The subconstituent algebra of a bipartite distance-regular graph; thin modules with endpoint two

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Abstract

We consider a bipartite distance-regular graph Γ with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers b_i, c_i, distance matrices A_i, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let X denote the vertex set of Γ and fix $x \in X$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where $A = A_1$ and E_i^* denotes the projection onto the i^{th} subconstituent of Γ with respect to x. T is called the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x. An irreducible T-module W is said to be *thin* whenever $\dim E_i^* W \leq 1$ for $0 \leq i \leq D$. By the *endpoint* of W we mean $\min\{i | E_i^* W \neq 0\}$. Assume W is thin with endpoint 2. Observe $E_2^* W$ is a 1-dimensional eigenspace for $E_2^* A_2 E_2^*$; let η denote the corresponding eigenvalue. It is known $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_d$ where $\tilde{\theta}_1 = -1 - b_2 b_3 (\theta_1^2 - b_2)^{-1}$, $\tilde{\theta}_d = -1 - b_2 b_3 (\theta_d^2 - b_2)^{-1}$, and $d = \lfloor D/2 \rfloor$. To describe the structure of W we distinguish four cases: (i) $\eta = \tilde{\theta}_1$; (ii) D is odd and $\eta = \tilde{\theta}_d$; (iii) D is even and $\eta = \tilde{\theta}_d$; (iv) $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. We investigated cases (i), (ii) in [28]. Here we investigate cases (iii), (iv) and obtain the following results. We show the dimension of W is $D - 1 - e$ where $e = 1$ in case (iii) and $e = 0$ in case (iv). Let v denote a nonzero vector in $E_2^* W$. We show W has a basis $E_i v$ ($i \in S$), where E_i denotes the primitive idempotent of A associated with θ_i and where the set S is $\{1, 2, \ldots, d-1\} \cup \{d+1, d+2, \ldots, D-1\}$ in case (iii) and $\{1, 2, \ldots, D-1\}$ in case (iv). We show this basis is orthogonal (with respect to the Hermitian dot product) and we compute the square-norm of each basis vector. We show W has a basis $E_{i+2}^* A_i v$ ($0 \leq i \leq D-2-e$), and we find the matrix representing A with respect to this basis. We show this basis is orthogonal and we compute the square-norm of each basis vector. We find the transition matrix relating our two bases for W.

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1 Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers a_i, b_i, c_i, and distance matrices A_i (see Section 2 for formal definitions). We recall the subconstituent algebra of Γ. Let X denote the vertex set of Γ and fix $x \in X$. We view x as a "base vertex." Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, E_0^*, E_1^*, \ldots, E_D^*$, where $A = A_1$ and E_i^* represents the projection onto the i^{th} subconstituent of Γ with respect to x. The algebra T is called the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [31]. Observe T has finite dimension. Moreover T is semi-simple; the reason is each of $A, E_0^*, E_1^*, \ldots, E_D^*$ is symmetric with real entries, so T is closed under the conjugate-transpose map [18, p. 157]. Since T is semi-simple, each T-module is a direct sum of irreducible T-modules. Describing the irreducible T-modules is an active area of research [4]–[17], [19]–[24], [26], [28]–[36].

In this paper we are concerned with the irreducible T-modules that possess a certain property. In order to define this property we make a few observations. Let W denote an irreducible T-module. Then W is the direct sum of the nonzero spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. There is a second decomposition of interest. To obtain it we make a definition. Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of A, and for

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$0 \leq i \leq D$ let E_i denote the primitive idempotent of A associated with θ_i. Then W is the direct sum of the nonzero spaces among $E_0W, E_1W, \ldots, E_DW$. If the dimension of E_i^*W is at most 1 for $0 \leq i \leq D$ then the dimension of E_iW is at most 1 for $0 \leq i \leq D$ [31, Lemma 3.9]; in this case we say W is *thin*. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i | 0 \leq i \leq D,\ E_i^*W \neq 0\}$. There exists a unique irreducible T-module with endpoint 0 [21, Proposition 8.4]. We call this module V_0. The module V_0 is thin; in fact $E_i^*V_0$ and E_iV_0 have dimension 1 for $0 \leq i \leq D$ [31, Lemma 3.6]. For a detailed description of V_0 see [9], [21].

For the rest of this section assume Γ is bipartite. There exists, up to isomorphism, a unique irreducible T-module with endpoint 1 [9, Corollary 7.7]. We call this module V_1. The module V_1 is thin; in fact each of $E_i^*V_1$, E_iV_1 has dimension 1 for $1 \leq i \leq D-1$ and $E_D^*V_1 = 0$, $E_0V_1 = 0$, $E_DV_1 = 0$. For a detailed description of V_1 see [9]. In this paper we are concerned with the thin irreducible T-modules with endpoint 2.

In order to describe the thin irreducible T-modules with endpoint 2 we define some parameters. Let $\Gamma_2^2 = \Gamma_2^2(x)$ denote the graph with vertex set $\breve{X}$ and edge set $\breve{R}$, where

$$\begin{aligned} \breve{X} &= \{y \in X \mid \partial(x,y) = 2\}, \\ \breve{R} &= \{yz \mid y, z \in \breve{X},\ \partial(y,z) = 2\}, \end{aligned}$$

and where ∂ is the path-length distance function for Γ. The graph Γ_2^2 has exactly k_2 vertices, where k_2 is the second valency of Γ. Also, Γ_2^2 is regular with valency p_{22}^2. We let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of the adjacency matrix of Γ_2^2. By [10, Theorem 11.7], these eigenvalues may be ordered such that $\eta_1 = p_{22}^2$ and $\eta_i = b_3 - 1$ $(2 \leq i \leq k)$.

Abbreviate $d = \lfloor D/2 \rfloor$. It is shown in [28, Theorem 11.4] that $\tilde{\theta}_1 \leq \eta_i \leq \tilde{\theta}_d$ for $k+1 \leq i \leq k_2$, where $\tilde{\theta}_1 = -1 - b_2b_3(\theta_1^2 - b_2)^{-1}$ and $\tilde{\theta}_d = -1 - b_2b_3(\theta_d^2 - b_2)^{-1}$. We remark $\theta_1^2 > b_2 > \theta_d^2$ by [27, Lemma 2.6], so $\tilde{\theta}_1 < -1$ and $\tilde{\theta}_d \geq 0$.

Let W denote a thin irreducible T-module with endpoint 2. Observe E_2^*W is a 1-dimensional eigenspace for $E_2^*A_2E_2^*$; let η denote the corresponding eigenvalue. It turns out η is among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$ so $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_d$. We call η the *local eigenvalue* of W. To describe the structure of W we distinguish four cases: (i) $\eta = \tilde{\theta}_1$; (ii) D is odd and $\eta = \tilde{\theta}_d$; (iii) D is even and $\eta = \tilde{\theta}_d$; (iv) $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. In [28] we investigated cases (i), (ii). In the present paper we investigate cases (iii), (iv).

Concerning cases (i), (ii) our results from [28] are summarized as follows. Choose $n \in \{1, d\}$ if D is odd, and let $n = 1$ if D is even. Define $\eta = \tilde{\theta}_n$. Let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η. Then W has dimension $D-3$. Let v denote a nonzero vector in E_2^*W. We showed W has a basis E_iv $(1 \leq i \leq D-1,\ i \neq n,\ i \neq D-n)$. We showed this basis is orthogonal (with respect to the Hermitian dot product) and we computed the square-norm of each basis vector. We showed W has a basis $E_{i+2}^*A_iv$ $(0 \leq i \leq D-4)$. We found the matrix representing A with respect to this basis. We showed this basis is orthogonal and we computed the square-norm of each basis vector. We found the transition matrix relating our two bases for W. We showed the following scalars are equal: (i) The multiplicity with which W appears in the standard module $\mathbb{C}^X$; (ii) The number of times η appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$.

Concerning case (iii) above, in the present paper we obtain the following results. Assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. We show the dimension of W is $D-2$. Let v denote a nonzero vector in E_2^*W. We show W has a basis E_iv $(1 \leq i \leq D-1,\ i \neq d)$. We show this basis is orthogonal and we compute the square-norm of each basis vector. We show W has a basis $E_{i+2}^*A_iv$ $(0 \leq i \leq D-3)$. We find the matrix representing A with respect to this basis. We show this basis is orthogonal and we compute the square-norm of each basis vector. We find the transition matrix relating our two bases for W.

Concerning case (iv) above, in the present paper we obtain the following results. Let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η $(\tilde{\theta}_1 < \eta < \tilde{\theta}_d)$. We show the dimension of

W is $D-1$. Let v denote a nonzero vector in E^*_2W. We show W has a basis E_iv $(1 \leq i \leq D-1)$. We show this basis is orthogonal and we compute the square-norm of each basis vector. We show W has a basis $E^*_{i+2}A_iv$ $(0 \leq i \leq D-2)$. We find the matrix representing A with respect to this basis. We show this basis is orthogonal and we compute the square-norm of each basis vector. We find the transition matrix relating our two bases for W.

For all $\eta \in \mathbb{R}$ let μ_η denote the multiplicity with which W appears in $\mathbb{C}^X$, where W is a thin irreducible T-module with endpoint 2 and local eigenvalue η. If no such W exists we interpret $\mu_\eta = 0$. We show μ_η is at most the number of times η appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. Concerning the case of equality, we show the following are equivalent: (i) For all $\eta \in \mathbb{R}$, μ_η is equal to the number of times η appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$; (ii) Every irreducible T-module with endpoint 2 is thin.

2 Preliminaries concerning distance-regular graphs

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1], [3], [25] or [31].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product $\langle\, ,\, \rangle$ which satisfies $\langle u, v\rangle = u^t\overline{v}$ for all $u, v \in V$, where t denotes transpose and $-$ denotes complex conjugation. We abbreviate $\|u\|^2 = \langle u, u\rangle$ for all $u \in V$. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. The following formula will be useful. For all $B \in \mathrm{Mat}_X(\mathbb{C})$ and for all $u, v \in V$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle. \tag{1}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D = \max\{\partial(x, y) \mid x, y \in X\}$. We refer to D as the *diameter* of Γ. Let $\lfloor D/2\rfloor$ denote the greatest integer at most $D/2$. Vertices $x, y \in X$ are called *adjacent* whenever xy is an edge. For an integer $k \geq 0$, we say Γ is *regular* with *valency* k whenever each vertex of Γ is adjacent to exactly k distinct vertices of Γ. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}| \tag{2}$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i = p^i_{1i-1}$ $(1 \leq i \leq D)$, $a_i = p^i_{1i}$ $(0 \leq i \leq D)$, and $b_i = p^i_{1i+1}$ $(0 \leq i \leq D-1)$. For notational convenience, we define $c_0 = 0$ and $b_D = 0$. We note $a_0 = 0$ and $c_1 = 1$.

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 3$.

By (2) and the triangle inequality,

$$p^h_{1j} = 0 \qquad \text{if} \qquad |h-j| > 1 \qquad\qquad (0 \leq h, j \leq D). \tag{3}$$

Observe Γ is regular with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for $0 \leq i \leq D$. Moreover $b_i > 0$ $(0 \leq i \leq D-1)$ and $c_i > 0$ $(1 \leq i \leq D)$. For $0 \leq i \leq D$ we abbreviate $k_i = p^0_{ii}$, and observe

$$k_i = |\{z \in X \mid \partial(x, z) = i\}|, \tag{4}$$

where x is any vertex in X. Apparently $k_0 = 1$ and $k_1 = k$. By [1, p.195] we have

$$k_i = \frac{b_0b_1\cdots b_{i-1}}{c_1c_2\cdots c_i} \qquad\qquad (0 \leq i \leq D). \tag{5}$$

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the i^{th} *distance matrix* of Γ. For convenience we define $A_i = 0$ for $i < 0$ and $i > D$. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{D} A_i = J$; (aiii) $\overline{A}_i = A_i$ $(0 \leq i \leq D)$; (aiv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (av) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \leq i, j \leq D)$, where I denotes the identity matrix and J denotes the all 1's matrix. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. Using (ai), (av) one can readily show $A_0, A_1, \ldots, A_D$ form a basis for M. We refer to M as the *Bose-Mesner algebra* of Γ. By [3, p.45] M has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1}J$; (eii) $\sum_{i=0}^{D} E_i = I$; (eiii) $\overline{E}_i = E_i$ $(0 \leq i \leq D)$; (eiv) $E_i^t = E_i$ $(0 \leq i \leq D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$. We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of Γ. We call E_0 the *trivial idempotent* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Combining this with (ev) we find $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. Using (aiii) and (eiii) we find $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since A generates M. By [2, Proposition 3.1] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$. Throughout this paper we assume $E_0, E_1, \ldots, E_D$ are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_D$. We refer to θ_i as the *eigenvalue* of Γ associated with E_i. We call θ_0 the *trivial eigenvalue* of Γ. For $0 \leq i \leq D$ let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i). From (ei) we find $m_0 = 1$. Using (eii)–(ev) we find

$$V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum).} \tag{6}$$

For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of A associated with θ_i. We observe the dimension of $E_i V$ is m_i. We now record a fact about the eigenvalues θ_1 and θ_D.

Lemma 2.1 *[27, Lemma 2.6] Let Γ denote a distance-regular graph with diameter $D \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Then (i) $-1 < \theta_1 < k$; (ii) $a_1 - k \leq \theta_D < -1$.*

Later in this paper we will discuss polynomials in one or two variables. We will use the following notation. Let λ denote an indeterminate. Let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{R}$. Let μ denote an indeterminate which commutes with λ. Let $\mathbb{R}[\lambda, \mu]$ denote the $\mathbb{R}$-algebra consisting of all polynomials in λ and μ that have coefficients in $\mathbb{R}$.

3 Bipartite distance-regular graphs

We now consider the case in which Γ is bipartite. We say Γ is *bipartite* whenever the vertex set X can be partitioned into two subsets, neither of which contains an edge. In the next few lemmas, we recall some routine facts concerning the case in which Γ is bipartite. To avoid trivialities, we will generally assume $D \geq 4$.

Lemma 3.1 *[3, Propositions 3.2.3, 4.2.2] Let Γ denote a distance-regular graph with diameter $D \geq 4$, valency k, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. The following are equivalent:*

(i) *Γ is bipartite.*

(ii) *$p_{ij}^h = 0$ if $h + i + j$ is odd* $\quad (0 \leq h, i, j \leq D)$.

(iii) *$a_i = 0$* $\quad (0 \leq i \leq D)$.

(iv) *$c_i + b_i = k$* $\quad (0 \leq i \leq D)$.

(v) *$\theta_{D-i} = -\theta_i$* $\quad (0 \leq i \leq D)$.

Lemma 3.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$.*

(i) *Assume D is even and let $d = D/2$. Then $\theta_d = 0$.*

(ii) *Assume D is odd and let $d = (D-1)/2$. Then $\theta_d > 0$ and $\theta_{d+1} = -\theta_d$.*

Proof. Immediate from Lemma 3.1(v). $\square$

Lemma 3.3 *[28, Lemma 3.4] Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Then $E_D = |X|^{-1}J'$, where*

$$J' = \sum_{i=0}^{D} (-1)^i A_i. \tag{7}$$

Lemma 3.4 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Then $\theta_1^2 > b_2 > \theta_d^2$, where $d = \lfloor D/2 \rfloor$.*

Proof. Apply Lemma 2.1 to the halved graph of Γ, and use [3, Proposition 4.2.3]. $\square$

4 Two families of polynomials

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$. In this section we recall two types of polynomials associated with Γ. To motivate things, we recall by (av) and the triangle inequality that

$$AA_i = b_{i-1}A_{i-1} + c_{i+1}A_{i+1} \qquad (0 \leq i \leq D), \tag{8}$$

where $b_{-1} = 0$ and $c_{D+1} = 0$. Let $f_0, f_1, \ldots, f_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying $f_0 = 1$ and

$$\lambda f_i = b_{i-1}f_{i-1} + c_{i+1}f_{i+1} \qquad (0 \leq i \leq D-1), \tag{9}$$

where $f_{-1} = 0$. For $0 \leq i \leq D$ the polynomial f_i has degree i, and the coefficient of λ^i is $(c_1 c_2 \cdots c_i)^{-1}$. Comparing (8) and (9) we find $f_i(A) = A_i$. By [1, p. 63] the polynomials $f_0, f_1, \ldots, f_D$ satisfy the orthogonality relation

$$\sum_{h=0}^{D} f_i(\theta_h) f_j(\theta_h) m_h = \delta_{ij}|X|k_i \qquad (0 \leq i, j \leq D).$$

We now recall some polynomials related to the f_i. Let $p_0, p_1, \ldots, p_D$ denote the polynomials in $\mathbb{R}[\lambda]$ satisfying

$$p_i = \begin{cases} f_0 + f_2 + f_4 + \cdots + f_i, & \text{if } i \text{ is even} \\ f_1 + f_3 + f_5 + \cdots + f_i, & \text{if } i \text{ is odd} \end{cases} \qquad (0 \leq i \leq D). \tag{10}$$

Observe $p_0 = 1$. For $0 \leq i \leq D$ the polynomial p_i has degree i, and the coefficient of λ^i is $(c_1 c_2 \cdots c_i)^{-1}$. Recalling $f_j(A) = A_j$ $(0 \leq j \leq D)$, we observe

$$p_D(A) + p_{D-1}(A) = J, \qquad p_D(A) - p_{D-1}(A) = (-1)^D J', \tag{11}$$

where J' is from (7). By [28, Theorem 4.2], we have

$$\lambda p_i = c_{i+1}p_{i+1} + b_{i+1}p_{i-1} \qquad (0 \leq i \leq D-1), \tag{12}$$

where $p_{-1} = 0$. We record a fact for later use.

Lemma 4.1 *[28, Lemma 4.3] Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $p_0, p_1, \ldots, p_D$ be as in (10). Then $p_{D-1}(\theta_h) = 0$ and $p_D(\theta_h) = 0$ for $1 \leq h \leq D-1$. Moreover,*

$$\sum_{h=0}^{D} p_i(\theta_h) p_j(\theta_h)(k^2 - \theta_h^2) m_h = \delta_{ij}|X|k_i b_i b_{i+1} \qquad (0 \leq i, j \leq D-2). \tag{13}$$

5 The polynomials Ψ_i

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. In the previous section we used Γ to define two families of polynomials in one variable. We called these polynomials the f_i and the p_i. Later in this paper we will use Γ to define a third family of polynomials in one variable. We will call these polynomials the g_i. To define and study the g_i it is convenient to first consider some polynomials Ψ_i in two variables.

Definition 5.1 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. For $0 \leq i \leq D-2$ let Ψ_i denote the polynomial in $\mathbb{R}[\lambda, \mu]$ given by

$$\Psi_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} p_h(\lambda) p_h(\mu) \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}}, \tag{14}$$

where the polynomials $p_0, p_1, \ldots, p_{D-2}$ are from (10). We observe $\Psi_0 = 1$ and $\Psi_1 = \lambda\mu$.

Lemma 5.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials p_i, Ψ_i be as in (10), (14), respectively. Then*

$$p_i(\lambda) p_i(\mu) = \Psi_i - \frac{b_i b_{i+1}}{c_i c_{i-1}} \Psi_{i-2} \qquad (2 \leq i \leq D-2).$$

Proof. Use Definition 5.1 and (5). $\square$

The following equation is a variation of the Christoffel-Darboux formula.

Lemma 5.3 *[28, Lemma 5.3] Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials p_i, Ψ_i be as in (10), (14) respectively. Then for $1 \leq i \leq D-1$,*

$$p_{i+1}(\lambda) p_{i-1}(\mu) - p_{i-1}(\lambda) p_{i+1}(\mu) = c_i^{-1} c_{i+1}^{-1} (\lambda^2 - \mu^2) \Psi_{i-1}.$$

Lemma 5.4 *[28, Lemma 5.4] Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials p_i, Ψ_i be as in (10), (14) respectively. Then for $0 \leq i, j \leq D-2$,*

$$\sum_{h=0}^{D} \Psi_i(\theta_h, \mu) \Psi_j(\theta_h, \mu)(k^2 - \theta_h^2)(\mu^2 - \theta_h^2) m_h \;=\; \delta_{ij} |X| p_i(\mu) p_{i+2}(\mu) k_i b_i b_{i+1} c_{i+1} c_{i+2}.$$

(We recall m_h denotes the multiplicity of θ_h for $0 \leq h \leq D$.)

Lemma 5.5 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials p_i be as in (10). Then the following (i), (ii) hold for all $\theta \in \mathbb{R}$:*

(i) *Suppose $\theta = \theta_1$. Then $p_i(\theta) > 0$ for $0 \leq i \leq D-2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.*

(ii) *Suppose $\theta > \theta_1$. Then $p_i(\theta) > 0$ for $0 \leq i \leq D$.*

Proof. Observe $p_{D-1}(\theta_1) = 0$, $p_D(\theta_1) = 0$ by Lemma 4.1. For notational convenience set $e = 0$ if $\theta > \theta_1$ and $e = 1$ if $\theta = \theta_1$. Suppose there exists an integer i $(0 \leq i \leq D-2e)$ such that $p_i(\theta) \leq 0$. Let us pick the minimal such i. Observe $i \geq 2$ since $p_0(\theta) = 1$, $p_1(\theta) = \theta$. Apparently $p_{i-2}(\theta) > 0$. We claim there exists an integer h $(1 + e \leq h \leq D - 1 - e)$ such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. To see this, observe by Definition 5.1 that $\Psi_{i-2}(\lambda, \theta)$ is a polynomial in λ with degree $i-2$. In this polynomial the coefficient of λ^{i-2} is $p_{i-2}(\theta)(c_1 c_2 \cdots c_{i-2})^{-1}$. Apparently this polynomial is not identically 0 so there exist at most $i-2$ integers h $(1 + e \leq h \leq D - 1 - e)$ such that $\Psi_{i-2}(\theta_h, \theta) = 0$. By this and since $i \leq D - 2e$, there exists at least one integer h $(1+e \leq h \leq D-1-e)$ such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. We have now proved our claim. We may now argue

$$\begin{aligned}
0 &< \sum_{h=1+e}^{D-1-e} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2) m_h \\
&= \sum_{h=0}^{D} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2) m_h \qquad \text{(by the definition of } e) \\
&= |X| p_{i-2}(\theta) p_i(\theta) k_{i-2} b_{i-2} b_{i-1} c_{i-1} c_i \qquad \text{(by Lemma 5.4)} \\
&\leq 0.
\end{aligned}$$

We now have a contradiction and the result follows. □

Lemma 5.6 *Let Γ denote a bipartite distance-regular graph with odd diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let d denote the integer satisfying $2d+1 = D$. Let the polynomials p_i be as in (10). Then the following (i), (ii) hold for all $\theta \in \mathbb{R}$:*

(i) *Suppose $\theta = \theta_d$. Then $(-1)^{\lfloor \frac{i}{2} \rfloor} p_i(\theta) > 0$ for $0 \leq i \leq D-2$, and $p_{D-1}(\theta) = 0$, $p_D(\theta) = 0$.*

(ii) *Suppose $0 < \theta < \theta_d$. Then $(-1)^{\lfloor \frac{i}{2} \rfloor} p_i(\theta) > 0$ for $0 \leq i \leq D$.*

Proof. Observe $p_{D-1}(\theta_d) = 0$, $p_D(\theta_d) = 0$ by Lemma 4.1. For notational convenience set $e = 0$ if $0 < \theta < \theta_d$ and $e = 1$ if $\theta = \theta_d$. Also for notational convenience we define the set S to be $\{1, 2, \ldots, D-1\}$ if $e = 0$, and $\{1, 2, \ldots, d-1\} \cup \{d+2, d+3, \ldots, D-1\}$ if $e = 1$. Suppose there exists an integer i $(0 \leq i \leq D-2e)$ such that $(-1)^{\lfloor \frac{i}{2} \rfloor} p_i(\theta) \leq 0$. Let us pick the minimal such i. Observe $i \geq 2$ since $p_0(\theta) = 1$, $p_1(\theta) = \theta$. Apparently $(-1)^{\lfloor \frac{i-2}{2} \rfloor} p_{i-2}(\theta) > 0$, so $p_{i-2}(\theta) p_i(\theta) \geq 0$. We claim there exists an integer $h \in S$ such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. To see this, observe by Definition 5.1 that $\Psi_{i-2}(\lambda, \theta)$ is a polynomial in λ with degree $i-2$. This polynomial is not identically zero, since the coefficient of λ^{i-2} is $p_{i-2}(\theta)(c_1 c_2 \cdots c_{i-2})^{-1}$ and since $p_{i-2}(\theta) \neq 0$ by construction. Therefore there exist at most $i-2$ integers $h \in S$ such that $\Psi_{i-2}(\theta_h, \theta) = 0$. By this and since $i \leq D - 2e$, there exists at least one integer $h \in S$ such that $\Psi_{i-2}(\theta_h, \theta) \neq 0$. We have now proved our claim. We may now argue

$$\begin{aligned}
0 &> \sum_{h \in S} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2) m_h \\
&= \sum_{h=0}^{D} \Psi_{i-2}^2(\theta_h, \theta)(k^2 - \theta_h^2)(\theta^2 - \theta_h^2) m_h \qquad \text{(by the definitions of } S \text{ and } e) \\
&= |X| p_{i-2}(\theta) p_i(\theta) k_{i-2} b_{i-2} b_{i-1} c_{i-1} c_i \qquad \text{(by Lemma 5.4)} \\
&\geq 0.
\end{aligned}$$

We now have a contradiction and the result follows. □

6 A variation of the p_i polynomials

In Section 4 we defined some polynomials p_i. In this section we define some closely related polynomials that we call the P_i. We do so for a technical reason that will become apparent later in the paper. We start with an observation. Recall that a polynomial in $\mathbb{R}[\lambda]$ is *even* (resp. *odd*) whenever the coefficient of λ^i is zero for all odd i (resp. all even i).

Lemma 6.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Then for $0 \leq i \leq D$ the polynomial p_i from (10) is even (resp. odd) if i is even (resp. odd).*

Proof. Routine using (12) and induction. □

In view of Lemma 6.1 we can make the following definition.

Definition 6.2 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. For $0 \leq i \leq D$ let P_i denote the polynomial in $\mathbb{R}[\lambda]$ such that

$$p_i(\lambda) = \begin{cases} P_i(\lambda^2), & \text{if } i \text{ is even} \\ \lambda P_i(\lambda^2), & \text{if } i \text{ is odd,} \end{cases} \tag{15}$$

where p_i is from (10). Observe the degree of P_i is $i/2$ if i is even and $(i-1)/2$ if i is odd. For notational convenience we define $P_{-1} = 0$.

Lemma 6.3 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let the polynomials $P_0, P_1, \ldots, P_D$ be as in Definition 6.2. Then the following (i), (ii) hold for $0 \leq i \leq D-1$:*

(i) *Suppose i is odd. Then $\lambda P_i = c_{i+1}P_{i+1} + b_{i+1}P_{i-1}$.*

(ii) *Suppose i is even. Then $P_i = c_{i+1}P_{i+1} + b_{i+1}P_{i-1}$.*

Proof. Routine using (12) and Definition 6.2. □

Referring to Lemma 6.3, in order to handle the cases of i odd and i even in a uniform fashion we introduce some notation.

Definition 6.4 For any integer i we define

$$s(i) = \begin{cases} 0, & \text{if } i \text{ is even} \\ 1, & \text{if } i \text{ is odd.} \end{cases}$$

Lemma 6.3 looks as follows in terms of $s(i)$.

Corollary 6.5 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, and let the polynomials $P_0, P_1, \ldots, P_D$ be as in Definition 6.2. Then for $0 \leq i \leq D-1$,*

$$\lambda^{s(i)} P_i = c_{i+1}P_{i+1} + b_{i+1}P_{i-1}. \tag{16}$$

Lemma 6.6 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $P_0, P_1, \ldots, P_D$ be as in Definition 6.2. Then the following (i)–(iii) hold for all $\psi \in \mathbb{R}$:*

(i) *Assume $\psi > \theta_1^2$. Then $P_i(\psi) > 0$ $(0 \leq i \leq D)$.*

(ii) *Assume D is odd and $\psi < \theta_d^2$, where $d = (D-1)/2$. Then $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) > 0$ $(0 \leq i \leq D)$.*

(iii) *Assume D is even and $\psi \leq 0$. Then $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) > 0$ $(0 \leq i \leq D-1)$. Moreover $(-1)^{\lfloor \frac{D}{2} \rfloor} P_D(\psi) > 0$ if $\psi < 0$ and $P_D(0) = 0$.*

Proof. (i). Since ψ is positive, there exists a positive real number α such that $\alpha^2 = \psi$. By the construction $\alpha > \theta_1$. For $0 \leq i \leq D$ we have $p_i(\alpha) > 0$ by Lemma 5.5(ii) so $P_i(\psi) > 0$ in view of Definition 6.2.
(ii). First assume $0 < \psi < \theta_d^2$. Again ψ is positive, so there exists a positive real number α such that $\alpha^2 = \psi$. By the construction $0 < \alpha < \theta_d$. For $0 \leq i \leq D$ we have $(-1)^{\lfloor \frac{i}{2} \rfloor} p_i(\alpha) > 0$ by Lemma 5.6(ii) so $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) > 0$ in view of Definition 6.2.

Now assume $\psi \leq 0$. Suppose there exists an integer i $(0 \leq i \leq D)$ such that $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) \leq 0$. Let us pick the minimal such i. Observe $i \geq 2$ since $P_0(\psi) = 1$, $P_1(\psi) = 1$. Setting $\lambda = \psi$ and replacing i by $i-1$ in (16) and then multiplying this equation by $(-1)^{\lfloor \frac{i}{2} \rfloor}$, we find

$$\begin{aligned} (-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) &= (-1)^{\lfloor \frac{i}{2} \rfloor} \psi^{s(i-1)} P_{i-1}(\psi) c_i^{-1} - (-1)^{\lfloor \frac{i}{2} \rfloor} P_{i-2}(\psi) b_i c_i^{-1} \\ &= (-\psi)^{s(i-1)} (-1)^{\lfloor \frac{i-1}{2} \rfloor} P_{i-1}(\psi) c_i^{-1} - (-1)^{\lfloor \frac{i}{2} \rfloor} P_{i-2}(\psi) b_i c_i^{-1} \\ &> 0, \end{aligned} \tag{17}$$

where the last inequality follows from the minimality of i and $\psi \leq 0$. We now have a contradiction and the result follows.
(iii). Similar to (ii). When $\psi = 0$, however, observe that the right side of (17) is 0 for $i = D$, and hence $P_D(0) = 0$. $\square$

Corollary 6.7 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let the polynomials $P_0, P_1, \ldots, P_D$ be as in Definition 6.2. Let θ denote a real number in the following range: For D odd, we assume $\theta > \theta_1^2$ or $\theta < \theta_d^2$, where $d = (D-1)/2$. For D even, we assume $\theta > \theta_1^2$ or $\theta \leq 0$. Then $P_i(\theta) \neq 0$ for $0 \leq i \leq D-1$.*

7 A third family of polynomials

In this section we will use the following notation.

Notation 7.1 Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_D$. Let $d = \lfloor D/2 \rfloor$. Let the polynomials p_i be as in (10), and let the polynomials P_i be as in Definition 6.2. Let θ denote a real number in the following range: For D odd, we assume $\theta > \theta_1^2$ or $\theta < \theta_d^2$. For D even, we assume $\theta > \theta_1^2$ or $\theta \leq 0$. We observe that in all cases $P_i(\theta) \neq 0$ for $0 \leq i \leq D-1$ by Corollary 6.7.

We now use Γ to define a family of polynomials in one variable. We call these polynomials the g_i.

Definition 7.2 With reference to Notation 7.1, for $0 \leq i \leq D-2$ we define the polynomial $g_i \in \mathbb{R}[\lambda]$ by

$$g_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{P_h(\theta)}{P_i(\theta)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h. \tag{18}$$

We emphasize g_i depends on θ as well as the intersection numbers of Γ.

Lemma 7.3 *With reference to Notation 7.1 and Definition 7.2,*

$$p_i = g_i - \frac{b_i b_{i+1}}{c_{i-1} c_i} \frac{P_{i-2}(\theta)}{P_i(\theta)} g_{i-2} \qquad (2 \leq i \leq D-2). \tag{19}$$

Proof. Routine using Definition 7.2 and (5). $\square$

Lemma 7.4 *With reference to Notation 7.1 and Definition 7.2, the following (i), (ii) hold for $0 \leq i \leq D-2$:*

(i) *The polynomial g_i has degree exactly i.*

(ii) *The coefficient of λ^i in g_i is $(c_1 c_2 \cdots c_i)^{-1}$.*

Proof. Routine. $\square$

We now present a three-term recurrence satisfied by the polynomials g_i.

Theorem 7.5 *With reference to Notation 7.1 and Definition 7.2, $g_0 = 1$ and*

$$\lambda g_i = c_{i+1} g_{i+1} + \omega_i g_{i-1} \tag{20}$$

for $0 \leq i \leq D-2$, where $g_{-1} = 0$, $\omega_0 = 0$, $g_{D-1} = p_{D-1}$, and

$$\omega_i = \frac{b_{i+1} c_{i+2}}{c_i} \frac{P_{i-1}(\theta) P_{i+2}(\theta)}{P_i(\theta) P_{i+1}(\theta)} \qquad (1 \leq i \leq D-2). \tag{21}$$

Proof. We find $g_0 = 1$ by Definition 7.2. We now prove (20) by induction on i. Line (20) holds for $i = 0, 1$ using Definition 7.2, (12), and Definition 6.2. Next assume $i \geq 2$ and by induction that

$$\lambda g_{i-2} = c_{i-1} g_{i-1} + \omega_{i-2} g_{i-3}. \tag{22}$$

Consider the right-hand side of (20). In this expression eliminate g_{i+1} using (19) if $i < D-2$ and $g_{D-1} = p_{D-1}$ if $i = D-2$. Also eliminate ω_i using (21) and simplify the result using (16) to get

$$c_{i+1} g_{i+1} + \omega_i g_{i-1} = c_{i+1} p_{i+1} + \frac{b_{i+1} \theta^{s(i+1)}}{c_i} \frac{P_{i-1}(\theta)}{P_i(\theta)} g_{i-1}. \tag{23}$$

Now consider the left-hand side of (20). Replacing g_i in this expression using (19), and eliminating λp_i, λg_{i-2} in the result using (12), (22), respectively, we find

$$\lambda g_i = c_{i+1} p_{i+1} + b_{i+1} p_{i-1} + \frac{b_i b_{i+1}}{c_{i-1} c_i} \frac{P_{i-2}(\theta)}{P_i(\theta)} (c_{i-1} g_{i-1} + \omega_{i-2} g_{i-3}). \tag{24}$$

If $i > 2$, in (24) we eliminate ω_{i-2} using (21) and then eliminate $b_{i-1} b_i P_{i-3}(\theta)(c_{i-2} c_{i-1} P_{i-1}(\theta))^{-1} g_{i-3}$ in the resulting equation using (19). If $i = 2$, in (24) we note $\omega_0 = 0$ and $p_1 = g_1$ in view of Definition 7.2. In either case we find

$$\lambda g_i = c_{i+1} p_{i+1} + b_{i+1} \frac{c_i P_i(\theta) + b_i P_{i-2}(\theta)}{c_i P_i(\theta)} g_{i-1}. \tag{25}$$

Observe the right-hand sides of (23), (25) are equal in view of (16) and Definition 6.4, and thus the left-hand sides are equal. We obtain (20) as desired. □

Lemma 7.6 *With reference to Notation 7.1 and Definition 7.2, for* $0 \leq i \leq D-2$ *we have*

$$c_{i+1}^{-1} c_{i+2}^{-1} (\lambda^2 - \theta) g_i = p_{i+2} - \frac{P_{i+2}(\theta)}{P_i(\theta)} p_i. \tag{26}$$

Proof. We show (26) by induction on i. Line (26) holds for $i = 0, 1$ by Definition 7.2, (12), and Definition 6.2. Next assume $i \geq 2$ and by induction that

$$c_{i-1}^{-1} c_i^{-1} (\lambda^2 - \theta) g_{i-2} = p_i - \frac{P_i(\theta)}{P_{i-2}(\theta)} p_{i-2}. \tag{27}$$

Repeatedly applying (12), we find

$$\lambda^2 p_i = c_{i+1} c_{i+2} p_{i+2} + (c_{i+1} b_{i+2} + b_{i+1} c_i) p_i + b_i b_{i+1} p_{i-2}. \tag{28}$$

Similarly, by repeatedly applying Lemma 6.3, we find

$$\theta P_i(\theta) = c_{i+1} c_{i+2} P_{i+2}(\theta) + (c_{i+1} b_{i+2} + b_{i+1} c_i) P_i(\theta) + b_i b_{i+1} P_{i-2}(\theta). \tag{29}$$

By (19), we find

$$g_i = p_i + \frac{b_i b_{i+1}}{c_{i-1} c_i} \frac{P_{i-2}(\theta)}{P_i(\theta)} g_{i-2}. \tag{30}$$

Using (30) to eliminate g_{i-2} in (27), and then applying (28), (29), we obtain (26). □

Theorem 7.7 *With reference to Notation 7.1 and Definition 7.2, for* $0 \leq i, j \leq D-2$ *we have*

$$\sum_{h=0}^{D} g_i(\theta_h) g_j(\theta_h)(k^2 - \theta_h^2)(\theta - \theta_h^2) m_h = \delta_{ij} |X| k_i b_i b_{i+1} c_{i+1} c_{i+2} \frac{P_{i+2}(\theta)}{P_i(\theta)}. \tag{31}$$

Proof. Without loss of generality, we may assume $i \leq j$. First we eliminate $g_i(\theta_h)$ and $g_j(\theta_h)(\theta - \theta_h^2)$ in the left-hand side of (31) by using Definition 7.2 and (26), respectively. Simplifying the resulting expression using (13) and the fact that $i \leq j$, we obtain the right-hand side of (31). The result follows. $\square$

We finish this section with a comment.

Lemma 7.8 *With reference to Notation 7.1 and Definition 7.2, assume D is even and $\theta = 0$. Then $g_{D-2}(\theta_h) = 0$ for $1 \leq h \leq D-1,\ h \neq d$.*

Proof. Recall $\theta_d = 0$ by Lemma 3.2. Setting $i = j = D-2$ and $\theta = 0$ in (31), we find

$$\sum_{\substack{h=1 \\ h\neq d}}^{D-1} \theta_h^2 g_{D-2}^2(\theta_h)(k^2 - \theta_h^2)m_h = -|X|k_{D-2}b_{D-2}b_{D-1}c_{D-1}c_D \frac{P_D(0)}{P_{D-2}(0)}. \qquad (32)$$

In (32) the right-hand side is zero by Lemma 6.6. In the left-hand side each summand is nonnegative so each summand is zero. In each summand the factor $\theta_h^2(k^2 - \theta_h^2)m_h$ is nonzero so the remaining factor $g_{D-2}(\theta_h)$ is zero. The result follows. $\square$

8 The subconstituent algebra and its modules

In this section we recall some definitions and basic concepts concerning the subconstituent algebra and its modules. For more information we refer the reader to [4], [9], [10], [23], [26], [31].

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. We recall the dual Bose-Mesner algebra of Γ. From now on we fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X). \qquad (33)$$

We call E_i^* the i^{th} *dual idempotent of* Γ *with respect to* x. We observe (di) $\sum_{i=0}^D E_i^* = I$; (dii) $\overline{E_i^*} = E_i^*$ $(0 \leq i \leq D)$; (diii) $E_i^{*t} = E_i^*$ $(0 \leq i \leq D)$; (div) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq D)$. Using (di) and (div) we find $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra of* Γ *with respect to* x. We recall the subconstituents of Γ. Using (33) we find

$$E_i^* V = \mathrm{span}\{\hat{y} \mid y \in X, \quad \partial(x,y) = i\} \qquad (0 \leq i \leq D). \qquad (34)$$

By (34) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \qquad \text{(orthogonal direct sum)}.$$

Combining (34) and (4) we find the dimension of $E_i^* V$ is k_i for $0 \leq i \leq D$. We call $E_i^* V$ the i^{th} *subconstituent of* Γ *with respect to* x.

We recall how M and M^* are related. By [31, Lemma 3.2],

$$E_h^* A_i E_j^* = 0 \quad \text{if and only if} \quad p_{ij}^h = 0 \qquad (0 \leq h, i, j \leq D). \qquad (35)$$

Combining (35) and (3) we find

$$E_i^* A E_j^* = 0 \qquad \text{if} \qquad |i - j| > 1 \qquad (0 \leq i, j \leq D). \qquad (36)$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra of* Γ *with respect to* x [31]. We observe T has finite dimension. Moreover T is semi-simple; the reason is that T is closed under the conjugate-transpose map [18, p. 157].

We now consider the modules for T. By a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. We refer to V itself as the *standard module* for T. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that

$$(\sigma B - B\sigma)W = 0 \qquad \text{for all } B \in T.$$

The modules W, W' are said to be *isomorphic as T-modules* whenever there exists an isomorphism of T-modules from W to W'.

Let W denote a T-module and let W' denote a T-module contained in W. Using (1) we find the orthogonal complement of W' in W is a T-module. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. We mention any two nonisomorphic irreducible T-modules are orthogonal [18, Chapter IV].

Let W denote an irreducible T-module. Using (di)–(div) above we find W is the direct sum of the nonzero spaces among $E_0^*W, E_1^*W, \ldots, E_D^*W$. Similarly using (eii)–(ev) we find W is the direct sum of the nonzero spaces among $E_0W, E_1W, \ldots, E_DW$. If the dimension of E_i^*W is at most 1 for $0 \leq i \leq D$ then the dimension of E_iW is at most 1 for $0 \leq i \leq D$ [31, Lemma 3.9]; in this case we say W is *thin.* Let W denote an irreducible T-module. By the *endpoint* of W we mean

$$\min\{i \mid 0 \leq i \leq D, \;\; E_i^*W \neq 0\}.$$

For the rest of the paper we adopt the following notational convention.

Definition 8.1 Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers b_i, c_i, distance matrices A_i, Bose-Mesner algebra M, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. For $0 \leq i \leq D$ we let E_i denote the primitive idempotent of Γ associated with θ_i. We define $d = \lfloor D/2 \rfloor$. We fix $x \in X$ and abbreviate $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $M^* = M^*(x)$, $T = T(x)$. We let V denote the standard module for Γ. We define

$$s_i = \sum_{\substack{y \in X \\ \partial(x,y)=i}} \hat{y} \qquad (0 \leq i \leq D). \tag{37}$$

9 The T-module of endpoint 0

With reference to Definition 8.1, there exists a unique irreducible T-module with endpoint 0 [21, Proposition 8.4]. We call this module V_0. The module V_0 is described in [9], [21]. We summarize some details below in order to motivate the results that follow.

The module V_0 is thin. In fact each of E_iV_0, $E_i^*V_0$ has dimension 1 for $0 \leq i \leq D$. We give two bases for V_0. The vectors $E_0\hat{x}, E_1\hat{x}, \ldots, E_D\hat{x}$ form a basis for V_0. These vectors are mutually orthogonal and $\|E_i\hat{x}\|^2 = m_i|X|^{-1}$ for $0 \leq i \leq D$. To motivate the second basis we make some comments. For $0 \leq i \leq D$ we have $s_i = A_i\hat{x}$. Moreover $s_i = E_i^*\delta$, where $\delta = \sum_{y \in X} \hat{y}$. The vectors $s_0, s_1, \ldots, s_D$ form a basis for V_0. These vectors are mutually orthogonal and $\|s_i\|^2 = k_i$ for $0 \leq i \leq D$. With respect to the basis $s_0, s_1, \ldots, s_D$ the matrix representing A is

$$\begin{pmatrix} 0 & b_0 & & & & \mathbf{0} \\ c_1 & 0 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{D-1} \\ \mathbf{0} & & & & c_D & 0 \end{pmatrix}.$$

The two bases for V_0 given above are related as follows. For $0 \leq i \leq D$ we have

$$s_i = \sum_{h=0}^{D} f_i(\theta_h) E_h \hat{x},$$

where the polynomial f_i is from (9).

10 The T-modules of endpoint 1

With reference to Definition 8.1, there exists, up to isomorphism, a unique irreducible T-module with endpoint 1 [9, Corollary 7.7]. We call this module V_1. The module V_1 is described in [9], [24]. We summarize some details below.

The module V_1 is thin with dimension $D-1$. We give two bases for V_1. Let v denote a nonzero vector in $E_1^* V_1$. The vectors

$$E_i v \qquad (1 \leq i \leq D-1) \tag{38}$$

form a basis for V_1 and $E_0 v = 0$, $E_D v = 0$. The vectors in (38) are mutually orthogonal and

$$||E_i v||^2 = \frac{m_i(k^2 - \theta_i^2)}{|X|k(k-1)} ||v||^2 \qquad\qquad (1 \leq i \leq D-1).$$

To motivate the second basis we make some comments. We have $E_{i+1}^* A_i v = p_i(A) v$ for $0 \leq i \leq D-1$, where the p_i are from (10). The vectors

$$E_{i+1}^* A_i v \qquad (0 \leq i \leq D-2) \tag{39}$$

form a basis for V_1 and $E_D^* A_{D-1} v = 0$. The vectors in (39) are mutually orthogonal and

$$||E_{i+1}^* A_i v||^2 = \frac{b_2 \cdots b_{i+1}}{c_1 \cdots c_i} ||v||^2 \qquad\qquad (0 \leq i \leq D-2).$$

With respect to the basis (39) the matrix representing A is

$$\begin{pmatrix} 0 & b_2 & & & & \mathbf{0} \\ c_1 & 0 & b_3 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{D-1} \\ \mathbf{0} & & & & c_{D-2} & 0 \end{pmatrix}.$$

The two bases for V_1 given above are related as follows. For $0 \leq i \leq D-2$ we have

$$E_{i+1}^* A_i v = \sum_{h=1}^{D-1} p_i(\theta_h) E_h v.$$

We comment that V_1 appears in V with multiplicity $k-1$. We will need the following result.

Corollary 10.1 *With reference to Definition 8.1, let W denote an irreducible T-module with endpoint 1. Observe $E_2^* W$ is an eigenspace for $E_2^* A_2 E_2^*$. The corresponding eigenvalue is $b_3 - 1$.*

Proof. The desired eigenvalue is the entry in the second row and second column of the matrix representing A_2 with respect to the basis (39). To compute this entry, first set $i = 1$ in (8) and observe that $c_2 A_2 = A^2 - kI$. Using this fact and the above matrix display of A, we verify the specified matrix entry is $b_3 - 1$. $\square$

11 The local eigenvalues

A bit later in this paper we will consider the thin irreducible T-modules with endpoint 2. In order to discuss these we recall the local eigenvalues.

Definition 11.1 With reference to Definition 8.1, we let $\Gamma_2^2 = \Gamma_2^2(x)$ denote the graph $(\breve{X}, \breve{R})$, where

$$\begin{aligned} \breve{X} &= \{y \in X \mid \partial(x,y) = 2\}, \\ \breve{R} &= \{yz \mid y, z \in \breve{X},\ \partial(y,z) = 2\}, \end{aligned}$$

where we recall ∂ denotes the path-length distance function for Γ. The graph Γ_2^2 has exactly k_2 vertices, where k_2 is the second valency of Γ. Also, Γ_2^2 is regular with valency p^2_{22}. We let $\breve{A}$ denote the adjacency matrix of Γ_2^2. The matrix $\breve{A}$ is symmetric with real entries; therefore $\breve{A}$ is diagonalizable with all eigenvalues real. We let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of $\breve{A}$. We call $\eta_1, \eta_2, \ldots, \eta_{k_2}$ the *local eigenvalues of* Γ *with respect to* x.

With reference to Definition 8.1, we consider the second subconstituent E_2^*V. We recall the dimension of E_2^*V is k_2. Observe E_2^*V is invariant under the action of $E_2^*A_2E_2^*$. To illuminate this action we make an observation. For an appropriate ordering of the vertices of Γ we have

$$E_2^*A_2E_2^* = \begin{pmatrix} \breve{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\breve{A}$ is from Definition 11.1. Apparently the action of $E_2^*A_2E_2^*$ on E_2^*V is essentially the adjacency map for Γ_2^2. In particular the action of $E_2^*A_2E_2^*$ on E_2^*V is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_{k_2}$. We observe the vector s_2 from (37) is contained in E_2^*V. One may easily show that s_2 is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue p^2_{22}. Let v denote a vector in E_2^*V. We observe the following are equivalent: (i) v is orthogonal to s_2; (ii) $E_0v = 0$; (iii) $Jv = 0$; (iv) $E_Dv = 0$; (v) $J'v = 0$. Let V_1 denote an irreducible T-module of endpoint 1, and let v denote a vector in $E_2^*V_1$. By Corollary 10.1, v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue $b_3 - 1$. Reordering the local eigenvalues if necessary, we have $\eta_1 = p^2_{22}$ and $\eta_i = b_3 - 1$ $(2 \le i \le k)$. For the rest of this paper we assume the local eigenvalues of Γ are ordered in this way.

We now need some notation.

Definition 11.2 With reference to Definition 8.1, let Y denote the subspace of V spanned by the irreducible T-modules with endpoint 1. We define U to be the orthogonal complement of $E_2^*V_0 + E_2^*Y$ in E_2^*V.

Definition 11.3 With reference to Definition 8.1, let Φ denote the set of distinct scalars among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$, where the η_i are from Definition 11.1. For $\eta \in \mathbb{R}$ we let mult_η denote the number of times η appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. We observe $\mathrm{mult}_\eta \neq 0$ if and only if $\eta \in \Phi$.

Using (1) we find U is invariant under $E_2^*A_2E_2^*$. Apparently the restriction of $E_2^*A_2E_2^*$ to U is diagonalizable with eigenvalues $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. For $\eta \in \mathbb{R}$ let U_η denote the set consisting of those vectors in U that are eigenvectors for $E_2^*A_2E_2^*$ with eigenvalue η. We observe U_η is a subspace of U with dimension mult_η. We emphasize the following are equivalent: (i) $\mathrm{mult}_\eta \neq 0$; (ii) $U_\eta \neq 0$; (iii) $\eta \in \Phi$. By (1) and since $E_2^*A_2E_2^*$ is symmetric with real entries we find

$$U = \sum_{\eta \in \Phi} U_\eta \qquad \text{(orthogonal direct sum).} \tag{40}$$

Definition 11.4 With reference to Definition 8.1, for all $z \in \mathbb{C} \cup \infty$ we define

$$\tilde{z} = \begin{cases} -1 - \frac{b_2 b_3}{z^2 - b_2}, & \text{if } z \neq \infty,\ z^2 \neq b_2 \\ \infty, & \text{if } z^2 = b_2 \\ -1, & \text{if } z = \infty. \end{cases}$$

Note 11.5 With reference to Definition 8.1, neither of θ_1^2, θ_d^2 is equal to b_2 by Lemma 3.4, so

$$\tilde{\theta}_1 = -1 - b_2 b_3 (\theta_1^2 - b_2)^{-1}, \qquad \tilde{\theta}_d = -1 - b_2 b_3 (\theta_d^2 - b_2)^{-1}. \tag{41}$$

By the data in Lemma 3.4 we have $\tilde{\theta}_1 < -1$. Moreover $\tilde{\theta}_d > b_3 - 1$ if D is odd and $\tilde{\theta}_d = b_3 - 1$ if D is even. In either case $\tilde{\theta}_d \geq 0$.

Lemma 11.6 *[28, Theorem 11.4] With reference to Definitions 8.1 and 11.1, we have* $\tilde{\theta}_1 \leq \eta_i \leq \tilde{\theta}_d$ *for* $k + 1 \leq i \leq k_2$.

We remark on the case of equality in the above lemma.

Lemma 11.7 *[28, Lemma 11.5] With reference to Definition 8.1, let v denote a nonzero vector in U. Then (i)–(vi) hold below:*

(i) $E_0 v = 0$ *and* $E_D v = 0$.

(ii) *For* $1 \leq i \leq D - 1$, $E_i v \neq 0$ *provided i is not among* $1, d, D - d, D - 1$.

(iii) $E_1 v = 0$ *if and only if* $v \in U_{\tilde{\theta}_1}$.

(iv) $E_{D-1} v = 0$ *if and only if* $v \in U_{\tilde{\theta}_1}$.

(v) $E_d v = 0$ *if and only if* $v \in U_{\tilde{\theta}_d}$.

(vi) $E_{D-d} v = 0$ *if and only if* $v \in U_{\tilde{\theta}_d}$.

Corollary 11.8 *[28, Corollary 11.6] With reference to Definition 8.1, let v denote a nonzero vector in U. Then (i)–(iv) hold below:*

(i) *If* $v \in U_{\tilde{\theta}_1}$ *then Mv has dimension* $D - 3$.

(ii) *If* $v \in U_{\tilde{\theta}_d}$ *and D is odd, then Mv has dimension* $D - 3$.

(iii) *If* $v \in U_{\tilde{\theta}_d}$ *and D is even, then Mv has dimension* $D - 2$.

(iv) *If* $v \notin U_{\tilde{\theta}_1}$ *and* $v \notin U_{\tilde{\theta}_d}$ *then Mv has dimension* $D - 1$.

Definition 11.9 With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2. Observe $E_2^* W$ is a 1-dimensional eigenspace for $E_2^* A_2 E_2^*$; let η denote the corresponding eigenvalue. We observe $E_2^* W$ is contained in $E_2^* V$ and is orthogonal to any irreducible T-module with endpoint 0 or 1, so $E_2^* W \subseteq U_\eta$. Apparently $U_\eta \neq 0$ so η is among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. We have $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_d$ by Lemma 11.6. We refer to η as the *local eigenvalue* of W.

With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η. In order to describe W we distinguish four cases: (i) $\eta = \tilde{\theta}_1$; (ii) D is odd and $\eta = \tilde{\theta}_d$; (iii) D is even and $\eta = \tilde{\theta}_d$; (iv) $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. For cases (i), (ii) the module W was described by the present authors in [28]; we summarize these results in the following section. For cases (iii), (iv) we describe W in Sections 14 and 16.

12 Some thin irreducible T-modules with endpoint 2

In this section we summarize some results from [28] concerning the thin irreducible T-modules with endpoint 2 and local eigenvalue η, where $\eta = \tilde{\theta}_1$, or $\eta = \tilde{\theta}_d$ with D odd.

With reference to Definition 8.1, choose $n \in \{1, d\}$ if D is odd, and let $n = 1$ if D is even. Define $\eta = \tilde{\theta}_n$. Let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η. The dimension of W is $D - 3$. For $0 \leq i \leq D$, $E_i^* W$ is zero if $i \in \{0, 1, D - 1, D\}$, and has dimension 1 if $i \notin \{0, 1, D - 1, D\}$.

Moreover E_iW is zero if $i \in \{0, n, D-n, D\}$, and has dimension 1 if $i \notin \{0, n, D-n, D\}$. Let v denote a nonzero vector in E_2^*W. Then $W = Mv$. The vectors

$$E_i v \qquad (1 \le i \le D-1, \ \ i \ne n, \ i \ne D-n) \tag{42}$$

form a basis for W, and each of $E_0v, E_nv, E_{D-n}v, E_Dv$ is zero. The vectors in (42) are mutually orthogonal and

$$\|E_iv\|^2 = \frac{m_i(\theta_i^2 - k^2)(\theta_i^2 - \theta_n^2)}{|X|kb_1(\theta_n^2 - b_2)}\|v\|^2 \qquad (1 \le i \le D-1, \ \ i \ne n, \ \ i \ne D-n).$$

We mention a second basis for W. To motivate things we remark

$$E_{i+2}^*A_iv = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{p_h(\theta_n)}{p_i(\theta_n)} \frac{k_ib_ib_{i+1}}{k_hb_hb_{h+1}} p_h(A)v \qquad (0 \le i \le D-2).$$

The vectors

$$E_{i+2}^*A_iv \qquad (0 \le i \le D-4) \tag{43}$$

form a basis for W, and $E_{D-1}^*A_{D-3}v = 0$, $E_D^*A_{D-2}v = 0$. The vectors in (43) are mutually orthogonal and

$$\|E_{i+2}^*A_iv\|^2 = \frac{k_ib_ib_{i+1}c_{i+1}c_{i+2}}{kb_1(\theta_n^2 - b_2)} \frac{p_{i+2}(\theta_n)}{p_i(\theta_n)}\|v\|^2 \qquad (0 \le i \le D-4).$$

With respect to the basis given in (43) the matrix representing A is

$$\begin{pmatrix} 0 & w_1 & & & & \mathbf{0} \\ c_1 & 0 & w_2 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & w_{D-4} \\ \mathbf{0} & & & & c_{D-4} & 0 \end{pmatrix},$$

where

$$w_i = \frac{b_{i+1}c_{i+2}}{c_i} \frac{p_{i-1}(\theta_n)p_{i+2}(\theta_n)}{p_i(\theta_n)p_{i+1}(\theta_n)} \qquad (1 \le i \le D-4).$$

The bases for W given in (42), (43) are related as follows. For $0 \le i \le D-4$ we have

$$E_{i+2}^*A_iv = \sum_{\substack{1 \le j \le D-1 \\ j \ne n, \, j \ne D-n}} \gamma_i(\theta_j)E_jv,$$

where

$$\gamma_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{p_h(\theta_n)}{p_i(\theta_n)} \frac{k_ib_ib_{i+1}}{k_hb_hb_{h+1}} p_h.$$

We finish this section with a comment.

Lemma 12.1 *[28, Theorem 12.9] With reference to Definition 8.1, let v denote a nonzero vector in U. Let $n \in \{1, d\}$ if D is odd, and let $n = 1$ if D is even. Assume v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue $\tilde{\theta}_n$. Then Mv is a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_n$.*

13 The space Mv when D is even and $v \in U_{\tilde{\theta}_d}$

With reference to Definition 8.1, assume D is even. One of our ultimate goals in this paper is to describe the thin irreducible T-modules with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Before we get to this, we find it illuminating to consider a more general type of space. Let v denote a nonzero vector in U and assume v is an eigenvector for $E_2^*A_2E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. In this section we investigate the space Mv. We present two orthogonal bases for Mv which we find attractive. Recall that since D is even, we have $\theta_d = 0$ and thus $\tilde{\theta}_d = b_3 - 1$.

Theorem 13.1 *With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. Then the vectors $E_i v$ $(1 \leq i \leq D-1,\ i \neq d)$ form a basis for Mv. Moreover $E_0 v = 0$, $E_d v = 0$, $E_D v = 0$.*

Proof. Recall $E_0, E_1, \ldots, E_D$ form a basis for M. Observe $E_0 v = 0$, $E_d v = 0$, $E_D v = 0$ by Lemma 11.7 so the vectors $E_i v$ $(1 \leq i \leq D-1,\ i \neq d)$ span Mv. These vectors are nonzero by Lemma 11.7 and mutually orthogonal by (6), so they are linearly independent. The result follows. $\square$

Theorem 13.2 *[28, Theorem 11.2] With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. Then the vectors $E_i v$ $(1 \leq i \leq D-1,\ i \neq d)$ are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

$$\|E_i v\|^2 = \frac{m_i (k-\theta_i)(k+\theta_i)\theta_i^2}{|X| k b_1 b_2} \|v\|^2 \qquad (1 \leq i \leq D-1,\ i \neq d).$$

(The scalar m_i denotes the multiplicity of θ_i.)

Referring to Theorem 13.1, we now consider a second basis for Mv.

Definition 13.3 With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. We define the vectors $v_0, v_1, \ldots, v_{D-2}$ by

$$v_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{P_h(0)}{P_i(0)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h(A) v \qquad (0 \leq i \leq D-2). \tag{44}$$

(The polynomials p_i are from (10), and the P_i are from (15).) The denominators in (44) are nonzero by Corollary 6.7.

Theorem 13.4 *With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. Then with reference to (44), the vectors $v_0, v_1, \ldots, v_{D-3}$ form a basis for Mv and $v_{D-2} = 0$.*

Proof. By Theorem 13.1 we find Mv has dimension $D-2$. By this and since A generates M, we find Mv has a basis $v, Av, \ldots, A^{D-3}v$. For $0 \leq i \leq D-3$ the vector v_i is contained in the span of $v, Av, \ldots, A^i v$ but not in the span of $v, Av, \ldots, A^{i-1}v$. It follows that $v_0, v_1, \ldots, v_{D-3}$ form a basis for Mv. To see that $v_{D-2} = 0$, first let g_{D-2} denote the polynomial from Definition 7.2, where $\theta = 0$. Comparing (18), (44) we find $v_{D-2} = g_{D-2}(A)v$. Using this and (eii) we routinely obtain $v_{D-2} = \sum_{j=0}^{D} g_{D-2}(\theta_j) E_j v$. Applying Lemma 7.8 and Theorem 13.1, we find $v_{D-2} = 0$. $\square$

With reference to Definition 13.3, we will show the vectors $v_0, v_1, \ldots, v_{D-3}$ are mutually orthogonal and we will compute their square-norms. To do this we need the following result.

Theorem 13.5 *With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. Let the vectors $v_0, v_1, \ldots, v_{D-3}$ be as in Definition 13.3. Then for $0 \leq i \leq D-3$ we have*

$$v_i = \sum_{\substack{j=1 \\ j \neq d}}^{D-1} g_i(\theta_j) E_j v, \tag{45}$$

where

$$g_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{P_h(0)}{P_i(0)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h. \tag{46}$$

Proof. Let the integer i be given. Comparing (44), (46) we find $v_i = g_i(A)v$. Using this and (eii) we routinely obtain $v_i = \sum_{j=0}^{D} g_i(\theta_j)E_j v$. Line (45) follows since $E_0 v = 0$, $E_d v = 0$, $E_D v = 0$ by Theorem 13.1. $\square$

Theorem 13.6 *With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. Then the vectors $v_0, v_1, \ldots, v_{D-3}$ from Definition 13.3 are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

$$\|v_i\|^2 = -\frac{k_i b_i b_{i+1} c_{i+1} c_{i+2}}{k b_1 b_2} \frac{P_{i+2}(0)}{P_i(0)} \|v\|^2 \qquad (0 \leq i \leq D-3). \tag{47}$$

Proof. Let the polynomials $g_0, g_1, \ldots, g_{D-3}$ be as in (46). Using in order Theorem 13.5, Theorem 13.2, and Theorem 7.7, we find that for $0 \leq i, j \leq D-3$,

$$\begin{aligned}
\langle v_i, v_j \rangle &= \sum_{\substack{h=1 \\ h \neq d}}^{D-1} g_i(\theta_h) g_j(\theta_h) \|E_h v\|^2 \\
&= \sum_{\substack{h=1 \\ h \neq d}}^{D-1} g_i(\theta_h) g_j(\theta_h) \frac{m_h (k-\theta_h)(k+\theta_h)\theta_h^2}{|X| k b_1 b_2} \|v\|^2 \\
&= -\delta_{ij} \frac{k_i b_i b_{i+1} c_{i+1} c_{i+2}}{k b_1 b_2} \frac{P_{i+2}(0)}{P_i(0)} \|v\|^2.
\end{aligned}$$

Apparently $v_0, v_1, \ldots, v_{D-3}$ are mutually orthogonal and satisfy (47). $\square$

Theorem 13.7 *With reference to Definition 8.1, assume D is even, and let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ with corresponding eigenvalue $\tilde{\theta}_d$. With respect to the basis $v_0, v_1, \ldots, v_{D-3}$ for Mv given in Definition 13.3 the matrix representing A is*

$$\begin{pmatrix}
0 & \omega_1 & & & & \mathbf{0} \\
c_1 & 0 & \omega_2 & & & \\
 & c_2 & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \\
 & & & \cdot & \cdot & \omega_{D-3} \\
\mathbf{0} & & & & c_{D-3} & 0
\end{pmatrix},$$

where

$$\omega_i = \frac{b_{i+1} c_{i+2}}{c_i} \frac{P_{i-1}(0) P_{i+2}(0)}{P_i(0) P_{i+1}(0)} \qquad (1 \leq i \leq D-3). \tag{48}$$

Proof. For $0 \leq i \leq D-2$ we define g_i as in Definition 7.2, where $\theta = 0$. Setting $\lambda = A$ and $\theta = 0$ in Theorem 7.5 we find

$$A g_i(A) = c_{i+1} g_{i+1}(A) + \omega_i g_{i-1}(A) \qquad (0 \leq i \leq D-3), \tag{49}$$

where $g_{-1} = 0$, $\omega_0 = 0$, and the ω_i are from (48). Observe $g_i(A)v = v_i$ for $0 \leq i \leq D-2$. Applying (49) to v, and simplifying the result using these comments, we find

$$A v_i = c_{i+1} v_{i+1} + \omega_i v_{i-1} \qquad (0 \leq i \leq D-3),$$

where $v_{-1} = 0$. The result follows from this and since $v_{D-2} = 0$ by Theorem 13.4. $\square$

14 The thin irreducible T-modules with endpoint 2 and local eigenvalue $\tilde{\theta}_d$, when D is even

With reference to Definition 8.1, assume D is even. We now describe the thin irreducible T-modules with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. This section contains some of our main results. Because of this we have tried to make it as self-contained as possible.

Theorem 14.1 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Let v denote a nonzero vector in E_2^*W. Then $W = Mv$. The vectors*

$$E_i v \qquad (1 \leq i \leq D-1,\ i \neq d) \tag{50}$$

form a basis for W and $E_0 v = 0$, $E_d v = 0$, $E_D v = 0$.

Proof. We first show $W = Mv$. From the construction Mv is nonzero and contained in W. Consequently in order to show $Mv = W$, it suffices to show Mv is a T-module. By construction Mv is closed under multiplication by M. We now show that Mv is closed under multiplication by M^*. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $\tilde{\theta}_d$. Observe that Mv has basis $v, Av, \ldots, A^{D-3}v$ by Definition 13.3 and Theorem 13.4. Using this and (36) we find $Mv \subseteq \sum_{h=2}^{D-1} E_h^* W$. Observe the dimension of Mv is $D-2$ and the dimension of $\sum_{h=2}^{D-1} E_h^* W$ is at most $D-2$. Therefore $Mv = \sum_{h=2}^{D-1} E_h^* W$. From this we find Mv is closed under multiplication by M^* as desired. We have shown that Mv is a nonzero T-submodule of W so $Mv = W$ by the irreducibility of W. The remaining assertions of the present theorem follow in view of Theorem 13.1. □

Theorem 14.2 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Then the basis vectors for W from (50) are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

$$\|E_i v\|^2 = \frac{m_i(k-\theta_i)(k+\theta_i)\theta_i^2}{|X| k b_1 b_2} \|v\|^2 \qquad (1 \leq i \leq D-1,\ i \neq d).$$

(The scalar m_i denotes the multiplicity of θ_i.)

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $\tilde{\theta}_d$. Applying Theorem 13.2 we obtain the result. □

Theorem 14.3 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Let v denote a nonzero vector in E_2^*W. Then*

$$E_{i+2}^* A_i v = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{P_h(0)}{P_i(0)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h(A) v \qquad (0 \leq i \leq D-2). \tag{51}$$

Moreover, each side of (51) is zero for $i = D-2$. (The polynomials p_i are from (10), and the P_i are from (15).)

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $\tilde{\theta}_d$. Let the vectors $v_0, v_1, \ldots, v_{D-2}$ be as in Definition 13.3. We show $E_{i+2}^* A_i v = v_i$ for $0 \leq i \leq D-2$. Using (36) we find $A^i v$ is contained in $E_2^* W + \cdots + E_{i+2}^* W$ for $0 \leq i \leq D-2$. Also for $0 \leq i \leq D-2$, v_i is a linear combination of $v, Av, \ldots, A^i v$, so v_i is contained in $E_2^* W + \cdots + E_{i+2}^* W$. By this and since $v_0, v_1, \ldots, v_{D-3}$ are linearly independent, we find

$$v_0, v_1, \ldots, v_i \quad \text{is a basis for} \quad E_2^* W + E_3^* W + \cdots + E_{i+2}^* W \qquad (0 \leq i \leq D-3). \tag{52}$$

For the rest of this proof, fix an integer i $(0 \leq i \leq D-2)$. We show v_i is contained in $E^*_{i+2}W$. To see this, recall $E^*_2W, \ldots, E^*_DW$ are mutually orthogonal. Therefore $E^*_{i+2}W$ is equal to the orthogonal complement of $E^*_2W + \cdots + E^*_{i+1}W$ in $E^*_2W + \cdots + E^*_{i+2}W$. Recall v_i is orthogonal to each of $v_0, v_1, \ldots, v_{i-1}$. By (52) the vectors $v_0, v_1, \ldots, v_{i-1}$ form a basis for $E^*_2W + \cdots + E^*_{i+1}W$ so v_i is orthogonal to $E^*_2W + \cdots + E^*_{i+1}W$. Apparently v_i is contained in $E^*_{i+2}W$ as desired. We show $E^*_{i+2}A_iv = v_i$. We mentioned the vector v_i is a linear combination of $v, Av, \ldots, A^iv$. In this combination the coefficient of A^iv is $(c_1c_2\cdots c_i)^{-1}$ in view of Lemma 7.4(ii). Similarly A_iv is a linear combination of $v, Av, \ldots, A^iv$, and in this combination the coefficient of A^iv is $(c_1c_2\cdots c_i)^{-1}$. Apparently $A_iv - v_i$ is a linear combination of $v, Av, \ldots, A^{i-1}v$. From this and our above comments $A_iv - v_i$ is contained in $E^*_2W + \cdots + E^*_{i+1}W$ so $E^*_{i+2}(A_iv - v_i)$ is zero. We already showed $v_i \in E^*_{i+2}W$ so $E^*_{i+2}v_i = v_i$. Now $E^*_{i+2}A_iv = v_i$ as desired. Recall $v_{D-2} = 0$ by Theorem 13.4, so both sides of (51) are zero for $i = D-2$. □

Theorem 14.4 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Let v denote a nonzero vector in E^*_2W. Then the vectors*

$$E^*_{i+2}A_iv \qquad (0 \leq i \leq D-3) \tag{53}$$

form a basis for W.

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E^*_2A_2E^*_2$ with eigenvalue $\tilde{\theta}_d$. Let the vectors $v_0, v_1, \ldots, v_{D-3}$ be as in Definition 13.3. By Theorem 13.4 the vectors $v_0, v_1, \ldots, v_{D-3}$ form a basis for Mv. Recall $Mv = W$ by Theorem 14.1 so $v_0, v_1, \ldots, v_{D-3}$ form a basis for W. By Theorem 14.3 $v_i = E^*_{i+2}A_iv$ for $0 \leq i \leq D-3$ and the result follows. □

Theorem 14.5 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Then the vectors in (53) are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

$$\|E^*_{i+2}A_iv\|^2 = -\frac{k_ib_ib_{i+1}c_{i+1}c_{i+2}}{kb_1b_2}\frac{P_{i+2}(0)}{P_i(0)}\|v\|^2 \qquad (0 \leq i \leq D-3).$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E^*_2A_2E^*_2$ with eigenvalue $\tilde{\theta}_d$. The result follows in view of Theorem 13.6 and Theorem 14.3. □

Theorem 14.6 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. With respect to the basis for W given in (53) the matrix representing A is*

$$\begin{pmatrix} 0 & \omega_1 & & & & \mathbf{0} \\ c_1 & 0 & \omega_2 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \omega_{D-3} \\ \mathbf{0} & & & & c_{D-3} & 0 \end{pmatrix},$$

where

$$\omega_i = \frac{b_{i+1}c_{i+2}}{c_i}\frac{P_{i-1}(0)P_{i+2}(0)}{P_i(0)P_{i+1}(0)} \qquad (1 \leq i \leq D-3). \tag{54}$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E^*_2A_2E^*_2$ with eigenvalue $\tilde{\theta}_d$. The result follows in view of Theorem 13.7 and Theorem 14.3. □

Theorem 14.7 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Let v denote a nonzero vector in E_2^*W. Then for $0 \leq i \leq D-3$ we have*

$$E_{i+2}^* A_i v = \sum_{\substack{j=1 \\ j \neq d}}^{D-1} g_i(\theta_j) E_j v,$$

where

$$g_i = \sum_{\substack{h=0 \\ i-h \ \text{even}}}^{i} \frac{P_h(0)}{P_i(0)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h.$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $\tilde{\theta}_d$. The result follows in view of Theorem 13.5 and Theorem 14.3. □

In summary we have the following theorem.

Theorem 14.8 *With reference to Definition 8.1, assume D is even, and let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue $\tilde{\theta}_d$. Then W has dimension $D-2$. For $0 \leq i \leq D$, E_i^*W is zero if $i \in \{0, 1, D\}$ and has dimension 1 if $2 \leq i \leq D-1$. Moreover E_iW is zero if $i \in \{0, d, D\}$ and has dimension 1 if $1 \leq i \leq D-1$, $i \neq d$.*

Proof. The dimension of W is $D-2$ by Theorem 14.1. Fix an integer i $(0 \leq i \leq D)$. From Theorem 14.4 we find E_i^*W is zero if $i \in \{0, 1, D\}$ and has dimension 1 if $2 \leq i \leq D-1$. From Theorem 14.1 we find E_iW is zero if $i \in \{0, d, D\}$ and has dimension 1 if $1 \leq i \leq D-1$, $i \neq d$. □

15 The space Mv for $v \in U_\eta \quad (\tilde{\theta}_1 < \eta < \tilde{\theta}_d)$

With reference to Definition 8.1, let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$, and let η denote the corresponding eigenvalue. Assume $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. Given these assumptions we will examine the space Mv.

Theorem 15.1 *With reference to Definition 8.1, let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ and let η denote the corresponding eigenvalue. Assume $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. Then the vectors $E_1 v, E_2 v, \ldots, E_{D-1} v$ form a basis for Mv. Moreover $E_0 v = 0$, $E_D v = 0$.*

Proof. Recall $E_0, E_1, \ldots, E_D$ form a basis for M. Observe $E_0 v = 0$, $E_D v = 0$ by Lemma 11.7 so $E_1 v, E_2 v, \ldots, E_{D-1} v$ span Mv. These vectors are nonzero by Lemma 11.7 and mutually orthogonal by (6), so they are linearly independent. The result follows. □

Theorem 15.2 *[28, Theorem 11.2] With reference to Definition 8.1, let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ and let η denote the corresponding eigenvalue. Assume $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. Then the vectors $E_1 v, E_2 v, \ldots, E_{D-1} v$ are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

(i) *Assume $\eta \neq -1$. Then*

$$\|E_i v\|^2 = \frac{m_i(\theta_i - k)(\theta_i + k)(\theta_i^2 - \psi)}{|X| k b_1 (\psi - b_2)} \|v\|^2 \qquad (1 \leq i \leq D-1), \tag{55}$$

where

$$\psi = b_2 \left(1 - \frac{b_3}{1+\eta}\right). \tag{56}$$

We remark the denominator in (55) is nonzero by (56).

(ii) *Assume* $\eta = -1$. *Then*

$$\|E_i v\|^2 = \frac{m_i(k-\theta_i)(k+\theta_i)}{|X|kb_1}\|v\|^2 \qquad (1 \le i \le D-1).$$

(The scalar m_i *denotes the multiplicity of* θ_i*.)*

As we proceed in this section, we will encounter scalars of the form $P_i(\psi)$ in the denominator of some rational expressions. To make it clear these scalars are nonzero we present the following result.

Lemma 15.3 *With reference to Definition 8.1, let* η *denote a real number such that* $\eta \neq -1$ *and* $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$*, and let* ψ *be as in (56). Then (i)–(iii) hold below:*

(i) *Assume* $\tilde{\theta}_1 < \eta < -1$. *Then* $\psi > \theta_1^2$ *and* $P_i(\psi) > 0$ *for* $0 \le i \le D$.

(ii) *Assume* $-1 < \eta < \tilde{\theta}_d$. *Then* $\psi < \theta_d^2$ *and* $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) > 0$ *for* $0 \le i \le D$.

(iii) $P_i(\psi) \neq 0$ *for* $0 \le i \le D$.

Proof. (i) Combining the inequalities $\tilde{\theta}_1 < \eta < -1$ with (41), (56), and using Lemma 3.4, we routinely find $\psi > \theta_1^2$. Thus $P_i(\psi) > 0 \;\; (0 \le i \le D)$ by Lemma 6.6(i).
(ii) Combining the inequalities $-1 < \eta < \tilde{\theta}_d$ with (41), (56), and using Lemma 3.4, we routinely find $\psi < \theta_d^2$. Thus $(-1)^{\lfloor \frac{i}{2} \rfloor} P_i(\psi) > 0 \;\; (0 \le i \le D)$ by Lemma 6.6(ii),(iii).
(iii) Immediate from (i), (ii) above. □

Referring to Theorem 15.1, we now consider a second basis for Mv.

Definition 15.4 With reference to Definition 8.1, let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ and let η denote the corresponding eigenvalue. Assume $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. We define the vectors $v_0, v_1, \ldots, v_{D-2}$ as follows:

(i) Suppose $\eta \neq -1$. Then

$$v_i = \sum_{\substack{h=0 \\ i-h \text{ even}}}^{i} \frac{P_h(\psi)}{P_i(\psi)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h(A) v \qquad (0 \le i \le D-2), \tag{57}$$

where ψ is from (56).

(ii) Suppose $\eta = -1$. Then $v_i = p_i(A)v$ for $0 \le i \le D-2$.

(The polynomials p_i are from (10), and the P_i are from (15).)

Theorem 15.5 *With reference to Definition 8.1, let* v *denote a nonzero vector in* U. *Assume* v *is an eigenvector for* $E_2^* A_2 E_2^*$ *and let* η *denote the corresponding eigenvalue. Assume* $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. *Then the vectors* $v_0, v_1, \ldots, v_{D-2}$ *from Definition 15.4 form a basis for* Mv.

Proof. By Theorem 15.1 we find Mv has dimension $D-1$. By this and since A generates M, we find Mv has a basis $v, Av, \ldots, A^{D-2}v$. For $0 \le i \le D-2$ the vector v_i is contained in the span of $v, Av, \ldots, A^i v$ but not in the span of $v, Av, \ldots, A^{i-1}v$. It follows that $v_0, v_1, \ldots, v_{D-2}$ form a basis for Mv. □

With reference to Definition 15.4, we will show that the vectors $v_0, v_1, \ldots, v_{D-2}$ are mutually orthogonal and we will compute their square-norms. To do this we need the following result.

Theorem 15.6 *With reference to Definition 8.1, let* v *denote a nonzero vector in* U. *Assume* v *is an eigenvector for* $E_2^* A_2 E_2^*$ *and let* η *denote the corresponding eigenvalue. Assume* $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. *Let the vectors* $v_0, v_1, \ldots, v_{D-2}$ *be as in Definition 15.4.*

(i) *Suppose* $\eta \neq -1$. *Then for* $0 \leq i \leq D-2$ *we have*

$$v_i = \sum_{j=1}^{D-1} g_i(\theta_j) E_j v, \tag{58}$$

where

$$g_i = \sum_{\substack{h=0 \\ i-h \ \ even}}^{i} \frac{P_h(\psi)}{P_i(\psi)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h \tag{59}$$

and ψ *is from (56).*

(ii) *Suppose* $\eta = -1$. *Then*

$$v_i = \sum_{j=1}^{D-1} p_i(\theta_j) E_j v \qquad\qquad (0 \leq i \leq D-2).$$

Proof. (i) Let the integer i be given. Comparing (57), (59) we find $v_i = g_i(A)v$. Using this and (eii) we routinely obtain $v_i = \sum_{j=0}^{D} g_i(\theta_j) E_j v$. Line (58) follows since $E_0 v = 0$, $E_D v = 0$ by Lemma 11.7(i).
(ii) Similar to the proof of (i) above. □

Theorem 15.7 *With reference to Definition 8.1, let* v *denote a nonzero vector in* U. *Assume* v *is an eigenvector for* $E_2^* A_2 E_2^*$ *and let* η *denote the corresponding eigenvalue. Assume* $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. *Then the vectors* $v_0, v_1, \ldots, v_{D-2}$ *from Definition 15.4 are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

(i) *Suppose* $\eta \neq -1$. *Then*

$$\|v_i\|^2 = \frac{k_i b_i b_{i+1} c_{i+1} c_{i+2}}{k b_1 (\psi - b_2)} \frac{P_{i+2}(\psi)}{P_i(\psi)} \|v\|^2 \qquad\qquad (0 \leq i \leq D-2), \tag{60}$$

where ψ *is from (56).*

(ii) *Suppose* $\eta = -1$. *Then*

$$\|v_i\|^2 = \frac{k_i b_i b_{i+1}}{k b_1} \|v\|^2 \qquad\qquad (0 \leq i \leq D-2).$$

Proof. (i) Let the polynomials $g_0, g_1, \ldots, g_{D-2}$ be as in (59). Using in order Theorem 15.6, Theorem 15.2, and Theorem 7.7, we find that for $0 \leq i, j \leq D-2$,

$$\begin{aligned} \langle v_i, v_j \rangle &= \sum_{h=1}^{D-1} g_i(\theta_h) g_j(\theta_h) \|E_h v\|^2 \\ &= \sum_{h=1}^{D-1} g_i(\theta_h) g_j(\theta_h) \frac{m_h(\theta_h - k)(\theta_h + k)(\theta_h^2 - \psi)}{|X| k b_1 (\psi - b_2)} \|v\|^2 \\ &= \delta_{ij} \frac{k_i b_i b_{i+1} c_{i+1} c_{i+2}}{k b_1 (\psi - b_2)} \frac{P_{i+2}(\psi)}{P_i(\psi)} \|v\|^2. \end{aligned}$$

Apparently $v_0, v_1, \ldots, v_{D-2}$ are mutually orthogonal and satisfy (60).

(ii) The argument is similar to (i) above, with the p_i taking the place of the g_i and Lemma 4.1 taking the place of Theorem 7.7. □

Theorem 15.8 *With reference to Definition 8.1, let v denote a nonzero vector in U. Assume v is an eigenvector for $E_2^* A_2 E_2^*$ and let η denote the corresponding eigenvalue. Assume $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$. With respect to the basis $v_0, v_1, \ldots, v_{D-2}$ for Mv given in Definition 15.4 the matrix representing A is*

$$\left(\begin{array}{cccccc} 0 & \omega_1 & & & & \mathbf{0} \\ c_1 & 0 & \omega_2 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \omega_{D-2} \\ \mathbf{0} & & & & c_{D-2} & 0 \end{array}\right),$$

where the ω_i are as follows:

(i) *Suppose $\eta \neq -1$. Then*

$$\omega_i = \frac{b_{i+1}c_{i+2}}{c_i}\frac{P_{i-1}(\psi)P_{i+2}(\psi)}{P_i(\psi)P_{i+1}(\psi)} \qquad (1 \leq i \leq D-2), \tag{61}$$

where ψ is from (56).

(ii) *Suppose $\eta = -1$. Then*

$$\omega_i = b_{i+1} \qquad (1 \leq i \leq D-2). \tag{62}$$

Proof. (i) For $0 \leq i \leq D-2$ we define g_i as in (59). Setting $\lambda = A$ and $\theta = \psi$ in Theorem 7.5 we find

$$Ag_i(A) = c_{i+1}g_{i+1}(A) + \omega_i g_{i-1}(A) \qquad (0 \leq i \leq D-2), \tag{63}$$

where $g_{-1} = 0$, $\omega_0 = 0$, $g_{D-1} = p_{D-1}$, and the ω_i are from (61). Observe $g_i(A)v = v_i$ for $0 \leq i \leq D-2$. Applying both equations in (11) to v and recalling $Jv = 0$, $J'v = 0$, we find $p_{D-1}(A)v = 0$. Applying (63) to v, and simplifying the result using these comments, we find

$$Av_i = c_{i+1}v_{i+1} + \omega_i v_{i-1} \qquad (0 \leq i \leq D-2),$$

where $v_{-1} = 0$ and $v_{D-1} = 0$. The result follows.
(ii) The argument is similar to (i) above, with the p_i taking the place of the g_i and (12) taking the place of Theorem 7.5. □

16 The thin irreducible T-modules with endpoint 2 and local eigenvalue η $(\tilde{\theta}_1 < \eta < \tilde{\theta}_d)$

With reference to Definition 8.1, we now describe the thin irreducible T-modules with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). This section contains some of our main results. Because of this we have tried to make it as self-contained as possible.

Theorem 16.1 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). Let v denote a nonzero vector in E_2^*W. Then $W = Mv$. The vectors*

$$E_1v, E_2v, \ldots, E_{D-1}v \tag{64}$$

form a basis for W and $E_0v = 0$, $E_Dv = 0$.

Proof. To see $W = Mv$, observe that W contains v and is invariant under M so $Mv \subseteq W$. We assume W is thin with endpoint 2, so the dimension of W is at most $D-1$. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue η. Now Theorem 15.1 applies. By that theorem Mv has dimension $D-1$ so $W = Mv$. The remaining assertions of the present theorem follow in view of Theorem 15.1. □

Theorem 16.2 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η $(\tilde{\theta}_1 < \eta < \tilde{\theta}_d)$. Then the basis vectors for W from (64) are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

(i) *Suppose $\eta \neq -1$. Then*

$$\|E_i v\|^2 = \frac{m_i(\theta_i - k)(\theta_i + k)(\theta_i^2 - \psi)}{|X| k b_1 (\psi - b_2)} \|v\|^2 \qquad (1 \leq i \leq D-1),$$

where

$$\psi = b_2 \left(1 - \frac{b_3}{1+\eta}\right). \tag{65}$$

(ii) *Suppose $\eta = -1$. Then*

$$\|E_i v\|^2 = \frac{m_i(k - \theta_i)(k + \theta_i)}{|X| k b_1} \|v\|^2 \qquad (1 \leq i \leq D-1).$$

(The scalar m_i denotes the multiplicity of θ_i.)

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue η. Applying Theorem 15.2 we obtain the result. $\square$

Theorem 16.3 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η $(\tilde{\theta}_1 < \eta < \tilde{\theta}_d)$. Let v denote a nonzero vector in $E_2^* W$.*

(i) *Suppose $\eta \neq -1$. Then*

$$E_{i+2}^* A_i v = \sum_{\substack{h=0 \\ i-h \ \ even}}^{i} \frac{P_h(\psi)}{P_i(\psi)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h(A) v \qquad (0 \leq i \leq D-2),$$

where ψ is from (65).

(ii) *Suppose $\eta = -1$. Then*

$$E_{i+2}^* A_i v = p_i(A) v \qquad (0 \leq i \leq D-2).$$

(The polynomials p_i are from (10), and the P_i are from (15).)

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue η. Let the vectors $v_0, v_1, \ldots, v_{D-2}$ be as in Definition 15.4. We show $E_{i+2}^* A_i v = v_i$ for $0 \leq i \leq D-2$. Using (36) we find $A^i v$ is contained in $E_2^* W + \cdots + E_{i+2}^* W$ for $0 \leq i \leq D-2$. Also for $0 \leq i \leq D-2$, v_i is a linear combination of $v, Av, \ldots, A^i v$, so v_i is contained in $E_2^* W + \cdots + E_{i+2}^* W$. By this and since $v_0, v_1, \ldots, v_{D-2}$ are linearly independent, we find

$$v_0, v_1, \ldots, v_i \quad \text{is a basis for} \quad E_2^* W + E_3^* W + \cdots + E_{i+2}^* W \qquad (0 \leq i \leq D-2). \tag{66}$$

For the rest of this proof, fix an integer i $(0 \leq i \leq D-2)$. We show that v_i is contained in $E_{i+2}^* W$. To see this, recall $E_2^* W, \ldots, E_D^* W$ are mutually orthogonal. Therefore $E_{i+2}^* W$ is equal to the orthogonal complement of $E_2^* W + \cdots + E_{i+1}^* W$ in $E_2^* W + \cdots + E_{i+2}^* W$. Recall v_i is orthogonal to each of $v_0, v_1, \ldots, v_{i-1}$. By (66) the vectors $v_0, v_1, \ldots, v_{i-1}$ form a basis for $E_2^* W + \cdots + E_{i+1}^* W$ so v_i is orthogonal to $E_2^* W + \cdots + E_{i+1}^* W$. Apparently v_i is contained in $E_{i+2}^* W$ as desired. We show that $E_{i+2}^* A_i v = v_i$. We mentioned that the vector v_i is a linear combination of $v, Av, \ldots, A^i v$. In this combination the coefficient of $A^i v$ is $(c_1 c_2 \cdots c_i)^{-1}$ in view of Lemma 7.4(ii). Similarly $A_i v$ is a linear combination of $v, Av, \ldots, A^i v$, and in this combination the coefficient of $A^i v$ is $(c_1 c_2 \cdots c_i)^{-1}$. Apparently $A_i v - v_i$ is a linear combination of $v, Av, \ldots, A^{i-1} v$. From this and our above comments $A_i v - v_i$ is contained in $E_2^* W + \cdots + E_{i+1}^* W$ so $E_{i+2}^*(A_i v - v_i)$ is zero. We already showed that $v_i \in E_{i+2}^* W$ so $E_{i+2}^* v_i = v_i$. Now $E_{i+2}^* A_i v = v_i$ as desired. $\square$

Theorem 16.4 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). Let v denote a nonzero vector in E_2^*W. Then the vectors*

$$E^*_{i+2}A_iv \qquad\qquad (0 \leq i \leq D-2) \tag{67}$$

form a basis for W.

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue η. Let the vectors $v_0, v_1, \ldots, v_{D-2}$ be as in Definition 15.4. By Theorem 15.5 the vectors $v_0, v_1, \ldots, v_{D-2}$ form a basis for Mv. Recall $Mv = W$ by Theorem 16.1 so $v_0, v_1, \ldots, v_{D-2}$ form a basis for W. By Theorem 16.3 $v_i = E^*_{i+2}A_iv$ for $0 \leq i \leq D-2$ and the result follows. □

Theorem 16.5 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). Then the vectors in (67) are mutually orthogonal. Moreover the square-norms of these vectors are given as follows:*

(i) *Suppose $\eta \neq -1$. Then*

$$\|E^*_{i+2}A_iv\|^2 = \frac{k_ib_ib_{i+1}c_{i+1}c_{i+2}}{kb_1(\psi - b_2)}\,\frac{P_{i+2}(\psi)}{P_i(\psi)}\,\|v\|^2 \qquad\qquad (0 \leq i \leq D-2),$$

where ψ is from (65).

(ii) *Suppose $\eta = -1$. Then*

$$\|E^*_{i+2}A_iv\|^2 = \frac{k_ib_ib_{i+1}}{kb_1}\,\|v\|^2 \qquad\qquad (0 \leq i \leq D-2).$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue η. The result follows in view of Theorem 15.7 and Theorem 16.3. □

Theorem 16.6 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_D$). With respect to the basis for W given in (67) the matrix representing A is*

$$\begin{pmatrix} 0 & \omega_1 & & & & \mathbf{0} \\ c_1 & 0 & \omega_2 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \omega_{D-2} \\ \mathbf{0} & & & & c_{D-2} & 0 \end{pmatrix},$$

where the ω_i are as follows.

(i) *Suppose $\eta \neq -1$. Then*

$$\omega_i = \frac{b_{i+1}c_{i+2}}{c_i}\,\frac{P_{i-1}(\psi)P_{i+2}(\psi)}{P_i(\psi)P_{i+1}(\psi)} \qquad\qquad (1 \leq i \leq D-2), \tag{68}$$

where ψ is from (65).

(ii) *Suppose $\eta = -1$. Then*

$$\omega_i = b_{i+1} \qquad\qquad (1 \leq i \leq D-2). \tag{69}$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^*A_2E_2^*$ with eigenvalue η. The result follows in view of Theorem 15.8 and Theorem 16.3. □

Theorem 16.7 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). Let v denote a nonzero vector in E_2^*W.*

(i) *Suppose $\eta \neq -1$. Then for $0 \leq i \leq D-2$ we have*

$$E_{i+2}^* A_i v = \sum_{j=1}^{D-1} g_i(\theta_j) E_j v,$$

where

$$g_i = \sum_{\substack{h=0 \\ i-h \ \ even}}^{i} \frac{P_h(\psi)}{P_i(\psi)} \frac{k_i b_i b_{i+1}}{k_h b_h b_{h+1}} p_h$$

and ψ is from (65).

(ii) *Suppose $\eta = -1$. Then*

$$E_{i+2}^* A_i v = \sum_{j=1}^{D-1} p_i(\theta_j) E_j v \qquad (0 \leq i \leq D-2).$$

Proof. By Definition 11.9 the vector v is contained in U. Moreover v is an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue η. The result follows in view of Theorem 15.6 and Theorem 16.3. $\square$

In summary we have the following theorem.

Theorem 16.8 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η ($\tilde{\theta}_1 < \eta < \tilde{\theta}_d$). Then W has dimension $D-1$. For $0 \leq i \leq D$, E_i^*W is zero if $i \in \{0,1\}$ and has dimension 1 if $2 \leq i \leq D$. Moreover E_iW is zero if $i \in \{0, D\}$ and has dimension 1 if $1 \leq i \leq D-1$.*

Proof. The dimension of W is $D-1$ by Theorem 16.1. Fix an integer i ($0 \leq i \leq D$). From Theorem 16.4 we find E_i^*W is zero if $i \in \{0,1\}$ and has dimension 1 if $2 \leq i \leq D$. From Theorem 16.1 we find E_iW is zero if $i \in \{0, D\}$ and has dimension 1 if $1 \leq i \leq D-1$. $\square$

17 Some multiplicities

With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η. In this section we show that the isomorphism class of W as a T-module is determined by η. We show that the multiplicity with which W appears in the standard module V is at most the number of times η appears among $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}$. We investigate the case of equality.

Theorem 17.1 *With reference to Definition 8.1, let W denote a thin irreducible T-module with endpoint 2 and local eigenvalue η. Let W' denote an irreducible T-module. Then the following (i), (ii) are equivalent:*

(i) *W and W' are isomorphic as T-modules.*

(ii) *W' is thin with endpoint 2 and local eigenvalue η.*

Proof. (i)⇒(ii) Clear.
(ii)⇒(i) First observe that η satisfies one of the cases (i)–(iv) mentioned below Definition 11.9. If η satisfies case (i) or case (ii) then statement (i) of the present theorem holds by [28, Theorem 14.1]. Now assume η satisfies case (iii) or case (iv). For notational convenience set $e = 1$ if η satisfies case (iii) and set $e = 0$ if η satisfies case (iv). We display an isomorphism of T-modules from W to W'. Observe E_2^*W and E_2^*W' are both nonzero. Let v (resp. v') denote a nonzero vector in E_2^*W (resp. E_2^*W'). By Theorem 14.4 or Theorem 16.4 the vectors

$$E_{i+2}^* A_i v \qquad (0 \leq i \leq D-2-e) \tag{70}$$

form a basis for W. Similarly the vectors

$$E^*_{i+2}A_i v' \qquad (0 \leq i \leq D-2-e) \tag{71}$$

form a basis for W'. Let $\sigma : W \to W'$ denote the isomorphism of vector spaces that sends $E^*_{i+2}A_i v$ to $E^*_{i+2}A_i v'$ for $0 \leq i \leq D-2-e$. We show σ is an isomorphism of T-modules. By Theorem 14.6 or Theorem 16.6 the matrix representing A with respect to the basis (70) is equal to the matrix representing A with respect to the basis (71). It follows $\sigma A - A\sigma$ vanishes on W. From the construction we find that for $0 \leq h \leq D$, the matrix representing E^*_h with respect to the basis (70) is equal to the matrix representing E^*_h with respect to the basis (71). It follows $\sigma E^*_h - E^*_h \sigma$ vanishes on W. The algebra T is generated by $A, E^*_0, E^*_1, \ldots, E^*_D$. It follows $\sigma B - B\sigma$ vanishes on W for all $B \in T$. We now see σ is an isomorphism of T-modules from W to W'. $\square$

Lemma 17.2 *With reference to Definition 8.1, for all $\eta \in \mathbb{R}$ we have*

$$U_\eta \supseteq E^*_2 H_\eta, \tag{72}$$

where H_η denotes the subspace of V spanned by all the thin irreducible T-modules with endpoint 2 and local eigenvalue η.

Proof. Observe $E^*_2 H_\eta$ is spanned by the $E^*_2 W$, where W ranges over all the thin irreducible T-modules with endpoint 2 and local eigenvalue η. For all such W the space $E^*_2 W$ is contained in U_η by Definition 11.9. The result follows. $\square$

We remark on the dimension of the right-hand side in (72). To do this we make a definition.

Definition 17.3 With reference to Definition 8.1, and from our discussion in Section 8, the standard module V can be decomposed into an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *multiplicity with which W appears in V*, we mean the number of irreducible T-modules in the above decomposition which are isomorphic to W.

Definition 17.4 With reference to Definition 8.1, for all $\eta \in \mathbb{R}$ we let μ_η denote the multiplicity with which W appears in V, where W is a thin irreducible T-module with endpoint 2 and local eigenvalue η. If no such W exists we interpret $\mu_\eta = 0$.

Theorem 17.5 *With reference to Definition 8.1, for all $\eta \in \mathbb{R}$ the following scalars are equal:*

(i) *The scalar μ_η from Definition 17.4.*

(ii) *The dimension of $E^*_2 H_\eta$, where H_η is from Lemma 17.2.*

Moreover

$$mult_\eta \geq \mu_\eta. \tag{73}$$

Proof. We first show that μ_η is equal to the dimension of $E^*_2 H_\eta$. Observe H_η is a T-module so it is an orthogonal direct sum of irreducible T-modules. More precisely

$$H_\eta = W_1 + W_2 + \cdots + W_m \qquad \text{(orthogonal direct sum)}, \tag{74}$$

where m is a nonnegative integer, and where $W_1, W_2, \ldots, W_m$ are thin irreducible T-modules with endpoint 2 and local eigenvalue η. Apparently m is equal to μ_η. We show m is equal to the dimension of $E^*_2 H_\eta$. Applying E^*_2 to (74) we find

$$E^*_2 H_\eta = E^*_2 W_1 + E^*_2 W_2 + \cdots + E^*_2 W_m \qquad \text{(orthogonal direct sum)}. \tag{75}$$

Observe each summand on the right in (75) has dimension 1. These summands are mutually orthogonal so m is equal to the dimension of $E^*_2 H_\eta$. Now μ_η is equal to the dimension of $E^*_2 H_\eta$. We mentioned earlier that the dimension of U_η is mult_η. Combining these facts with Lemma 17.2 we obtain (73). $\square$

We are interested in the case of equality in (72) and (73). We begin with a result which is a routine consequence of Lemma 12.1.

Lemma 17.6 *[28, Lemma 14.2] With reference to Definition 8.1, choose $n \in \{1, d\}$ if D is odd, and let $n = 1$ if D is even. Let $\eta = \tilde{\theta}_n$. Then $U_\eta = E_2^* H_\eta$ and $mult_\eta = \mu_\eta$.*

Lemma 17.7 *With reference to Definition 8.1, let L denote the subspace of V spanned by the nonthin irreducible T-modules with endpoint 2. Then*

$$U = E_2^* L + \sum_{\eta \in \Phi} E_2^* H_\eta \qquad \textit{(orthogonal direct sum)}. \tag{76}$$

Proof. Let S denote the subspace of V spanned by all irreducible T-modules with endpoint 2, thin or not. Then

$$S = L + \sum_{\eta \in \Phi} H_\eta \qquad \text{(orthogonal direct sum)}. \tag{77}$$

Applying E_2^* to each term in (77) and using $E_2^* S = U$ we obtain (76). □

Theorem 17.8 *With reference to Definition 8.1, the following (i)–(iii) are equivalent:*

(i) *Equality holds in (72) for all $\eta \in \mathbb{R}$.*

(ii) *Equality holds in (73) for all $\eta \in \mathbb{R}$.*

(iii) *Every irreducible T-module with endpoint 2 is thin.*

Proof. (i)⇔(ii) Recall mult_η (resp. μ_η) is the dimension of U_η (resp. $E_2^* H_\eta$).
(i)⇒(iii) Let W denote an irreducible T-module with endpoint 2. We show W is thin. Suppose not. Then W is contained in the space L from Lemma 17.7. Observe $E_2^* W \neq 0$ since W has endpoint 2, so $E_2^* L \neq 0$. We show $E_2^* L = 0$ to get a contradiction. We assume $U_\eta = E_2^* H_\eta$ for all $\eta \in \mathbb{R}$; combining this with (40) we find $U = \sum_{\eta \in \Phi} E_2^* H_\eta$. From this and Lemma 17.7 we find $E_2^* L = 0$. We now have a contradiction and it follows W is thin.
(iii)⇒(i) There does not exist a nonthin irreducible T-module with endpoint 2, so $L = 0$. Setting $L = 0$ in (76) we find $U = \sum_{\eta \in \Phi} E_2^* H_\eta$. Combining this with (40) and Lemma 17.2 we routinely find $U_\eta = E_2^* H_\eta$ for all $\eta \in \Phi$. For any real number η that is not in Φ the spaces U_η and H_η are both 0. Now $U_\eta = E_2^* H_\eta$ for all $\eta \in \mathbb{R}$. □

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