\beta q^{n+1},\ q^{-x},\ \gamma\delta q^{x+1} \\ \alpha q,\ \beta\delta q,\ \gamma q \end{matrix} ;\ q;\ q \right),$$

  with

$$z(x) = q^{-x} + \gamma\delta q^{x+1} \qquad \text{and} \qquad q^{-d} \in \{\alpha q,\ \beta\delta q,\ \gamma q\}.$$

  Using q-difference relations for q-Racah polynomials, one can routinely check that $A_iP_i = P_iA_i^*$ and $A_i^*P_i = P_iA_i$ for $i = 1, 2$. This implies that conjugation by P_i converts the pair (A_i, A_i^*) to the matrix pair (A_i^*, A_i), and condition *(ii)* of Definition 1.1 is satisfied. (As we see, both Leonard pairs are self-dual.)

The conclusion is that both (A_1, A_1^*) and (A_2, A_2^*) are Leonard pairs, and they satisfy the same Askey-Wilson relations (4).

Table 1 predicts 5 Leonard pairs satisfying (4). Indeed, the 5 Leonard pairs are

$$(A_1, A_1^*), \quad (A_2, A_2^*), \quad (A_2, -A_2^*), \quad (-A_2, A_2^*), \quad (-A_2, -A_2^*). \tag{12}$$

The last 4 Leonard pairs give an example of non-isomorphic Leonard pairs related by affine transformations (which are scalings by -1) and satisfying the same Askey-Wilson relations. The same affine scalings of (A_1, A_1^*) are isomorphic to (A_1, A_1^*). Surely, the affine scalings leave the Askey-Wilson relations invariant.

Changing the sign of the square roots C_1 or C_2 has the effect of rescaling both matrices by -1. Note that the substitution $q \mapsto 1/q$ preserves the Askey-Wilson relations; it has the same affine scaling action on (A_1, A_1^*), and it leaves (A_2, A_2^*) invariant.

3 Leonard pairs and parameter arrays

Leonard pairs are represented and classified by parameter arrays. More precisely, parameter arrays are in one-to-one correspondence with *Leonard systems* [Ter04, Definition 3.2], and to each Leonard pair one associates 4 Leonard systems or parameter arrays. From now on, let d be a non-negative integer, and let V be a vector space with dimension $d+1$ over $\mathbb{K}$.

Definition 3.1 By a *parameter array* over $\mathbb{K}$, of diameter d, we mean a sequence

$$(\theta_0, \theta_1, \ldots, \theta_d;\ \theta_0^*, \theta_1^*, \ldots, \theta_d^*;\ \varphi_1, \ldots, \varphi_d;\ \phi_1, \ldots, \phi_d) \tag{13}$$

of scalars taken from $\mathbb{K}$, that satisfy the following conditions:

1. $\theta_i \neq \theta_j$ and $\theta_1^* \neq \theta_j^*$ if $i \neq j$, for $0 \leq i, j \leq d$.
2. $\varphi_i \neq 0$ and $\phi_i \neq 0$, for $1 \leq i, j \leq d$.
3. $\varphi_i = \phi_1 \sum_{j=0}^{i-1} \dfrac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)\,(\theta_{i-1} - \theta_d)$, for $1 \leq i, j \leq d$.
4. $\phi_i = \varphi_1 \sum_{j=0}^{i-1} \dfrac{\theta_j - \theta_{d-j}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)\,(\theta_{d-i+1} - \theta_0)$, for $1 \leq i, j \leq d$.
5. The expressions
$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$
are equal and independent of i, for $2 \leq i \leq d-1$.

To get a Leonard pair from parameter array (13), one must choose a basis for V and define the two linear transformations by the following matrices (with respect to that basis):

$$\begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{14}$$

Alternatively, the following two matrices define an isomorphic Leonard pair on V:

$$\begin{pmatrix} \theta_d & & & & \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & \phi_1 & & & \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ & & & & \theta_d^* \end{pmatrix}. \tag{15}$$

Conversely, if (A, A^*) is a Leonard pair on V, there exists [Ter04, Section 21] a basis for V with respect to which the matrices for A, A^* have the bidiagonal forms in (14), respectively. There exists other basis for V with respect to which the matrices for A, A^* have the bidiagonal forms in (15), respectively, with the same scalars

$\theta_0, \theta_1, \ldots, \theta_d; \theta^*_0, \theta^*_1, \ldots, \theta^*_d$. Then the following 4 sequences are parameter arrays of diameter d:

$$(\theta_0, \theta_1, \ldots, \theta_d;\ \theta^*_0, \theta^*_1, \ldots, \theta^*_d;\ \varphi_1, \ldots, \varphi_d;\ \phi_1, \ldots, \phi_d), \tag{16}$$

$$(\theta_0, \theta_1, \ldots, \theta_d;\ \theta^*_d, \ldots, \theta^*_1, \theta^*_0;\ \phi_d, \ldots, \phi_1;\ \varphi_d, \ldots, \varphi_1), \tag{17}$$

$$(\theta_d, \ldots, \theta_1, \theta_0;\ \theta^*_0, \theta^*_1, \ldots, \theta^*_d;\ \phi_1, \ldots, \phi_d;\ \varphi_1, \ldots, \varphi_d), \tag{18}$$

$$(\theta_d, \ldots, \theta_1, \theta_0;\ \theta^*_d, \ldots, \theta^*_1, \theta^*_0;\ \varphi_d, \ldots, \varphi_1;\ \phi_d, \ldots, \phi_1). \tag{19}$$

Up to isomorphism of Leonard pairs, each of these parameter arrays gives back (A, A^*) by the construction above. There are no other parameter arrays with this property, hence we associate precisely the parameter arrays in (16)–(19) to (A, A^*). Obviously, $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are the eigenvalues of A and A^*, respectively.

We call the parameter arrays in (16)–(19) *relatives* of each other. They are connected by permutations, which form the group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that the group action is without fixed points, since the eigenvalues θ_i's (or θ^*_i's) are distinct. Let $\downarrow$, $\Downarrow$ and $\downarrow\Downarrow$ denote the permutations which transform (16) into (17), (18) and (19) respectively. To be consistent with [Ter04, Section 4], we nominate the 4 parameter arrays associated to the Leonard pair (A^*, A) as relatives of (16)–(19) as well.

Parameter arrays are classified in [Ter04, Section 35] and in [Ter02]. For each parameter array, certain orthogonal polynomials naturally occur in entries of the transformation matrix between two bases characterized in Definition 1.1 for the corresponding Leonard pair. Terwilliger's classification largely mimics the terminating branch of orthogonal polynomials in the Askey-Wilson scheme [KS94]. Specifically, the classification comprises Racah, Hahn, Krawtchouk polynomials and their q-versions, plus Bannai-Ito and orphan polynomials. Classes of parameter arrays can be identified by the type of corresponding orthogonal polynomials; we refer to them as *Askey-Wilson types*. The type of a parameter array is unambiguously defined if $d \geq 3$. We recapitulate Terwilliger's classification in Section 4 by giving general normalized parameter arrays of each type.

By inspecting Terwilliger's general parameter arrays [Ter04, Section 35], one can observe that the relation operators $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$ do not change the Askey-Wilson type of a parameter array (but only the free parameters such as q, h, h^*, s there), except that the $\Downarrow$ and $\downarrow\Downarrow$ relations mix up the quantum q-Krawtchouk and affine q-Krawtchouk types. Consequently, given a Leonard pair, all 4 associated parameter arrays have the same type, except when parameter arrays of the quantum q-Krawtchouk or affine q-Krawtchouk type occur. Therefore we can use the same classifying terminology for Leonard pairs, except that we have to merge the quantum q-Krawtchouk and affine q-Krawtchouk types.

Expressions for Askey-Wilson coefficients in terms of parameter arrays are given in [TV04, Theorem 4.5 and Theorem 5.3] and [Vid05, formulas (11)-(23)]. For example, we have

$$\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}, \qquad \text{for } i = 2, \ldots, d-1; \tag{20}$$

$$
\begin{aligned}
\gamma &= \theta_{i-1} - \beta\theta_i + \theta_{i+1}, && \text{for } i = 1, \ldots, d-1; && (21)\\
\gamma^* &= \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}, && \text{for } i = 1, \ldots, d-1; && (22)\\
\varrho &= \theta_i^2 - \beta\,\theta_i\,\theta_{i-1} + \theta_{i-1}^2 - \gamma\,(\theta_i + \theta_{i-1}), && \text{for } i = 1, \ldots, d; && (23)\\
\varrho^* &= \theta_i^{*\,2} - \beta\theta^*_i\theta^*_{i-1} + \theta^{*2}_{i-1} - \gamma^*(\theta^*_i + \theta^*_{i-1}), && \text{for } i = 1, \ldots, d. && (24)
\end{aligned}
$$

In principle, these equations can be used to compute parameter arrays (and consequently, Leonard pairs) satisfying fixed Askey-Wilson relations. For instance, one can use (21)–(22) to eliminate consequently $\theta_2, \ldots, \theta_d$ and $\theta^*_2, \ldots, \theta^*_d$. Each solution of obtained equations represents a parameter array in general. Since we are interested in counting Leonard pairs rather than parameter arrays, we would get $5 \times 4 = 20$ solutions in general. To get an equation system whose solutions correspond directly to Leonard pairs, one should find $\Downarrow$-$\downarrow$-invariant equations and rewrite them in terms of invariants of the $\Downarrow$-$\downarrow$-action. Examples of such invariants are, for $i = 0, 1, \ldots, \lfloor \frac{d-1}{2} \rfloor$:

$$
\theta_i + \theta_{d-i}, \quad \theta_i\theta_{d-i}, \quad \varphi_i(\theta_{d-i+1} - \theta_{d-i}) + \varphi_{d-i+1}(\theta_i - \theta_{i-1}), \quad \theta^*_i + \theta^*_{d-i}.
$$

The Askey-Wilson coefficients are invariants as well. These direct equations can be investigated and solved if d is fixed and small. In general, it seems that one cannot avoid use of explicit solutions of recurrence relations such as (21)–(22), which basically leads to classification of parameter arrays. Therefore we openly use Terwilliger's classification. In Section 4 we present general normalized parameter arrays and Askey-Wilson relations for them.

Remark 3.2 For special instances of Askey-Wilson relations, the number of distinct Leonard pairs may be different from the respective generic number in Table 1, usually smaller. Within intersection theory (or moduli space) philosophy, there may be following "reasons" for this:

- Some solutions of a defining equation system are degenerate. They do not represent actual Leonard pairs, but "degenerate" objects. In our problem, degenerations are represented by "parameter arrays" which do not satisfy conditions 1 and 2 in Definition 3.1.
- Some Leonard pairs (in parametrized families) are supposed to be generically different and non-isomorphic, but they coincide or are isomorphic for special instances of Askey-Wilson relations. In these situations one tries to assign a *multiplicity* to each solution so that multiplicities of all solutions add up to the generic number. Multiplicities should be defined by considering the defining equation system locally, or by an appropriate infinitesimal deformation of the coefficients.

Example 5.3 here below presents instances of these situations. More generally, we may expect other two standard complications:

- Some missing solutions are at the "infinity", that is, on the compactified "space" of possible Leonard pairs. We do not need this interpretation within each Askey-Wilson type, unless we wish to have the most generic 5 Leonard pairs each time.

- A specialized defining equation system defines an algebraic variety of positive dimension. Then there are infinitely many solutions, continuous families of them. This situation is not relevant to us. (Lemma [Vid05, Lemma 4.1] suggests it for Askey-Wilson relations with $\beta = 2$, $\gamma = 0$, $\gamma^* = 0$, $\omega^2 = \varrho\varrho^*$, but all solutions are degenerate if $d \geq 3$.)

4 Normalized Leonard pairs

Let (A, A^*) denote a Leonard pair, and let c, c^*, t, t^* denote scalars in $\mathbb{K}$. It is easy to see that if t and t^* are non-zero, then $(t\,A + c,\; t^* A^* + c^*)$ is a Leonard pair again. We identify here affine transformations (3) acting on Leonard pairs. A corresponding action on parameter arrays is the following:

$$\theta_i \mapsto t\,\theta_i + c, \qquad \theta^*_i \mapsto t^*\theta^*_i + c^*, \qquad \varphi_i \mapsto t\,t^*\varphi_i, \qquad \phi_i \mapsto t\,t^*\phi_i. \tag{25}$$

Using affine transformations we can normalize a parameter array into a convenient form. We use the following normalizations.

Lemma 4.1 *The general parameter arrays in* [Ter04, Examples 35.2–35.13] *can be normalized by affine transformations* (25) *to the following forms:*

- *The q-Racah case:* $\theta_i = s\,q^{d-2i} + \dfrac{q^{2i-d}}{s}, \quad \theta^*_i = s^* q^{d-2i} + \dfrac{q^{2i-d}}{s^*}.$

$$\begin{aligned}
\varphi_i &= \frac{q^{2d+2-4i}}{s\,s^* r}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(s\,s^* - r\,q^{2i-d-1}\right)\left(s\,s^* r - q^{2i-d-1}\right),\\
\phi_i &= \frac{q^{2d+2-4i}}{s\,s^* r}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(s^* r - s\,q^{2i-d-1}\right)\left(s^* - s\,r\,q^{2i-d-1}\right).
\end{aligned}$$

- *The q-Hahn case:* $\theta_i = r\,q^{d-2i}, \quad \theta^*_i = s^* q^{d-2i} + \dfrac{q^{2i-d}}{s^*},$

$$\begin{aligned}
\varphi_i &= \frac{q^{2d+2-4i}}{r}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(s^* r^2 - q^{2i-d-1}\right),\\
\phi_i &= -\frac{q^{d+1-2i}}{r\,s^*}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(s^* - r^2 q^{2i-d-1}\right).
\end{aligned}$$

- *The dual q-Hahn case:* $\theta_i = s\,q^{d-2i} + \dfrac{q^{2i-d}}{s}, \quad \theta^*_i = r\,q^{d-2i},$

$$\begin{aligned}
\varphi_i &= \frac{q^{2d+2-4i}}{r}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(s\,r^2 - q^{2i-d-1}\right),\\
\phi_i &= \frac{q^{2d+2-4i}}{r\,s}\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(r^2 - s\,q^{2i-d-1}\right).
\end{aligned}$$

- *The q-Krawtchouk:* $\theta_i = q^{d-2i}, \quad \theta^*_i = s^* q^{d-2i} + \dfrac{q^{2i-d}}{s^*}$,

$$\begin{aligned}\varphi_i &= s^*\, q^{2d+2-4i}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right),\\ \phi_i &= \frac{1}{s^*}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right).\end{aligned}$$

- *The dual q-Krawtchouk:* $\theta_i = s\, q^{d-2i} + \dfrac{q^{2i-d}}{s}, \quad \theta^*_i = q^{d-2i}$,

$$\begin{aligned}\varphi_i &= s\, q^{2d+2-4i}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right),\\ \phi_i &= \frac{q^{2d+2-4i}}{s}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right).\end{aligned}$$

- *The quantum q-Krawtchouk:* $\theta_i = r\, q^{2i-d}, \; \theta^*_i = r\, q^{d-2i}$,

$$\begin{aligned}\varphi_i &= -\frac{q^{d+1-2i}}{r}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right),\\ \phi_i &= \frac{q^{2d+2-4i}}{r}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(r^3-q^{2i-d-1}\right).\end{aligned}$$

- *The affine q-Krawtchouk:* $\theta_i = r\, q^{d-2i}, \; \theta^*_i = r\, q^{d-2i}$,

$$\begin{aligned}\varphi_i &= \frac{q^{2d+2-4i}}{r}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right)\left(r^3-q^{2i-d-1}\right),\\ \phi_i &= -\frac{q^{d+1-2i}}{r}\,\left(1-q^{2i}\right)\left(1-q^{2i-2d-2}\right).\end{aligned}$$

- *The Racah case:* $\theta_i = (i+u)(i+u+1)$, $\theta^*_i = (i+u^*)(i+u^*+1)$,

$$\begin{aligned}\varphi_i &= i\,(i-d-1)\,(i+u+u^*+v)\,(i+u+u^*+d+1-v),\\ \phi_i &= i\,(i-d-1)\,(i-u+u^*-v)\,(i-u+u^*-d-1+v).\end{aligned}$$

- *The Hahn case:* $\theta_i = i+v-\frac{d}{2}$, $\theta^*_i = (i+u^*)(i+u^*+1)$,

$$\begin{aligned}\varphi_i &= i\,(i-d-1)\,(i+u^*+2v),\\ \phi_i &= -i\,(i-d-1)\,(i+u^*-2v).\end{aligned}$$

- *The dual Hahn case:* $\theta_i = (i+u)(i+u+1)$, $\theta^*_i = i+v-\frac{d}{2}$,

$$\begin{aligned}\varphi_i &= i\,(i-d-1)\,(i+u+2v),\\ \phi_i &= i\,(i-d-1)\,(i-u+2v-d-1).\end{aligned}$$

- *The Krawtchouk case:* $\theta_i = i-\frac{d}{2}$, $\theta^*_i = i-\frac{d}{2}$,

$$\begin{aligned}\varphi_i &= v\, i\,(i-d-1),\\ \phi_i &= (v-1)\, i\,(i-d-1).\end{aligned}$$

- *The Bannai-Ito case:* $\theta_i = (-1)^i \left(i + u - \frac{d}{2}\right)$, $\theta^*_i = (-1)^i \left(i + u^* - \frac{d}{2}\right)$,

$$\varphi_i = \begin{cases} -i\left(i + u + u^* + v - \frac{d+1}{2}\right), & \text{for } i \text{ even, } d \text{ even.} \\ -(i-d-1)\left(i + u + u^* - v - \frac{d+1}{2}\right), & \text{for } i \text{ odd, } d \text{ even.} \\ -i\,(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\ v^2 - \left(i + u + u^* - \frac{d+1}{2}\right)^2, & \text{for } i \text{ odd, } d \text{ odd.} \end{cases}$$

$$\phi_i = \begin{cases} i\left(i - u + u^* - v - \frac{d+1}{2}\right), & \text{for } i \text{ even, } d \text{ even.} \\ (i-d-1)\left(i - u + u^* + v - \frac{d+1}{2}\right), & \text{for } i \text{ odd, } d \text{ even.} \\ -i\,(i-d-1), & \text{for } i \text{ even, } d \text{ odd.} \\ v^2 - \left(i - u + u^* - \frac{d+1}{2}\right)^2, & \text{for } i \text{ odd, } d \text{ odd.} \end{cases}$$

In each case, q, s, s^*, r are non-zero scalar parameters, or u, u^*, v are scalar parameters, such that $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta_j$ for $0 \le i < j \le d$, and $\varphi_i \neq 0$, $\phi_i \neq 0$ for $1 \le i \le d$.

Proof. The results are identical to Lemmas 6.1 and Lemmas 7.1 in [Vid05]. In particular, to get the normalized q-Racah parameter array from the general parameter array (6)–(9) one may replace $q \mapsto q^2$, $s \mapsto 1/s^2q^{d+1}$, $s^* \mapsto 1/s^{*2}q^{d+1}$, $r \mapsto r/ss^*q^{d+1}$ and adjust $\theta_0, \theta_0, h, h^*$ by affine scalings. □

Affine transformations (3) act on Askey-Wilson relations as well. They do not change the number of Leonard pairs with the same Askey-Wilson relations. Hence it is enough to consider our problem for a set of *normalized* Askey-Wilson relations. Possible normalizations are discussed in [Vid05, Sections 4 and 8]. Askey-Wilson relations satisfied by at least one Leonard pair can be normalized as follows.

Lemma 4.2 *Let $AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$ denote a pair of Askey-Wilson relations satisfied by a Leonard pair. The relations can be uniquely normalized by affine translation $(A, A^*) \mapsto (A + c,\ A^* + c^*)$ as follows:*

1. *If $\beta \neq 2$, we can set $\gamma = 0$, $\gamma^* = 0$.*
2. *If $\beta = 2$, $\gamma \neq 0$, $\gamma^* \neq 0$, we can set $\varrho = 0$, $\varrho^* = 0$.*
3. *If $\beta = 2$, $\gamma = 0$, $\gamma^* \neq 0$, we can set $\varrho^* = 0$, $\omega = 0$.*
4. *If $\beta = 2$, $\gamma^* = 0$, $\gamma \neq 0$, we can set $\varrho = 0$, $\omega = 0$.*
5. *If $\beta = 2$, $\gamma = 0$, $\gamma^* = 0$ we can set $\eta = 0$, $\eta^* = 0$.*

After the translation normalization, each of the two sequences

$$(\gamma,\ \varrho,\ \eta,\ \eta^*) \quad \text{and} \quad (\gamma^*, \varrho^*, \eta^*, \eta) \tag{26}$$

contains a non-zero Askey-Wilson coefficient. By affine scaling $(A, A^*) \mapsto (tA,\ t^*A^*)$ one can put the first non-zero coefficients in both sequences to any convenient non-zero values.

Proof. The normalization by affine translations follows from [Vid05, Lemma 4.1 and Part 3 of Theorem 8.1]. Note that parts 6 and 7 of [Vid05, Lemma 4.1] do not apply. Normalization by affine scaling follows from [Vid05, Lemma 5.2 (or Lemma 6.2) and Lemma 7.2]. □

The Askey-Wilson relations for the parameter arrays of Lemma 4.1 are normalized according to the specifications of Lemma 4.2. The following Lemma presents those Askey-Wilson relations. The first non-zero parameters in the two sequences (26) are normalized to the following values:

$$\begin{array}{ll} \lambda, \lambda^* : & 2 \text{ (if } \beta = 2); \\ \varrho, \varrho^* : & \begin{cases} 4-\beta^2, & \text{if } \beta \neq \pm 2, \\ 1, & \text{if } \beta = \pm 2; \end{cases} \\ \eta, \eta^* : & \begin{cases} \sqrt{\beta+2}\,(\beta-2), & \text{if } \eta\eta^* \neq 0 \text{ or } \omega = 0, \\ \sqrt{\beta+2}\,(\beta-2)\,Q_{d+1}, & \text{if } \eta\eta^* = 0 \text{ and } \omega \neq 0. \end{cases} \end{array} \tag{27}$$

We should identify $\sqrt{\beta+2} = q + 1/q$. This normalization of Askey-Wilson relations is not unique, and (in the q-cases) there may be two alternative normalizations with different signs of $\sqrt{\beta+2}$; see [Vid05, Section 9].

Lemma 4.3 *As in the previous Lemma, let q, s, s^*, r denote non-zero scalar parameters, and u, u^*, v denote scalar parameters. We use the following notations:*

$$Q_j = q^j + q^{-j}, \qquad Q^*_j = q^j - q^{-j}, \quad \text{for } j = 1, 2, \ldots, \tag{28}$$

$$S = s + \frac{1}{s}, \qquad S^* = s^* + \frac{1}{s^*}, \qquad R = r + \frac{1}{r}. \tag{29}$$

The Askey-Wilson relations for the parameter arrays of Lemma 4.1 are:

- *For the q-Racah case:*

$$AW\big(Q_2,\, 0,\, 0,\, -Q_2^{*2},\, -Q_2^{*2},\, -Q_1^{*2}(S\,S^* + Q_{d+1}R)\,,\; Q_1Q_1^{*2}(S\,R + Q_{d+1}S^*)\,,\, Q_1Q_1^{*2}(S^*R + Q_{d+1}S)\big). \tag{30}$$

- *For the q-Hahn case:*

$$AW\big(Q_2,\, 0,\, 0,\, 0,\, -Q_2^{*2},\, -Q_1^{*2}\big(S^*r + Q_{d+1}r^{-1}\big)\,,\; Q_1Q_1^{*2},\, Q_1Q_1^{*2}\big(S^*r^{-1} + Q_{d+1}r\big)\big). \tag{31}$$

- *For the dual q-Hahn case:*

$$AW\big(Q_2,\, 0,\, 0,\, -Q_2^{*2},\, 0,\, -Q_1^{*2}\big(S\,r + Q_{d+1}r^{-1}\big)\,,\; Q_1Q_1^{*2}\big(S\,r^{-1} + Q_{d+1}r\big)\,,\, Q_1Q_1^{*2}\big). \tag{32}$$

- *For the q-Krawtchouk case:*
$$AW\big(Q_2,\, 0,\, 0,\, 0,\, -Q_2^{*2},\, -Q_1^{*2}S^*,\, 0,\, Q_1Q_1^{*2}Q_{d+1}\big). \tag{33}$$
- *For the dual q-Krawtchouk case:*
$$AW\big(Q_2,\, 0,\, 0,\, -Q_2^{*2},\, 0,\, -Q_1^{*2}S,\, Q_1Q_1^{*2}Q_{d+1},\, 0\big). \tag{34}$$
- *For the quantum q-Krawtchouk and affine q-Krawtchouk cases:*
$$AW\big(Q_2,\, 0,\, 0,\, 0,\, 0,\, -Q_1^{*2}\big(r^2+Q_{d+1}r^{-1}\big)\,,\, Q_1Q_1^{*2},\, Q_1Q_1^{*2}\big). \tag{35}$$
- *For the Racah case:*
$$\begin{aligned} AW\big(&2,\, 2,\, 2,\, 0,\, 0, -2u^2-2u^{*2}-2v^2-2(d+1)(u+u^*+v)-2d^2-4d,\\ &2\,u\,(u+d+1)\,(v-u^*)\,(v+u^*+d+1)\,,\\ &2\,u^*(u^*+d+1)\,(v-u)\,(v+u+d+1)\,\big). \end{aligned} \tag{36}$$
- *For the Hahn case:*
$$AW\big(2,\, 0,\, 2,\, 1,\, 0,\, 0,\, -(u^*+1)(u^*+d)-2v^2-\tfrac{d^2}{2},\, -4u^*(u^*+d+1)v\big). \tag{37}$$
- *For the dual Hahn case:*
$$AW\big(2,\, 2,\, 0,\, 0,\, 1,\, 0,\,\, -4u(u+d+1)\,v,\, -(u+1)(u+d)-2v^2-\tfrac{d^2}{2}\big). \tag{38}$$
- *For the Krawtchouk case:*
$$AW\big(2,\, 0,\, 0,\, 1,\, 1,\, 2v-1,\, 0,\, 0\big). \tag{39}$$
- *For the Bannai-Ito case, if d is even:*
$$AW\big(-2,\, 0,\, 0,\, 1,\, 1,\, 4uu^*-2(d+1)\,v,\, 2uv-(d+1)\,u^*,\, 2u^*v-(d+1)\,u\big). \tag{40}$$
- *For the Bannai-Ito case, if d is odd:*
$$\begin{aligned} AW\big(&-2,\, 0,\, 0,\, 1,\, 1,\, -2u^2-2u^{*2}+2v^2+\tfrac{(d+1)^2}{2},\\ &-u^2+u^{*2}-v^2+\tfrac{(d+1)^2}{4},\, u^2-u^{*2}-v^2+\tfrac{(d+1)^2}{4}\big). \end{aligned} \tag{41}$$

Proof. The results are identical to Lemmas 6.2 and Lemmas 7.2 in [Vid05]. □

The Askey-Wilson type can be defined for Askey-Wilson relations so that type nominations for Leonard pairs and Askey-Wilson relations are consistent; see [Vid05, Section 8]. The classification of Askey-Wilson relations is largely recapitulated by the first and third columns of Table 1.

An important question for us is the following. If we take concrete Askey-Wilson relations normalized according to Lemma 4.2 and formulas (27), are all Leonard pairs satisfying them representable by parameter arrays of Lemma 4.1? The following Lemma settles this question.

Lemma 4.4 *1. All Leonard pairs satisfying normalized Askey-Wilson relations can be represented by parameter arrays of Lemma* (4.1)*, except when the Askey-Wilson type is Bannai-Ito, and d is odd.*

2. Suppose that d is odd. Let (B, B^*) denote the Leonard pair represented by the parameter array in Lemma 4.1 *of the Bannai-Ito type. Then a general normalized Leonard pair of the Bannai-Ito type (with odd d) has one of the following forms:*

$$(B, B^*), \quad (-B, B^*), \quad (B, -B^*), \quad (-B, -B^*). \tag{42}$$

Parameter arrays for these 4 *Leonard pairs cannot be transformed to each other by change of the parameters u, u^*, v or the relation operations $\downarrow$, $\Downarrow$, $\downarrow\Downarrow$.*

Proof. These are results of Lemmas 9.5 in [Vid05]. Notice that in the parameter array of Lemma (4.1) for the Bannai-Ito case, the even-indexed θ_i's and θ_i^*'s form increasing sequences; when d is odd, the relations operations $\Downarrow$, $\downarrow$ preserve this property. □

5 Correctness of Table 1

This Section gives a proof of Table 1. Besides, we give a few more examples of Askey-Wilson relations with all Leonard pairs that satisfy them.

Recall that we assume $d \geq 3$. By part 1 of [Vid05, Theorem 8.1], all Leonard pairs satisfy Askey-Wilson relations of the same Askey-Wilson type. Therefore we can consider Askey-Wilson relations of different types independently, and in each case look only for Leonard pairs of the same Askey-Wilson type.

As mentioned just before Lemma 4.2, it is enough to consider only normalized Askey-Wilson relations. By Lemma 4.4, all Leonard pairs satisfying normalized Askey-Wilson relations are representable by parameter arrays of Lemma 4.1, except when the Askey-Wilson type is Bannai-Ito and d is odd. In most cases, we just have to assume free values of non-normalized coefficients in the Askey-Wilson relations of Lemma 4.3, equate to those free values the coefficient expressions in free parameters (such as s, s^*, r or u, u^*, v) and count solutions of obtained algebraic equations.

If $\beta \neq \pm 2$, we have 4 possibilities for q. They are related by the substitutions $q \mapsto -q$, $q \mapsto 1/q$ and $q \mapsto -1/q$. We may consider q fixed, because Tables 3 and 4 in [Vid05] show the following. If a Leonard pair is represented by a q-parameter array of Lemma 4.1, then it can be represented by a parameter array of Lemma 4.1 with q replaced by $1/q$ as well. In some cases, the same holds for the substitution $q \mapsto -q$. In other cases, the substitution $q \mapsto -q$ leads to alternatively normalized Askey-Wilson relations (with the other sign of $\sqrt{\beta + 2}$). In any case, it is enough to count parameter arrays for one q-possibility.

We should take into account substitutions of the free parameters which preserve Leonard pairs; their either preserve parameter arrays or transform them to the $\Downarrow$-$\downarrow$-relatives. Discarding transformations which change q, these parameter substitutions are given in Table 2. The equations for free values of non-normalized Askey-Wilson

Askey-Wilson type	Parameter array stays invariant	Conversion to relatives	
		$\Downarrow$	$\downarrow$
q-Racah	$r \mapsto 1/r$	$s \mapsto 1/s$	$s^* \mapsto 1/s^*$
q-Hahn	—	—	$s^* \mapsto 1/s^*$
Dual q-Hahn	—	$s \mapsto 1/s$	—
q-Krawtchouk	—	—	$s^* \mapsto 1/s^*$
Dual q-Krawtchouk	—	$s \mapsto 1/s$	—
Racah	$v \mapsto -v-d-1$	$u \mapsto -u-d-1$	$u^* \mapsto -u^*-d-1$
Hahn	—	—	$u^* \mapsto -u^*-d-1$
Dual Hahn	—	$u \mapsto -u-d-1$	—
Bannai-Ito, d odd	$v \mapsto -v$	$u \mapsto -u$	$u^* \mapsto -u^*$

Table 2: Reparametrizations preserving Leonard pairs

coefficients (in normalized Askey-Wilson relations) should be rewritten in invariants of the transformations preserving Leonard pairs. Examples of these invariants (for some cases) are S, S^*, R as in (29).

In each Askey-Wilson case we ought to check whether solutions are generally non-degenerate. For this, one can check generic irreducibility (over the ring generated by free parameters) of the equation systems, or check that degenerate solutions form lower-dimensional subvarieties. For fixed $q \neq \pm 2$, generically degenerate Leonard pairs occur only if $q^{2j} = 1$ for some $j \in \{1, 2, \ldots, d\}$, or equivalently, $\beta = 2\cos \pi/j$.

Now we consider all normalized Askey-Wilson relations case by case. We use the notation of Lemma 4.3, and also

$$U = \left(u + \tfrac{d+1}{2}\right)^2, \qquad U^* = \left(u^* + \tfrac{d+1}{2}\right)^2, \qquad V = \left(v + \tfrac{d+1}{2}\right)^2. \tag{43}$$

In the q-Racah case, we introduce the following indeterminants:

$$x = \frac{S}{Q}, \qquad y = \frac{S^*}{Q}, \qquad z = \frac{R}{Q}. \tag{44}$$

They are invariant under the relevant transformations of Table 2. Equating the non-normalized Askey-Wilson parameters gives the equations

$$\begin{aligned} xy + z &= C_1, \\ xz + y &= C_2, \\ yz + x &= C_3, \end{aligned} \tag{45}$$

where

$$C_1 = -\frac{\omega}{Q^2\, K^*}, \quad C_2 = \frac{\eta}{(q+1)\, Q^2\, K^*}, \quad C_3 = \frac{\eta^*}{(q+1)\, Q^2\, K^*}.$$

Elimination of y, z from (45) gives the degree 5 equation

$$(x - C_3)(x^2 - 1)^2 + C_1C_2\,(x^2 - 1) - (C_1^2 + C_2^2)\,x = 0. \tag{46}$$

Each solution gives exactly one Leonard pair satisfying $AW\left(q + q^{-1}, 0, 0, K, K, \omega, \eta, \eta^*\right)$. There are more solutions in terms of (s, s^*, r), but distinct Leonard pairs come from distinct (x, y, z). Polynomial (46) in x does not have multiple roots in general. Hence the general number of Leonard pairs is 5.

In the q-Hahn case, invariant variables are S^*, r, and free Askey-Wilson coefficients are ω, η^*. Elimination of S^* gives a polynomial of degree 4 in r, without multiple roots in general. The general number of Leonard pairs is 4. The dual q-Hahn case is similar.

In the q-Krawtchouk case, we have the equation $\omega = -K^*S^*$ which obviously has one solution in S^*. The dual q-Krawtchouk case is similar.

Suppose not that $\beta = q + q^{-1}$ with $q \neq \pm 1$, and that after the normalization $\gamma = 0$, $\gamma^* = 0$ we have $\varrho = 0$, $\varrho^* = 0$, $\eta\eta^* \neq 0$. We can rescale so that $\eta = (q{+}1)K^*$ and $\eta^* = (q{+}1)K^*$. Corresponding parameter arrays can be of the quantum q-Krawtchouk or the affine q-Krawtchouk types. Parameter arrays of these two types are related by the $\Downarrow$ operation, so corresponding Leonard pairs can be represented by parameter arras of either type. Whatever type we choose, we get 3 solutions in general.

In the Racah case, we use (43) and rewrite the equations as

$$\begin{aligned}
\omega &= -2\,U - 2\,U^* - 2\,V - \tfrac{(d-1)(d+3)}{2},\\
\eta &= 2\left(U - \tfrac{(d+1)^2}{4}\right)(V - U^*)\,,\\
\eta^* &= 2\left(U^* - \tfrac{(d+1)^2}{4}\right)(V - U)\,.
\end{aligned} \tag{47}$$

Here U, U^*, V are invariants by Table 2. Elimination of two invariants confirms that the general number of solutions is $4 = 1 \cdot 2 \cdot 2$.

In the Hahn case, we have the equations

$$\begin{aligned}
\eta &= -U^* - 2v^2 - \tfrac{d^2+2d-1}{4},\\
\eta^* &= -4v\left(U^* - \tfrac{(d+1)^2}{4}\right).
\end{aligned} \tag{48}$$

The invariants are U^* and v. Elimination of U^* gives a cubic equation in v:

$$v^3 + \left(\tfrac{\eta}{2} + \tfrac{d(d+2)}{4}\right) v - \eta^* = 0.$$

Hence there are 3 Leonard pairs in general. The dual Hahn case is similar.

In the Krawtchouk case, we obviously have one solution.

In the Bannai-Ito case for even d, after setting

$$x = \frac{2u}{d+1}, \qquad y = \frac{2u^*}{d+1}, \qquad z = \frac{2v}{d+1}, \tag{49}$$

we arrive at an equation system of the same form as in (45), so the general number of solutions is 5 as well.

In the Bannai-Ito case for odd d we have to keep in mind part 2 of Lemma 4.4. A Leonard pair is determined by values of the invariants u^2, u^{*2}, v^2, see Table 2. The corresponding expressions for ω, η, η^* in Lemma 4.3 are linear in these invariants, so there is only one solution representable by a parameter array of Lemma 4.1. Part 2 of Lemma 4.4 implies that there are 4 Leonard pairs in total.

Correctness of Table 1 is proved. $\square$

Example 5.1 The example in Section 2 gives the 5 Leonard pairs satisfying normalized Askey-Wilson relations of the q-Racah case for $\omega = 0$, $\eta = 0$, $\eta^* = 0$. Then the system (45) has the following solutions:

$$(x, y, z) \in \{(0,0,0), (1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}.$$

These solutions correspond to the Leonard pairs in (12), respectively. We should replace q by q^2 in Section 2 to get the form of Lemma 4.1.

For normalized Askey-Wilson relations of the Bannai-Ito type for even d, with $\omega = 0$, $\eta = 0$, $\eta^* = 0$, we have the same solutions in terms of (49).

For the same Askey-Wilson relations of the Bannai-Ito type for odd d, one solution is given by the corresponding parameter array of Lemma 4.1 with

$$u = \frac{d+1}{2}, \qquad u^* = \frac{d+1}{2}, \qquad v = \frac{d+1}{2}.$$

The other 3 solutions can be obtained by multiplying one or both components of the Leonard pair by -1.

Example 5.2 The 4 Leonard pairs satisfying normalized Askey-Wilson relations of the Racah type with $\omega = 0$, $\eta = 0$, $\eta^* = 0$ are defined by:

$$\begin{aligned}(U, U^*, V) \quad \in \quad & \Big\{ \Big(\tfrac{(d+1)^2}{4}, \tfrac{(d+1)^2}{4}, \tfrac{1-6d-3d^2}{4} \Big), \\ & \Big(\tfrac{(d+1)^2}{4}, \tfrac{1-6d-3d^2}{4}, \tfrac{(d+1)^2}{4} \Big), \\ & \Big(\tfrac{1-6d-3d^2}{4}, \tfrac{(d+1)^2}{4}, \tfrac{(d+1)^2}{4} \Big), \\ & \Big(-\tfrac{(d-1)(d+3)}{12}, -\tfrac{(d-1)(d+3)}{12}, -\tfrac{(d-1)(d+3)}{12} \Big) \Big\}.\end{aligned}$$

Diagonal-tridiagonal matrix entries for these normalized Leonard pairs are not rational numbers.

Example 5.3 Here we consider Askey-Wilson relations of the Hahn type with $\eta^* = 0$. There must be solutions with $v = 0$ and with $U^* = \frac{(d+1)^2}{4}$. We want all entries in the representing matrices to be in $\mathbb{Q}$, so we must have $\big(u^* + \frac{d+1}{2}\big)^2 - 2v^2 = \frac{(d+1)^2}{4}$. Rational solutions of this equation can be parametrized with $v = \frac{(d+1)t}{t^2-2}$, which gives the following family of Askey-Wilson relations:

$$AW\left(2, 0, 2, 1, 0, 0, \frac{1}{2} - \frac{(d+1)^2(t^4+4)}{2\,(t^2-2)^2}, 0\right).$$

The 3 Leonard pairs can represented by parameter arrays of Lemma 4.1 with

$$(u^*, v) \in \left\{ \left(\frac{2\,(d+1)}{t^2-2}, 0 \right), \left(0, \frac{(d+1)\,t}{t^2-2} \right), \left(0, -\frac{(d+1)\,t}{t^2-2} \right) \right\}. \tag{50}$$

For $t = 1$ we have 3 solutions, as expected. They are representable (after the $\downarrow$ operation) by

$$(u^*, v) \in \{(d+1, 0), (0, d+1), (0, -d-1)\}.$$

For $t = 3$, we have

$$u^* \pm 2v \in \left\{ \frac{2(d{+}1)}{7}, -\frac{9(d{+}1)}{7}, \frac{6(d{+}1)}{7}, -\frac{13(d{+}1)}{7}, -\frac{6(d{+}1)}{7}, -\frac{(d{+}1)}{7} \right\}.$$

If $d + 1$ is divisible by 7, then we have only one Leonard pair solution, because two other solutions have $\varphi_i \phi_i = 0$ for $i = \frac{(d+1)}{7}$ or $i = \frac{6(d+1)}{7}$ so they are degenerate. A similar statement holds for $t = 4$.

For $t = 0$, all three solutions in (50) give the Leonard pair representable by the parameter array of Lemma 4.1 with $(u^*, v) = (0, 0)$. So we have just one solution of "multiplicity 3".

Example 5.4 Suppose that $\xi \in \mathbb{C}$ satisfies $\xi^{d+1} = 2$, and consider the Askey-Wilson relations

$$AW \left(\xi^2 + \xi^{-2}, 0, 0, 0, 0, -\frac{21(\xi - \xi^{-1})^2}{4}, (\xi + \xi^{-1})(\xi - \xi^{-1})^2, (\xi + \xi^{-1})(\xi - \xi^{-1})^2 \right).$$

Leonard pairs satisfying these relations can be represented by parameter arrays of the quantum q-Krawtchouk of the affine q-Krawtchouk types. There are 3 such Leonard pairs. To get affine q-Krawtchouk parameter arrays, one may take $q = \xi$ so that $Q_{d+1} = \frac{5}{2}$. The cubic equation is then $r^3 + \frac{5}{2} = \frac{21}{4} r$. The solutions have $r \in \{2, \frac{1}{2}, -\frac{5}{2}\}$.

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