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## Characterization and Enumeration of Toroidal $K_{3,3}$ -Subdivision-Free Graphs

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### Abstract

We describe the structure of 2-connected non-planar toroidal graphs with no  $K_{3,3}$ -subdivisions, using an appropriate substitution of planar networks into the edges of certain graphs called toroidal cores. The structural result is based on a refinement of the algorithmic results for graphs containing a fixed  $K_5$ -subdivision in [A. Gagarin and W. Kocay, "Embedding graphs containing  $K_5$ -subdivisions", Ars Combin. **64** (2002), 33-49]. It allows to recognize these graphs in linear-time and makes possible to enumerate labelled 2-connected toroidal graphs containing no  $K_{3,3}$ -subdivisions and having minimum vertex degree two or three by using an approach similar to [A. Gagarin, G. Labelle, and P. Leroux, "Counting labelled projective-planar graphs without a  $K_{3,3}$ -subdivision", submitted, arXiv:math.CO/ 0406140, (2004)].

### 1 Introduction

We use basic graph-theoretic terminology from Bondy and Murty [5] and Diestel [6], and deal with undirected simple graphs. Graph embeddings on a surface are important in VLSI design and in statistical mechanics. We are interested in non-planar graphs that can be embedded on the torus or on the projective plane. By Kuratowski's theorem [13], a graph G is non-planar if and only if it contains a subdivision of  $K_5$  or  $K_{3,3}$  (see Figure 1). In this paper we characterize (and enumerate) the 2-connected toroidal graphs with no  $K_{3,3}$ -subdivisions, following an analogous work for projective-planar graphs ([9]). The next step in this research would be to characterize toroidal and projective-planar graphs containing a  $K_{3,3}$ -subdivision (with or without a  $K_5$ -subdivision).

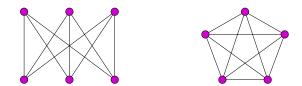


Figure 1: Minimal non-planar graphs  $K_{3,3}$  and  $K_5$ .

We assume that G is a 2-connected non-planar graph. A graph containing no  $K_{3,3}$ subdivisions will be called  $K_{3,3}$ -subdivision-free. A general recursive decomposition of nonplanar  $K_{3,3}$ -subdivision-free graphs is described in [16] and [12]. A local decomposition of non-planar graphs containing a  $K_5$ -subdivision of a special type is described in [7] and [8] (some  $K_{3,3}$ -subdivisions are allowed), that is used later in [8] to detect a projectiveplanar or toroidal graph. The results of [8] provide a toroidality criterion for graphs containing a given  $K_5$ -subdivision and avoiding certain  $K_{3,3}$ -subdivisions by examining the embeddings of  $K_5$  on the torus. The torus is an orientable surface of genus one which can be represented as a rectangle with two pairs of opposite sides identified. The graph  $K_5$  has six different embeddings on the torus shown in Figure 2. Notice that the hatched region of each of the embeddings  $E_1$  and  $E_2$  forms a single face F.

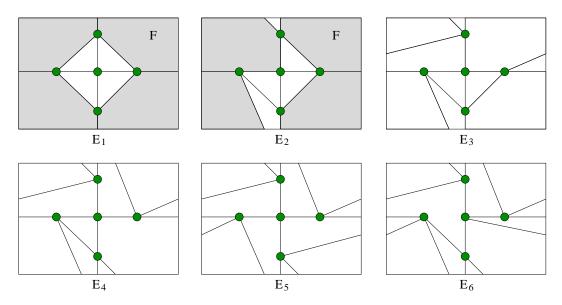


Figure 2: Embeddings of  $K_5$  on the torus.

In [9] we prove the uniqueness of the decomposition of [8] for 2-connected non-planar projective-planar graphs with no  $K_{3,3}$ -subdivisions that gives a characterization of these graphs. In the present paper we state and prove an analogous structure theorem for the class  $\mathcal{T}$  of 2-connected non-planar toroidal graphs with no  $K_{3,3}$ -subdivisions, involving certain "circular crowns" of  $K_5 \setminus e$  networks and substitution of strongly planar networks for edges. The structure theorem provides a practical algorithm to recognize the toroidal graphs with no  $K_{3,3}$ -subdivisions in linear-time. Here we use the structure theorem to

enumerate the labelled graphs in  $\mathcal{T}$  by using the counting techniques of [9] and [17] and improve known bounds for their number of edges. Finally, we enumerate the labelled graphs in  $\mathcal{T}$  having no vertex of degree two. Tables can be found at the end of the paper.

### 2 The structure theorem

A network is a connected graph N with two distinguished vertices a and b, such that the graph  $N \cup ab$  is 2-connected. The vertices a and b are called the *poles* of N. The vertices of a network that are not poles are called *internal*. A network N is strongly planar if the graph  $N \cup ab$  is planar. We denote by  $\mathcal{N}_P$  the class of strongly planar networks.

The substitution of a network N for an edge e = uv is done in the following way: choose an arbitrary orientation, say  $\vec{e} = u\vec{v}$  of the edge, identify the pole a of N with the vertex u and b with v, and disregard the orientation of e and the poles a and b. Note that both orientations of e should be considered. It is assumed that the underlying set of N is disjoint from  $\{u, v\}$ . The set of one or two resulting graphs is denoted by  $e \uparrow N$ . More generally, given a graph  $G_0$  with k edges,  $E = \{e_1, e_2, \ldots, e_k\}$ , and a sequence  $(N_1, N_2, \ldots, N_k)$  of disjoint networks, we define the composition  $G_0 \uparrow (N_1, N_2, \ldots, N_k)$ as the set of graphs that can be obtained by substituting the network  $N_j$  for the edge  $e_j$  of  $G_0$ ,  $j = 1, 2, \ldots, k$ . The graph  $G_0$  is called the core, and the  $N_i$ 's are called the components of the resulting graphs. For a class of graphs  $\mathcal{G}$  and a class of networks  $\mathcal{N}$ , we denote by  $\mathcal{G} \uparrow \mathcal{N}$  the class of graphs obtained as compositions  $G_0 \uparrow (N_1, N_2, \ldots, N_k)$  with  $G_0 \in \mathcal{G}$  and  $N_i \in \mathcal{N}$ ,  $i = 1, 2, \ldots, k$ . We say that the composition  $\mathcal{G} \uparrow \mathcal{N}$  is canonical if for any graph  $G \in \mathcal{G} \uparrow \mathcal{N}$ , there is a unique core  $G_0 \in \mathcal{G}$  and unique (up to orientation) components  $N_1, N_2, \ldots, N_k \in \mathcal{N}$  that yield G.

In [9] we prove the uniqueness of the representation  $K_5 \uparrow \mathcal{N}_P$  for  $K_{3,3}$ -subdivision-free projective-planar graphs. This gives an example of a canonical composition.

**Theorem 1 ([8, 9])** A 2-connected non-planar graph G without a  $K_{3,3}$ -subdivision is projective-planar if and only if  $G \in K_5 \uparrow \mathcal{N}_P$ . Moreover, the composition  $K_5 \uparrow \mathcal{N}_P$  is canonical.

**Definition 1** Given two  $K_5$ -graphs, the graph obtained by identifying an edge of one of the  $K_5$ 's with an edge of the other is called an *M*-graph (see Figure 3a)), and, when the edge of identification is deleted, an  $M^*$ -graph (see Figure 3b)).

**Definition 2** A network obtained from  $K_5$  by removing the edge *ab* between two poles is called a  $K_5 \setminus e$ -network. A circular crown is a graph obtained from a cycle  $C_i$ ,  $i \geq 3$ , by substituting  $K_5 \setminus e$ -networks for some edges of  $C_i$  in such a way that no pair of unsubstituted edges of  $C_i$  are adjacent (see Figure 4).

**Definition 3** A toroidal core is a graph H which is isomorphic to either  $K_5$ , an M-graph, an  $M^*$ -graph, or a circular crown. We denote by  $\mathcal{T}_C$  the class of toroidal cores.

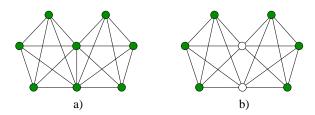


Figure 3: a) M-graph, b)  $M^*$ -graph.

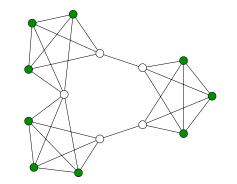


Figure 4: A circular crown obtained from  $C_5$ .

The main result of this paper is the following structure theorem. The proof is given in Section 4.

**Theorem 2** A 2-connected non-planar  $K_{3,3}$ -subdivision-free graph G is toroidal if and only if  $G \in \mathcal{T}_C \uparrow \mathcal{N}_P$ . Moreover, the composition  $\mathcal{T} = \mathcal{T}_C \uparrow \mathcal{N}_P$  is canonical.

This theorem is used in Section 5 for the enumeration of labelled graphs in  $\mathcal{T}$ . In the future we hope to use Theorem 2 to enumerate unlabelled graphs in  $\mathcal{T}$  as well.

### 3 Related known results

This section gives an overview of the structural results for toroidal graphs described in [8]. Following Diestel [6], a  $K_5$ -subdivision is denoted by  $TK_5$ . The vertices of degree 4 in  $TK_5$  are the *corners* and the vertices of degree 2 are the *inner vertices* of  $TK_5$ . For a pair of corners a and b, the path  $P_{ab}$  between a and b with all other vertices inner vertices is called a *side* of the  $K_5$ -subdivision.

Let G be a non-planar graph containing a fixed  $K_5$ -subdivision  $TK_5$ . A path p in G with one endpoint an inner vertex of  $TK_5$ , the other endpoint on a different side of  $TK_5$ , and all other vertices and edges in  $G \setminus TK_5$ , is called a *short cut* of the  $K_5$ -subdivision. A vertex  $u \in G \setminus TK_5$  is called a 3-*corner vertex* with respect to  $TK_5$  if  $G \setminus TK_5$  contains internally disjoint paths connecting u with at least three corners of the  $K_5$ -subdivision.

**Proposition 1** ([1, 7, 8]) Let G be a non-planar graph with a  $K_5$ -subdivision  $TK_5$  for which there is either a short cut or a 3-corner vertex. Then G contains a  $K_{3,3}$ -subdivision.

**Proposition 2** ([7, 8]) Let G be a 2-connected graph with a  $TK_5$  having no short cut or 3-corner vertex. Let K denote the set of corners of  $TK_5$ . Then any connected component C of  $G \setminus K$  contains inner vertices of at most one side of  $TK_5$  and C is connected in G to exactly two corners of  $TK_5$ .

Given a graph G satisfying the hypothesis of Proposition 2, a side component of  $TK_5$  is defined as the subgraph of G induced by a pair of corners a and b in K and the connected components of  $G \setminus K$  which are connected to both a and b in G. Notice that side components of G can contain  $K_{3,3}$ -subdivisions.

**Corollary 1** ([7, 8]) For a 2-connected graph G with a  $TK_5$  having no short cut or 3corner vertex, two side components of  $TK_5$  in G have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of  $TK_5$ .

Thus we see that a graph G satisfying the hypothesis of Proposition 2 can be decomposed into side components corresponding to the sides of  $TK_5$ . Each side component S contains exactly two corners a and b corresponding to a side of  $TK_5$ . If the edge ab between the corners is not in S, we can add it to S to obtain  $S \cup ab$ . Otherwise  $S \cup ab = S$ . We call  $S \cup ab$  an augmented side component of  $TK_5$ . Side components of a subdivision of an M-graph are defined by analogy with the side components of a  $K_5$ -subdivision by considering pairs of adjacent vertices of the M-graph.

A planar side component S of  $TK_5$  in G with two corners a and b is called *cylindrical* if the edge  $ab \notin S$  and the augmented side component  $S \cup ab$  is non-planar. Notice that a planar side component  $S = S \setminus ab$  is embeddable in a cylindrical section of the torus. A cylindrical section is provided by the face F of the embeddings  $E_1$  and  $E_2$  of  $K_5$  on the torus shown in Figure 2. Toroidal graphs described in [8] can contain  $K_{3,3}$ -subdivisions because of a cylindrical side component S. An example of an embedding of the cylindrical side component  $S = K_{3,3} \setminus e$  of a  $TK_5$  on the torus is shown in Figure 6 where the graph G of Figure 5 is embedded by completing the embedding  $E_1$  of  $K_5$  shown in Figure 2.

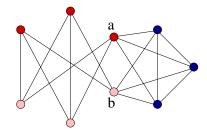


Figure 5: A toroidal graph G containing subdivisions of  $K_{3,3}$  and of  $K_5$ .

If a graph G has no  $K_{3,3}$ -subdivisions, then Proposition 2 can be applied, in virtue of Proposition 1. In this case, a result of [8] can be summarized as follows.

**Proposition 3 ([8])** A 2-connected non-planar  $K_{3,3}$ -subdivision-free graph G containing a  $K_5$ -subdivision  $TK_5$  is toroidal if and only if:

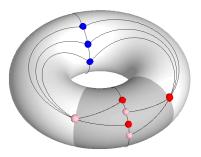


Figure 6: Embedding of the cylindrical side component  $K_{3,3} \setminus e$ .

(i) all the augmented side components of  $TK_5$  in G are planar graphs, or

(ii) nine augmented side components of  $TK_5$  in G are planar, and the remaining side component S is cylindrical, or

(iii) G contains a subdivision TM of an M-graph, and all the augmented side components of TM in G are planar.

Further analysis of the cylindrical side component S of Proposition 3(ii) will provide a proof of Theorem 2. Notice that graphs with 6 or more vertices satisfying Propositon 3 are not 3-connected. Therefore a 3-connected non-planar graph different from  $K_5$  must contain a  $K_{3,3}$ -subdivision (see also [1]).

### 4 Proof of the structure theorem

A side component S having two corners a and b can be considered as a network. We use the notation Int(S) to denote the *interior* of S, that is the subgraph  $Int(S) = S \setminus (\{a\} \cup \{b\})$ obtained by removing the two vertices a and b. A network S is called *cylindrical* if  $ab \notin S$ , S is a planar graph, but  $S \cup ab$  is non-planar. Recall that a network S is called strongly planar if  $S \cup ab$  is planar.

A block is a maximal 2-connected subgraph of a graph. A description of the blockcutvertex tree decomposition of a connected graph can be found in [6]. We consider blocks  $G_i$  having two distinguished vertices  $a_i$  and  $b_i$ . The distinguished vertices are called *poles* of the block.

**Proposition 4** Let G be a 2-connected non-planar toroidal  $K_{3,3}$ -subdivision-free graph satisfying Proposition 3(ii) with the cylindrical side component S having corners a and b. Then the block-cutvertex decomposition of S forms a path of blocks  $S_1, S_2, \ldots, S_k, k \ge 1$ , as in Figure 7, and at least one of the blocks  $S_1, S_2, \ldots, S_k, k \ge 1$ , is a cylindrical network. Moreover, every block  $S_i$ ,  $i = 1, 2, \ldots, k$ , of S is either a strongly planar network, or a cylindrical network of the form  $K_5 \setminus e \uparrow (N_1, N_2, \ldots, N_9)$ , where  $e = a_i b_i$  and the  $N_j$ 's are strongly planar networks.

*Proof.* Since G is 2-connected, each cut-vertex of S belongs to exactly two blocks and lies on the corresponding side  $P_{ab}$  of  $TK_5$ . Therefore the blocks of S form a path as in Figure 7.

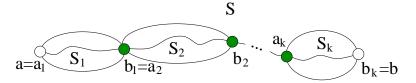


Figure 7: Block-cutvertex decomposition for the cylindrical side component S.

Suppose each block  $S_i$  of S, i = 1, 2, ..., k, remains planar when the edge  $a_i b_i$  is added to  $S_i$ . Then, clearly,  $S \cup ab$  remains planar as well. Hence the fact that S is cylindrical implies that at least one of the blocks  $S_i$ , i = 1, 2, ..., k, is itself a cylindrical network.

Suppose a block  $S_m$ ,  $1 \le m \le k$ , of S is cylindrical. Then, by Kuratowski's theorem,  $S_m \cup a_m b_m$  contains a  $K_5$ -subdivision  $TK'_5$ . Clearly,  $a_m b_m \in TK'_5$ ,  $TK'_5$  has no short-cut or 3-corner vertex in G and  $a_m$  and  $b_m$  are two corners of the  $TK'_5$ . The edge  $a_m b_m$  of  $TK'_5$  can be replaced by a path  $P_{a_m b_m}$  in  $G \setminus \text{Int}(S_m)$  and we can decompose G into the side components of  $TK'_5$ .

Since G is toroidal and the side component  $G \setminus \text{Int}(S_m)$  of  $TK'_5$  is cylindrical, all the other side components of  $TK'_5$  in G must be strongly planar networks by Proposition 3(ii). Therefore  $S_m$  is a cylindrical network of the form  $K_5 \setminus e \uparrow (N_1, N_2, \ldots, N_9)$ , with  $e = a_m b_m$  and  $N_j \in \mathcal{N}_P, j = 1, 2, \ldots, 9$ .

Now we are ready to prove the structure Theorem 2 using Propositions 3 and 4.

**Proof of Theorem 2.** (Sufficiency) Suppose G is a graph in  $\mathcal{T}_C \uparrow \mathcal{N}_P$ , i.e.  $G = H \uparrow (N_1, N_2, \ldots, N_k)$ , where H is a toroidal core having k edges and  $N_i$ 's,  $i = 1, 2, \ldots, k$ , are strongly planar networks. If  $H = K_5$  or H = M, then G can be decomposed into the side components of  $TK_5$  or TM respectively and the augmented side components are planar graphs. Therefore, by Proposition 3(i) or 3(iii) respectively, G is toroidal  $K_{3,3}$ -subdivision-free.

If  $H = M^*$  or H is a circular crown, then we can choose a  $K_5 \setminus e$ -network N in Hand find a path  $P_{ab}$  connecting a and b in the complementary part  $H \setminus \text{Int}(N)$ . This determines a subdivision  $TK_5$  in G such that nine augmented side components of  $TK_5$ in G are planar, and the remaining side component S defined by the corners a and b of  $TK_5$  is cylindrical. Therefore, by Proposition 3(ii), G is toroidal  $K_{3,3}$ -subdivision-free.

(Necessity and Uniqueness) Let G be a 2-connected non-planar  $K_{3,3}$ -subdivision-free toroidal graph G. By Kuratowski's theorem, G contains a  $K_5$ -subdivision  $TK_5$ . Let us prove that  $G \in \mathcal{T}_C \uparrow \mathcal{N}_P$  by using Propositions 3 and 4. The fact that the composition  $H \uparrow \mathcal{N}_P, H \in \mathcal{T}_C$ , of G is canonical will follow from the uniqueness of the sets of corner vertices in Proposition 3.

Clearly, the sets of graphs corresponding to the cases (i), (ii) and (iii) of Proposition 3 are mutually disjoint. Suppose G contains a subdivision  $TK_5$  or TM and all the augmented side components of  $TK_5$  or TM, respectively, in G are planar graphs as in Proposition 3(i, iii). Then  $G = K_5 \uparrow (N_1, N_2, \ldots, N_{10})$  or  $G = M \uparrow (N_1, N_2, \ldots, N_{19})$ , respectively,  $K_5, M \in \mathcal{T}_C$  and all the  $N_i$ 's are in  $\mathcal{N}_P$ . The uniqueness of the decomposition in cases (i) and (iii) of Proposition 3 can be proved by analogy with Theorem 3 in [9]: the set of corners of the  $K_5$ -subdivision in Proposition 3(i) and the set of corners of the M-graph subdivision in Proposition 3(ii) are uniquely defined. This covers toroidal cores  $K_5$  and the M-graph.

Suppose S is the unique cylindrical side component of  $TK_5$  in G as in Proposition 3(ii). Notice that  $G \setminus \text{Int}(S)$  itself is a cylindrical network of the form  $K_5 \setminus e \uparrow (N_1, N_2, \ldots, N_9)$ , where e = ab and  $N_j \in \mathcal{N}_P, j = 1, 2, \ldots, 9$ . By Proposition 4, the block-cutvertex decomposition of S forms a path of blocks  $S_1, S_2, \ldots, S_k, k \ge 1$ , as in Figure 7, and at least one of the blocks  $S_1, S_2, \ldots, S_k, k \geq 1$ , is a cylindrical network. In this path we can regroup maximal series of consecutive strongly planar networks into single strongly planar networks so that at most one strongly planar network N' is separating two cylindrical networks in the resulting path, and the poles of the strongly planar network N' are uniquely defined by maximality. By Proposition 4, the cylindrical networks in the path are of the form  $K_5 \setminus e \uparrow (N_1, N_2, \ldots, N_9)$ , where  $N_j \in \mathcal{N}_P, j = 1, 2, \ldots, 9$ , and the corners a' and b', e = a'b', are uniquely defined with respect to the corresponding  $K_5$ -subdivision  $TK'_5$  in G. Therefore the unique set of corners completely defines a toroidal core  $M^*$  or a circular crown H having k edges and the set of corresponding strongly planar networks  $N_1, N_2, \ldots, N_k$ , such that  $G = M^* \uparrow (N_1, N_2, \ldots, N_{18})$  or  $G = H \uparrow (N_1, N_2, \ldots, N_k)$ , respectively. 

Theorems 1 and 2 imply that a projective-planar graph with no  $K_{3,3}$ -subdivisions is toroidal. However an arbitrary projective-planar graph can be non-toroidal. The characterizations of Theorems 1 and 2 can be used to detect projective-planar or toroidal graphs with no  $K_{3,3}$ -subdivisions in linear time. The implementation of this algorithm can be derived from [8] by using a breadth-first or depth-first search technique for the decomposition and by doing a linear-time planarity testing. The linear-time complexity follows from the linear-time complexity of the decomposition and from the fact that each vertex of the initial graph can appear in at most 7 different components.

A corollary to Euler's formula for the plane says that a planar graph with  $n \geq 3$  vertices can have at most 3n - 6 edges (see, for example, [5] and [6]). Let us state this for 2-connected planar graphs with n vertices and m edges as follows:

$$m \le \begin{cases} 3n-5 & \text{if } n=2\\ 3n-6 & \text{if } n \ge 3 \end{cases}.$$
 (1)

In fact, m = 3n - 5 = 1 if n = 2. The generalized Euler formula (see, for example, [15]) implies that a toroidal graph G with n vertices can have up to 3n edges. An arbitrary graph G without a  $K_{3,3}$ -subdivision is known to have at most 3n - 5 edges (see [1]). The following proposition shows that toroidal graphs with no  $K_{3,3}$ -subdivisions satisfy a stronger relation, which is analogous to planar graphs.

**Proposition 5** The number m of edges of a non-planar  $K_{3,3}$ -subdivision-free toroidal n-vertex graph G satisfies  $m \leq 3n-5$  if n=5 or 8, and

$$m \le 3n - 6, \text{ if } n \ge 6 \text{ and } n \ne 8.$$

Proof. Clearly, toroidal graphs satisfying Theorem 2 also satisfy Proposition 3. By Proposition 3(i, ii), each side component  $S_i$  of  $TK_5$  in G, i = 1, 2, ..., 10, satisfies the condition (1) with  $n = n_i$ , the number of vertices, and  $m = m_i$ , the number of edges of  $S_i$ , i = 1, 2, ..., 10. Since each corner of  $TK_5$  is in precisely 4 side components, we have  $\sum_{i=1}^{10} n_i = n + 15$  and we obtain, by summing these 10 inequalities,

$$m = \sum_{i=1}^{10} m_i \le \begin{cases} 3\sum_{i=1}^{10} n_i - 50 = 3(n+15) - 50 = 3n-5 & \text{if } n = 5\\ \\ 3\sum_{i=1}^{10} n_i - 51 = 3(n+15) - 51 = 3n-6 & \text{if } n \ge 6 \end{cases},$$

since n = 5 iff  $n_i = 2, i = 1, 2, ..., 10$ , and  $n \ge 6$  if and only if at least one  $n_j \ge 3$ , j = 1, 2, ..., 10.

Similarly, by Proposition 3(*iii*), each side component  $S_i$  of TM in G, i = 1, 2, ..., 19, satisfies the condition (1) with  $n = n_i$ , the number of vertices, and  $m = m_i$ , the number of edges of  $S_i$ , i = 1, 2, ..., 19. Since 2 vertices of TM are in precisely 7 side components, 6 vertices of TM are in precisely 4 side components, and all the other vertices of G are in a unique side component, we have  $\sum_{i=1}^{19} n_i = n + 30$  and we obtain, by summing these 19 inequalities,

$$m = \sum_{i=1}^{19} m_i \le \begin{cases} 3\sum_{i=1}^{19} n_i - 95 = 3(n+30) - 95 = 3n-5 & \text{if } n = 8\\ \\ 3\sum_{i=1}^{19} n_i - 96 = 3(n+30) - 96 = 3n-6 & \text{if } n \ge 9 \end{cases},$$

since n = 8 iff  $n_i = 2, i = 1, 2, ..., 19$ , and  $n \ge 9$  if and only if at least one  $n_j \ge 3$ , j = 1, 2, ..., 19.

An analogous result for the projective-planar graphs can be found in [9]. Also note that Corollary 8.3.5 of [6] implies that graphs with no  $K_5$ -minors can have at most 3n-6 edges.

# 5 Counting labelled $K_{3,3}$ -subdivision-free toroidal graphs

Now let us consider the question of the labelled enumeration of toroidal graphs with no  $K_{3,3}$ -subdivisions according to the numbers of vertices and edges. First, we review some basic notions and terminology of labelled enumeration together with the counting methods and technique used in [17, 9]. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). For example, see [2], [11], [14], or [18].

By a *labelled* graph, we mean a simple graph G = (V, E) where the set of vertices V = V(G) is itself the set of labels and the labelling function is the identity function. V is called the *underlying set* of G. An edge e of G then consists of an unordered pair e = uv of elements of V and E = E(G) denotes the set of edges of G. If W is another

set and  $\sigma: V \to W$  is a bijection, then any graph G = (V, E) on V, can be transformed into a graph  $G' = \sigma(G) = (W, \sigma(E))$ , where  $\sigma(E) = \{\sigma(e) = \sigma(u)\sigma(v) \mid e \in E\}$ . We say that G' is obtained from G by vertex relabelling and that  $\sigma$  is a graph isomorphism  $G \to G'$ . An unlabelled graph is then seen as an isomorphism class  $\gamma$  of labelled graphs. We write  $\gamma = \gamma(G)$  if  $\gamma$  is the isomorphism class of G. By the number of ways to label an unlabelled graph  $\gamma(G)$ , where G = (V, E), we mean the number of distinct graphs G'on the underlying set V which are isomorphic to G. Recall that this number is given by  $n!/|\operatorname{Aut}(G)|$ , where n = |V| and  $\operatorname{Aut}(G)$  denotes the automorphism group of G.

A species of graphs is a class of labelled graphs which is closed under vertex relabellings. Thus any class  $\mathcal{G}$  of unlabelled graphs gives rise to a species, also denoted by  $\mathcal{G}$ , by taking the set union of the isomorphism classes in  $\mathcal{G}$ . For any species  $\mathcal{G}$  of graphs, we introduce its (exponential) generating function  $\mathcal{G}(x, y)$  as the formal power series

$$\mathcal{G}(x,y) = \sum_{n\geq 0} g_n(y) \frac{x^n}{n!}, \quad \text{with} \quad g_n(y) = \sum_{m\geq 0} g_{n,m} y^m, \tag{3}$$

where  $g_{n,m}$  is the number of graphs in  $\mathcal{G}$  with m edges over a given set of vertices  $V_n$  of size n. Here y is a formal variable which acts as an edge counter. For example, for the species  $\mathcal{G} = K = \{K_n\}_{n\geq 0}$  of complete graphs, we have

$$K(x,y) = \sum_{n \ge 0} y^{\binom{n}{2}} x^n / n!,$$
(4)

while for the species  $\mathcal{G} = \mathcal{G}_a$  of all simple graphs, we have  $\mathcal{G}_a(x, y) = K(x, 1 + y)$ .

A species of graphs is *molecular* if it contains only one isomorphism class. For a molecular species  $\gamma = \gamma(G)$ , where G has n vertices and m edges, we have  $\gamma(x, y) = \frac{y^m n!}{|\operatorname{Aut}(G)|} x^n / n! = y^m x^n / |\operatorname{Aut}(G)|$ . For example,

$$K_5(x,y) = \frac{x^5 y^{10}}{5!}.$$
(5)

Also, for the graphs M and  $M^*$  described in Section 2, we have

$$M(x,y) = 280 \frac{x^8 y^{19}}{8!}, \quad M^*(x,y) = 280 \frac{x^8 y^{18}}{8!}, \tag{6}$$

since  $|Aut(M)| = |Aut(M^*)| = 144$ .

For the enumeration of networks, we consider that the poles a and b are not labelled, or, in other words, that only the internal vertices form the underlying set. Hence the generating function of a class (or species)  $\mathcal{N}$  of networks is defined by

$$\mathcal{N}(x,y) = \sum_{n \ge 0} \nu_n(y) \frac{x^n}{n!}, \text{ with } \nu_n(y) = \sum_{m \ge 0} \nu_{n,m} y^m, \tag{7}$$

where  $\nu_{n,m}$  is the number of networks in  $\mathcal{N}$  with m edges and a given set of internal vertices  $V_n$  of size n. For example, we have

$$(K_5 \setminus e)(x, y) = \frac{x^3 y^9}{3!},$$
 (8)

A species  $\mathcal{N}$  of networks is called *symmetric* if for any  $\mathcal{N}$ -network N (i.e. N in  $\mathcal{N}$ ), the *opposite network*  $\tau \cdot N$ , obtained by interchanging the poles a and b, is also in  $\mathcal{N}$ . Examples of symmetric species of networks are the classes  $\mathcal{N}_P$ , of strongly planar networks, and  $\mathcal{R}$ , of series-parallel networks (see [17, 9]).

**Lemma 1** (T. Walsh [17, 9]) Let  $\mathcal{G}$  be a species of graphs and  $\mathcal{N}$  be a symmetric species of networks such that the composition  $\mathcal{G} \uparrow \mathcal{N}$  is canonical. Then the following generating function identity holds:

$$(\mathcal{G}\uparrow\mathcal{N})(x,y) = \mathcal{G}(x,\mathcal{N}(x,y)).$$
(9)

By Theorem 2 and Lemma 1, we have the following proposition.

**Proposition 6** The generating function  $\mathcal{T}(x, y)$  of labelled non-planar  $K_{3,3}$ -subdivision-free toroidal graphs is given by

$$\mathcal{T}(x,y) = (\mathcal{T}_C \uparrow \mathcal{N}_P)(x,y) = \mathcal{T}_C(x,\mathcal{N}_P(x,y)), \tag{10}$$

where  $\mathcal{T}_C$  denotes the class of toroidal cores (see Definition 3).

Let P denote the species of 2-connected planar graphs. Then the generating function of  $\mathcal{N}_P$ , the associated class of strongly planar networks, is given by

$$\mathcal{N}_P(x,y) = (1+y)\frac{2}{x^2}\frac{\partial}{\partial y}P(x,y) - 1 \tag{11}$$

(see [17, 9]). Methods for computing the generating function P(x, y) of labelled 2connected planar graphs are described in [3] and [4]. Formula (11) can then be used to compute  $\mathcal{N}_P(x, y)$ . Therefore there remains only to compute the generating function  $\mathcal{T}_C(x, y)$  for toroidal cores. Recall that  $\mathcal{T}_C = K_5 + M + M^* + CC$ , where CC denotes the class of circular crowns. Circular crowns can be enumerated as follows using matching polynomials.

**Proposition 7** The mixed generating series CC(x, y) of circular crowns is given by

$$CC(x,y) = -\frac{12x^4y^9 + 12x^5y^{10} + x^8y^{18} + 72\ln(1 - \frac{x^4y^9}{6} - \frac{x^5y^{10}}{6})}{144}.$$
 (12)

*Proof*. Recall that a *matching*  $\mu$  of a finite graph G is a set of disjoint edges of G. We define the *matching polynomial* of G as

$$M_G(y) = \sum_{\mu \in \mathcal{M}(G)} y^{|\mu|},\tag{13}$$

where  $\mathcal{M}(G)$  denotes the set of matchings of G. In particular, the matching polynomials  $U_n(y)$  and  $T_n(y)$  for paths and cycles of size n are well known (see [10]). They are closely related to the Chebyshev polynomials. To be precise, let  $P_n$  denote the path graph (V, E)

with  $V = [n] = \{1, 2, ..., n\}$  and  $E = \{\{i, i+1\} | i = 1, 2, ..., n-1\}$  and  $C_n$  denote the cycle graph with V = [n] and  $E = \{\{i, i+1 \pmod{n}\} | i = 1, 2, ..., n\}$ . Then we have

$$U_n(y) = \sum_{\mu \in \mathcal{M}(P_n)} y^{|\mu|}, \quad T_n(y) = \sum_{\mu \in \mathcal{M}(C_n)} y^{|\mu|}.$$
 (14)

The dichotomy caused by the membership of the edge  $\{n-1, n\}$  in the matchings of the path  $P_n$  leads to the recurrence relation

$$U_n(y) = yU_{n-2}(y) + U_{n-1}(y), (15)$$

for  $n \ge 2$ , with  $U_0(y) = U_1(y) = 1$ . It follows that the ordinary generating function of the matching polynomials  $U_n(y)$  is rational. In fact, it is easily seen that

$$\sum_{n \ge 0} U_n(y) x^n = \frac{1}{1 - x - yx^2}.$$
(16)

Now, the dichotomy caused by the membership of the edge  $\{1, n\}$  in the matchings of the cycle  $C_n$  leads to the relation

$$T_n(y) = yU_{n-2}(y) + U_n(y),$$
(17)

for  $n \geq 3$ . It is then a simple matter, using (16) and (17) to compute their ordinary generating function, denoted by G(x, y). We find

$$G(x,y) = \sum_{n \ge 3} T_n(y) x^n = \frac{x^3(1+3y+yx+2y^2x)}{1-x-yx^2}.$$
 (18)

In fact, we also need to consider the homogeneous matchings polynomials

$$T_n(y,z) = z^n T_n(\frac{y}{z}) = \sum_{\mu \in \mathcal{M}(C_n)} y^{|\mu|} z^{n-|\mu|},$$
(19)

where the variable z marks the edges which are not selected in the matchings, whose generating function  $G(x, y, z) = \sum_{n \ge 3} T_n(y, z) x^n$  is given by

$$G(x, y, z) = G(xz, \frac{y}{z}) = \frac{x^3 z^2 (z + 3y + xyz + 2xy^2)}{1 - xz - x^2 yz}.$$
(20)

We now introduce the species BC of pairs  $(c, \mu)$ , where c is a cycle of length  $n \ge 3$  and  $\mu$  is a matching of c, with weight  $y^{|\mu|}z^{n-|\mu|}$ . Since there are  $\frac{(n-1)!}{2}$  non-oriented cycles on a set of size  $n \ge 3$ , and all these cycles admit the same homogeneous matching polynomial

 $T_n(y, z)$ , the mixed generating function of labelled *BC*-structures is

$$BC(x, y, z) = \sum_{n \ge 3} \frac{(n-1)!}{2} T_n(y, z) \frac{x^n}{n!}$$
$$= \frac{1}{2} \sum_{n \ge 3} T_n(y, z) \frac{x^n}{n}$$
$$= \frac{1}{2} \int_0^x \frac{1}{t} G(t, y, z) dt$$
$$= -\frac{2xz + 2x^2 zy + x^2 z^2 + 2\ln(1 - xz - x^2 yz)}{4}.$$
(21)

Notice that in a circular crown, the unsubstituted edges are not adjacent, by definition, and hence form a matching of the underlying cycle, while the substituted edges are replaced by  $K_5 \$ e-networks. We can thus write

$$CC = BC \uparrow_z (K_5 \backslash e), \tag{22}$$

where the notation  $\uparrow_z$  means that only the edges marked by z are replaced by  $K_5 \setminus e$ networks. Moreover the decomposition (22) is canonical and we have

$$CC(x,y) = BC(x,y,(K_5 \setminus e)(x,y)),$$
(23)

which implies (12) using (8).

=

A substitution of the generating function  $\mathcal{N}_P(x, y)$  of (11) counting the strongly planar networks for the variable y in (6), (5), and (12) gives the generating function for labelled 2-connected non-planar toroidal graphs with no  $K_{3,3}$ -subdivision, i.e.

$$\mathcal{T}(x,y) = K_5(x,\mathcal{N}_P(x,y)) + M(x,\mathcal{N}_P(x,y)) + M^*(x,\mathcal{N}_P(x,y)) + CC(x,\mathcal{N}_P(x,y)).$$
(24)

Notice that the term  $K_5(x, \mathcal{N}_P(x, y))$  in (24) also enumerates non-planar 2-connected  $K_{3,3}$ -subdivision-free projective-planar graphs and that corresponding tables are given in [9]. Here we present the computational results just for labelled graphs in  $\mathcal{T}$  that are not projective-planar. Numerical results are presented in Tables 1 and 2, where  $\mathcal{T}(x,y) - K_5(x,\mathcal{N}_P(x,y)) = \sum_{n\geq 8} \sum_m t_{n,m} x^n y^m / n!$  and  $t_n = \sum_m t_{n,m}$  count labelled non-projective-planar graphs in  $\mathcal{T}$ .

The homeomorphically irreducible non-projective-planar graphs in  $\mathcal{T}$ , i.e. the graphs having no vertex of degree two, can be counted by using several methods described in detail in Section 4 of [9]. We used the approach of Proposition 8 of [9] to obtain the numerical data presented in Tables 3 and 4 for labelled homeomorphically irreducible graphs in  $\mathcal{T}$  that are not projective-planar.

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| n  | m  | $t_{n,m}$    | n  | m  | $t_{n,m}$         | n  | m  | $t_{n,m}$             |
|----|----|--------------|----|----|-------------------|----|----|-----------------------|
| 8  | 18 | 280          | 13 | 23 | 1838008972800     | 15 | 25 | 5973529161600000      |
| 8  | 19 | 280          | 13 | 24 | 12383684913600    | 15 | 26 | 60679359861120000     |
| 9  | 19 | 50400        | 13 | 25 | 36576568828800    | 15 | 27 | 280619124786000000    |
| 9  | 20 | 93240        | 13 | 26 | 61986597472800    | 15 | 28 | 785755439324856000    |
| 9  | 21 | 47880        | 13 | 27 | 66199273620480    | 15 | 29 | 1496142328612932000   |
| 10 | 20 | 5292000      | 13 | 28 | 46419992138520    | 15 | 30 | 2068477720590481200   |
| 10 | 21 | 15044400     | 13 | 29 | 22180672954440    | 15 | 31 | 2175937397296462800   |
| 10 | 22 | 15510600     | 13 | 30 | 7737403073400     | 15 | 32 | 1810128996903427200   |
| 10 | 23 | 5972400      | 13 | 31 | 2053743892200     | 15 | 33 | 1223242124356652400   |
| 10 | 24 | 239400       | 13 | 32 | 348540192000      | 15 | 34 | 673154380612513800    |
| 11 | 21 | 426888000    | 13 | 33 | 27935107200       | 15 | 35 | 293316332440131000    |
| 11 | 22 | 1700899200   | 14 | 24 | 107217190080000   | 15 | 36 | 96295664217753000     |
| 11 | 23 | 2724044400   | 14 | 25 | 896474952172800   | 15 | 37 | 22260497063805000     |
| 11 | 24 | 2136842400   | 14 | 26 | 3359265613704000  | 15 | 38 | 3218036781960000      |
| 11 | 25 | 773295600    | 14 | 27 | 7460402644094400  | 15 | 39 | 218263565520000       |
| 11 | 26 | 94386600     | 14 | 28 | 10948159170748800 | 16 | 26 | 322570574726400000    |
| 11 | 27 | 7900200      | 14 | 29 | 11253868616390400 | 16 | 27 | 3914073525922560000   |
| 12 | 22 | 29455272000  | 14 | 30 | 8467602606022560  | 16 | 28 | 21877169871997440000  |
| 12 | 23 | 155542464000 | 14 | 31 | 4876995169606560  | 16 | 29 | 75157668529175232000  |
| 12 | 24 | 348414066000 | 14 | 32 | 2222245323698400  | 16 | 30 | 178928606393593056000 |
| 12 | 25 | 424294516800 | 14 | 33 | 785187373370400   | 16 | 31 | 316283670286218835200 |
| 12 | 26 | 297599563800 | 14 | 34 | 197208318106800   | 16 | 32 | 435483254883942064800 |
| 12 | 27 | 118905448200 | 14 | 35 | 31064455422000    | 16 | 33 | 484253520685973438400 |
| 12 | 28 | 27683548200  | 14 | 36 | 2294786894400     | 16 | 34 | 445576710488584474800 |
| 12 | 29 | 4821201000   |    |    |                   | 16 | 35 | 341998556200139638800 |
| 12 | 30 | 410810400    | 1  |    |                   | 16 | 36 | 216864722075241240000 |
|    |    |              | -  |    |                   | 16 | 37 | 111029372376938215200 |
|    |    |              |    |    |                   | 16 | 38 | 44479356838490574000  |
|    |    |              |    |    |                   | 16 | 39 | 13374653821603074000  |
|    |    |              |    |    |                   | 16 | 40 | 2831094029443680000   |
|    |    |              |    |    |                   | 16 | 41 | 375386906774880000    |

Table 1: The number of labelled non-planar non-projective-planar toroidal 2-connected graphs without a  $K_{3,3}$ -subdivision (having *n* vertices and *m* edges).

16 42

23417178744960000

| n  | $t_n$                            |
|----|----------------------------------|
| 8  | 560                              |
| 9  | 191520                           |
| 10 | 42058800                         |
| 11 | 7864256400                       |
| 12 | 1407126890400                    |
| 13 | 257752421166240                  |
| 14 | 50607986220311520                |
| 15 | 10995419195575214400             |
| 16 | 2692773804667509763200           |
| 17 | 747221542837742897724800         |
| 18 | 233698171655650029030743040      |
| 19 | 81472765051132560093387934080    |
| 20 | 31268587126068905034073041062400 |

Table 2: The number of labelled non-planar non-projective-planar toroidal 2-connected  $K_{3,3}$ -subdivision-free graphs (having *n* vertices).

| n  | m  | $t_{n,m}$    | n  | m  | $t_{n,m}$         | n  | m  | $t_{n,m}$              |
|----|----|--------------|----|----|-------------------|----|----|------------------------|
| 8  | 18 | 280          | 14 | 26 | 6054048000        | 16 | 29 | 5811886080000          |
| 8  | 19 | 280          | 14 | 27 | 285751065600      | 16 | 30 | 621544891968000        |
| 9  | 19 | 5040         | 14 | 28 | 3361812854400     | 16 | 31 | 11935943091072000      |
| 10 | 20 | 25200        | 14 | 29 | 17840270448000    | 16 | 32 | 101350194001056000     |
| 10 | 22 | 226800       | 14 | 30 | 55133382704400    | 16 | 33 | 499371733276416000     |
| 10 | 23 | 466200       | 14 | 31 | 108994658572800   | 16 | 34 | 1611221546830896000    |
| 10 | 24 | 239400       | 14 | 32 | 141179453415000   | 16 | 35 | 3605404135132800000    |
| 11 | 23 | 10256400     | 14 | 33 | 118498240060200   | 16 | 36 | 5738963267481444000    |
| 11 | 24 | 30492000     | 14 | 34 | 61801664324400    | 16 | 37 | 6540526990277280000    |
| 11 | 25 | 43520400     | 14 | 35 | 18158435895600    | 16 | 38 | 5293490794557966000    |
| 11 | 26 | 31185000     | 14 | 36 | 2294786894400     | 16 | 39 | 2967845927880834000    |
| 11 | 27 | 7900200      | 15 | 28 | 1961511552000     | 16 | 40 | 1095216458944608000    |
| 12 | 24 | 189604800    | 15 | 29 | 57537672192000    | 16 | 41 | 239190441890400000     |
| 12 | 25 | 1079416800   | 15 | 30 | 557188343712000   | 16 | 42 | 23417178744960000      |
| 12 | 26 | 3044487600   | 15 | 31 | 2827950253128000  | 17 | 31 | 3903916528512000       |
| 12 | 27 | 5080614000   | 15 | 32 | 8936155496268000  | 17 | 32 | 174648084811200000     |
| 12 | 28 | 4776294600   | 15 | 33 | 18886100303070000 | 17 | 33 | 2606052624215040000    |
| 12 | 29 | 2261536200   | 15 | 34 | 27395286118200000 | 17 | 34 | 20178959825344320000   |
| 12 | 30 | 410810400    | 15 | 35 | 27296971027326000 | 17 | 35 | 97287841256493888000   |
| 13 | 25 | 1686484800   | 15 | 36 | 18324093378591000 | 17 | 36 | 319780940570307216000  |
| 13 | 26 | 22875652800  | 15 | 37 | 7906712877063000  | 17 | 37 | 751384930811218704000  |
| 13 | 27 | 126680954400 | 15 | 38 | 1978851858984000  | 17 | 38 | 1292496613555066920000 |
| 13 | 28 | 382608626400 | 15 | 39 | 218263565520000   | 17 | 39 | 1642597679422623924000 |
| 13 | 29 | 700723623600 |    |    |                   | 17 | 40 | 1539140405659676820000 |
| 13 | 30 | 788388400800 |    |    |                   | 17 | 41 | 1049167407329489448000 |
| 13 | 31 | 525156231600 |    |    |                   | 17 | 42 | 505608857591934096000  |
| 13 | 32 | 188324136000 |    |    |                   | 17 | 43 | 163183484418946992000  |
| 13 | 33 | 27935107200  | J  |    |                   | 17 | 44 | 31635477128166912000   |
|    |    |              |    |    |                   | 17 | 45 | 2784602773016064000    |

Table 3: The number of labelled non-planar non-projective-planar toroidal 2-connected  $K_{3,3}$ -subdivision-free graphs with no vertex of degree 2 (having *n* vertices and *m* edges).

| n  | $t_n$                          |
|----|--------------------------------|
| 8  | 560                            |
| 9  | 5040                           |
| 10 | 957600                         |
| 11 | 123354000                      |
| 12 | 16842764400                    |
| 13 | 2764379217600                  |
| 14 | 527554510282800                |
| 15 | 114387072405606000             |
| 16 | 27728561968887780000           |
| 17 | 7418031804967840056000         |
| 18 | 2167306256125914230527200      |
| 19 | 685709965521372865035362400    |
| 20 | 233306923207078035272369412000 |

Table 4: The number of labelled non-planar non-projective-planar toroidal 2-connected  $K_{3,3}$ -subdivision-free graphs with no vertex of degree 2 (having *n* vertices).