# Characterization and Enumeration of Toroidal $K_{3,3}$-Subdivision-Free Graphs 

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#### Abstract

We describe the structure of 2-connected non-planar toroidal graphs with no $K_{3,3}$-subdivisions, using an appropriate substitution of planar networks into the edges of certain graphs called toroidal cores. The structural result is based on a refinement of the algorithmic results for graphs containing a fixed $K_{5}$-subdivision in [A. Gagarin and W. Kocay, "Embedding graphs containing $K_{5}$-subdivisions", Ars Combin. 64 (2002), 33-49]. It allows to recognize these graphs in linear-time and makes possible to enumerate labelled 2 -connected toroidal graphs containing no $K_{3,3}$-subdivisions and having minimum vertex degree two or three by using an approach similar to [A. Gagarin, G. Labelle, and P. Leroux, "Counting labelled projective-planar graphs without a $K_{3,3}$-subdivision", submitted, arXiv:math.CO/ 0406140, (2004)].


## 1 Introduction

We use basic graph-theoretic terminology from Bondy and Murty [5] and Diestel [6], and deal with undirected simple graphs. Graph embeddings on a surface are important in VLSI design and in statistical mechanics. We are interested in non-planar graphs that can be embedded on the torus or on the projective plane. By Kuratowski's theorem [13, a graph $G$ is non-planar if and only if it contains a subdivision of $K_{5}$ or $K_{3,3}$ (see Figure 1). In this paper we characterize (and enumerate) the 2-connected toroidal graphs with no $K_{3,3}$-subdivisions, following an analogous work for projective-planar graphs ( 9$]$ ). The next step in this research would be to characterize toroidal and projective-planar graphs containing a $K_{3,3}$-subdivision (with or without a $K_{5}$-subdivision).


Figure 1: Minimal non-planar graphs $K_{3,3}$ and $K_{5}$.

We assume that $G$ is a 2 -connected non-planar graph. A graph containing no $K_{3,3^{-}}$ subdivisions will be called $K_{3,3}$-subdivision-free. A general recursive decomposition of nonplanar $K_{3,3}$-subdivision-free graphs is described in 16] and 12]. A local decomposition of non-planar graphs containing a $K_{5}$-subdivision of a special type is described in [7] and [8] (some $K_{3,3}$-subdivisions are allowed), that is used later in [8] to detect a projectiveplanar or toroidal graph. The results of [8] provide a toroidality criterion for graphs containing a given $K_{5}$-subdivision and avoiding certain $K_{3,3}$-subdivisions by examining the embeddings of $K_{5}$ on the torus. The torus is an orientable surface of genus one which can be represented as a rectangle with two pairs of opposite sides identified. The graph $K_{5}$ has six different embeddings on the torus shown in Figure 2. Notice that the hatched region of each of the embeddings $E_{1}$ and $E_{2}$ forms a single face $F$.


Figure 2: Embeddings of $K_{5}$ on the torus.
In [9] we prove the uniqueness of the decomposition of [8] for 2-connected non-planar projective-planar graphs with no $K_{3,3}$-subdivisions that gives a characterization of these graphs. In the present paper we state and prove an analogous structure theorem for the class $\mathcal{T}$ of 2 -connected non-planar toroidal graphs with no $K_{3,3}$-subdivisions, involving certain "circular crowns" of $K_{5} \backslash e$ networks and substitution of strongly planar networks for edges. The structure theorem provides a practical algorithm to recognize the toroidal graphs with no $K_{3,3}$-subdivisions in linear-time. Here we use the structure theorem to
enumerate the labelled graphs in $\mathcal{T}$ by using the counting techniques of [9] and [17] and improve known bounds for their number of edges. Finally, we enumerate the labelled graphs in $\mathcal{T}$ having no vertex of degree two. Tables can be found at the end of the paper.

## 2 The structure theorem

A network is a connected graph $N$ with two distinguished vertices $a$ and $b$, such that the graph $N \cup a b$ is 2 -connected. The vertices $a$ and $b$ are called the poles of $N$. The vertices of a network that are not poles are called internal. A network $N$ is strongly planar if the graph $N \cup a b$ is planar. We denote by $\mathcal{N}_{P}$ the class of strongly planar networks.

The substitution of a network $N$ for an edge $e=u v$ is done in the following way: choose an arbitrary orientation, say $\vec{e}=\overrightarrow{u v}$ of the edge, identify the pole $a$ of $N$ with the vertex $u$ and $b$ with $v$, and disregard the orientation of $e$ and the poles $a$ and $b$. Note that both orientations of $e$ should be considered. It is assumed that the underlying set of $N$ is disjoint from $\{u, v\}$. The set of one or two resulting graphs is denoted by $e \uparrow N$. More generally, given a graph $G_{0}$ with $k$ edges, $E=\left\{e_{1}, e_{2}, \ldots, e_{k}\right\}$, and a sequence $\left(N_{1}, N_{2}, \ldots, N_{k}\right)$ of disjoint networks, we define the composition $G_{0} \uparrow\left(N_{1}, N_{2}, \ldots, N_{k}\right)$ as the set of graphs that can be obtained by substituting the network $N_{j}$ for the edge $e_{j}$ of $G_{0}, j=1,2, \ldots, k$. The graph $G_{0}$ is called the core, and the $N_{i}$ 's are called the components of the resulting graphs. For a class of graphs $\mathcal{G}$ and a class of networks $\mathcal{N}$, we denote by $\mathcal{G} \uparrow \mathcal{N}$ the class of graphs obtained as compositions $G_{0} \uparrow\left(N_{1}, N_{2}, \ldots, N_{k}\right)$ with $G_{0} \in \mathcal{G}$ and $N_{i} \in \mathcal{N}, i=1,2, \ldots, k$. We say that the composition $\mathcal{G} \uparrow \mathcal{N}$ is canonical if for any graph $G \in \mathcal{G} \uparrow \mathcal{N}$, there is a unique core $G_{0} \in \mathcal{G}$ and unique (up to orientation) components $N_{1}, N_{2}, \ldots, N_{k} \in \mathcal{N}$ that yield $G$.

In [9] we prove the uniqueness of the representation $K_{5} \uparrow \mathcal{N}_{P}$ for $K_{3,3}$-subdivision-free projective-planar graphs. This gives an example of a canonical composition.

Theorem 1 ([8, [9]) A 2-connected non-planar graph $G$ without a $K_{3,3}$-subdivision is projective-planar if and only if $G \in K_{5} \uparrow \mathcal{N}_{P}$. Moreover, the composition $K_{5} \uparrow \mathcal{N}_{P}$ is canonical.

Definition 1 Given two $K_{5}$-graphs, the graph obtained by identifying an edge of one of the $K_{5}$ 's with an edge of the other is called an $M$-graph (see Figure 3a)), and, when the edge of identification is deleted, an $M^{*}$-graph (see Figure 3 b )).

Definition 2 A network obtained from $K_{5}$ by removing the edge $a b$ between two poles is called a $K_{5} \backslash e$-network. A circular crown is a graph obtained from a cycle $C_{i}, i \geq$ 3, by substituting $K_{5} \backslash e$-networks for some edges of $C_{i}$ in such a way that no pair of unsubstituted edges of $C_{i}$ are adjacent (see Figure 4).

Definition 3 A toroidal core is a graph $H$ which is isomorphic to either $K_{5}$, an $M$-graph, an $M^{*}$-graph, or a circular crown. We denote by $\mathcal{T}_{C}$ the class of toroidal cores.


Figure 3: a) $M$-graph, b) $M^{*}$-graph.


Figure 4: A circular crown obtained from $C_{5}$.

The main result of this paper is the following structure theorem. The proof is given in Section 4.

Theorem 2 A 2-connected non-planar $K_{3,3}$-subdivision-free graph $G$ is toroidal if and only if $G \in \mathcal{T}_{C} \uparrow \mathcal{N}_{P}$. Moreover, the composition $\mathcal{T}=\mathcal{T}_{C} \uparrow \mathcal{N}_{P}$ is canonical.

This theorem is used in Section 5 for the enumeration of labelled graphs in $\mathcal{T}$. In the future we hope to use Theorem 2 to enumerate unlabelled graphs in $\mathcal{T}$ as well.

## 3 Related known results

This section gives an overview of the structural results for toroidal graphs described in [8]. Following Diestel [6] a $K_{5}$-subdivision is denoted by $T K_{5}$. The vertices of degree 4 in $T K_{5}$ are the corners and the vertices of degree 2 are the inner vertices of $T K_{5}$. For a pair of corners $a$ and $b$, the path $P_{a b}$ between $a$ and $b$ with all other vertices inner vertices is called a side of the $K_{5}$-subdivision.

Let $G$ be a non-planar graph containing a fixed $K_{5}$-subdivision $T K_{5}$. A path $p$ in $G$ with one endpoint an inner vertex of $T K_{5}$, the other endpoint on a different side of $T K_{5}$, and all other vertices and edges in $G \backslash T K_{5}$, is called a short cut of the $K_{5}$-subdivision. A vertex $u \in G \backslash T K_{5}$ is called a 3-corner vertex with respect to $T K_{5}$ if $G \backslash T K_{5}$ contains internally disjoint paths connecting $u$ with at least three corners of the $K_{5}$-subdivision.

Proposition $1\left([\mathbf{1}, \mathbf{7},[\mathbf{8}])\right.$ Let $G$ be a non-planar graph with a $K_{5}$-subdivision $T K_{5}$ for which there is either a short cut or a 3 -corner vertex. Then $G$ contains a $K_{3,3}$-subdivision.

Proposition $2([7, ~ 8])$ Let $G$ be a 2-connected graph with a $T K_{5}$ having no short cut or 3 -corner vertex. Let $K$ denote the set of corners of $T K_{5}$. Then any connected component $C$ of $G \backslash K$ contains inner vertices of at most one side of $T K_{5}$ and $C$ is connected in $G$ to exactly two corners of $T K_{5}$.

Given a graph $G$ satisfying the hypothesis of Proposition 2, a side component of $T K_{5}$ is defined as the subgraph of $G$ induced by a pair of corners $a$ and $b$ in $K$ and the connected components of $G \backslash K$ which are connected to both $a$ and $b$ in $G$. Notice that side components of $G$ can contain $K_{3,3}$-subdivisions.

Corollary 1 ( $[\mathbf{7}, \mathbf{8}]$ ) For a 2-connected graph $G$ with a $T K_{5}$ having no short cut or 3corner vertex, two side components of $T K_{5}$ in $G$ have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of $T K_{5}$.

Thus we see that a graph $G$ satisfying the hypothesis of Proposition 2 can be decomposed into side components corresponding to the sides of $T K_{5}$. Each side component $S$ contains exactly two corners $a$ and $b$ corresponding to a side of $T K_{5}$. If the edge $a b$ between the corners is not in $S$, we can add it to $S$ to obtain $S \cup a b$. Otherwise $S \cup a b=S$. We call $S \cup a b$ an augmented side component of $T K_{5}$. Side components of a subdivision of an $M$-graph are defined by analogy with the side components of a $K_{5}$-subdivision by considering pairs of adjacent vertices of the $M$-graph.

A planar side component $S$ of $T K_{5}$ in $G$ with two corners $a$ and $b$ is called cylindrical if the edge $a b \notin S$ and the augmented side component $S \cup a b$ is non-planar. Notice that a planar side component $S=S \backslash a b$ is embeddable in a cylindrical section of the torus. A cylindrical section is provided by the face $F$ of the embeddings $E_{1}$ and $E_{2}$ of $K_{5}$ on the torus shown in Figure 2. Toroidal graphs described in [8] can contain $K_{3,3^{3}}$-subdivisions because of a cylindrical side component $S$. An example of an embedding of the cylindrical side component $S=K_{3,3} \backslash e$ of a $T K_{5}$ on the torus is shown in Figure 6 where the graph $G$ of Figure 5 is embedded by completing the embedding $E_{1}$ of $K_{5}$ shown in Figure 2.


Figure 5: A toroidal graph $G$ containing subdivisions of $K_{3,3}$ and of $K_{5}$.
If a graph $G$ has no $K_{3,3}$-subdivisions, then Proposition 2 can be applied, in virtue of Proposition In In this case, a result of [8] can be summarized as follows.

Proposition 3 ([8]) A 2-connected non-planar $K_{3,3}$-subdivision-free graph $G$ containing a $K_{5}$-subdivision $T K_{5}$ is toroidal if and only if:


Figure 6: Embedding of the cylindrical side component $K_{3,3} \backslash e$.
(i) all the augmented side components of $T K_{5}$ in $G$ are planar graphs, or
(ii) nine augmented side components of $T K_{5}$ in $G$ are planar, and the remaining side component $S$ is cylindrical, or
(iii) $G$ contains a subdivision $T M$ of an $M$-graph, and all the augmented side components of $T M$ in $G$ are planar.

Further analysis of the cylindrical side component $S$ of Proposition 3(ii) will provide a proof of Theorem 2. Notice that graphs with 6 or more vertices satisfying Propositon 3 are not 3 -connected. Therefore a 3-connected non-planar graph different from $K_{5}$ must contain a $K_{3,3}$-subdivision (see also [1]).

## 4 Proof of the structure theorem

A side component $S$ having two corners $a$ and $b$ can be considered as a network. We use the notation $\operatorname{Int}(S)$ to denote the interior of $S$, that is the subgraph $\operatorname{Int}(S)=S \backslash(\{a\} \cup\{b\})$ obtained by removing the two vertices $a$ and $b$. A network $S$ is called cylindrical if $a b \notin S$, $S$ is a planar graph, but $S \cup a b$ is non-planar. Recall that a network $S$ is called strongly planar if $S \cup a b$ is planar.

A block is a maximal 2-connected subgraph of a graph. A description of the blockcutvertex tree decomposition of a connected graph can be found in [6]. We consider blocks $G_{i}$ having two distinguished vertices $a_{i}$ and $b_{i}$. The distinguished vertices are called poles of the block.

Proposition 4 Let $G$ be a 2-connected non-planar toroidal $K_{3,3}$-subdivision-free graph satisfying Proposition 3(ii) with the cylindrical side component $S$ having corners a and $b$. Then the block-cutvertex decomposition of $S$ forms a path of blocks $S_{1}, S_{2}, \ldots, S_{k}, k \geq 1$, as in Figure 7, and at least one of the blocks $S_{1}, S_{2}, \ldots, S_{k}, k \geq 1$, is a cylindrical network. Moreover, every block $S_{i}, i=1,2, \ldots, k$, of $S$ is either a strongly planar network, or a cylindrical network of the form $K_{5} \backslash e \uparrow\left(N_{1}, N_{2}, \ldots, N_{9}\right)$, where $e=a_{i} b_{i}$ and the $N_{j}$ 's are strongly planar networks.

Proof. Since $G$ is 2-connected, each cut-vertex of $S$ belongs to exactly two blocks and lies on the corresponding side $P_{a b}$ of $T K_{5}$. Therefore the blocks of $S$ form a path as in Figure 7.


Figure 7: Block-cutvertex decomposition for the cylindrical side component $S$.

Suppose each block $S_{i}$ of $S, i=1,2, \ldots, k$, remains planar when the edge $a_{i} b_{i}$ is added to $S_{i}$. Then, clearly, $S \cup a b$ remains planar as well. Hence the fact that $S$ is cylindrical implies that at least one of the blocks $S_{i}, i=1,2, \ldots, k$, is itself a cylindrical network.

Suppose a block $S_{m}, 1 \leq m \leq k$, of $S$ is cylindrical. Then, by Kuratowski's theorem, $S_{m} \cup a_{m} b_{m}$ contains a $K_{5}$-subdivision $T K_{5}^{\prime}$. Clearly, $a_{m} b_{m} \in T K_{5}^{\prime}, T K_{5}^{\prime}$ has no short-cut or 3 -corner vertex in $G$ and $a_{m}$ and $b_{m}$ are two corners of the $T K_{5}^{\prime}$. The edge $a_{m} b_{m}$ of $T K_{5}^{\prime}$ can be replaced by a path $P_{a_{m} b_{m}}$ in $G \backslash \operatorname{Int}\left(S_{m}\right)$ and we can decompose $G$ into the side components of $T K_{5}^{\prime}$.

Since $G$ is toroidal and the side component $G \backslash \operatorname{Int}\left(S_{m}\right)$ of $T K_{5}^{\prime}$ is cylindrical, all the other side components of $T K_{5}^{\prime}$ in $G$ must be strongly planar networks by Proposition 3(ii). Therefore $S_{m}$ is a cylindrical network of the form $K_{5} \backslash e \uparrow\left(N_{1}, N_{2}, \ldots, N_{9}\right)$, with $e=a_{m} b_{m}$ and $N_{j} \in \mathcal{N}_{P}, j=1,2, \ldots, 9$.

Now we are ready to prove the structure Theorem 2 using Propositions 3 and 4.
Proof of Theorem 2. (Sufficiency) Suppose $G$ is a graph in $\mathcal{T}_{C} \uparrow \mathcal{N}_{P}$, i.e. $G=H \uparrow$ $\left(N_{1}, N_{2}, \ldots, N_{k}\right)$, where $H$ is a toroidal core having $k$ edges and $N_{i}$ 's, $i=1,2, \ldots k$, are strongly planar networks. If $H=K_{5}$ or $H=M$, then $G$ can be decomposed into the side components of $T K_{5}$ or $T M$ respectively and the augmented side components are planar graphs. Therefore, by Proposition 3(i) or 3(iii) respectively, $G$ is toroidal $K_{3,3^{-}}$ subdivision-free.

If $H=M^{*}$ or $H$ is a circular crown, then we can choose a $K_{5} \backslash e$-network $N$ in $H$ and find a path $P_{a b}$ connecting $a$ and $b$ in the complementary part $H \backslash \operatorname{Int}(N)$. This determines a subdivision $T K_{5}$ in $G$ such that nine augmented side components of $T K_{5}$ in $G$ are planar, and the remaining side component $S$ defined by the corners $a$ and $b$ of $T K_{5}$ is cylindrical. Therefore, by Proposition 3(ii), $G$ is toroidal $K_{3,3}$-subdivision-free.
(Necessity and Uniqueness) Let $G$ be a 2-connected non-planar $K_{3,3}$-subdivision-free toroidal graph $G$. By Kuratowski's theorem, $G$ contains a $K_{5}$-subdivision $T K_{5}$. Let us prove that $G \in \mathcal{T}_{C} \uparrow \mathcal{N}_{P}$ by using Propositions 3 and 4. The fact that the composition $H \uparrow \mathcal{N}_{P}, H \in \mathcal{T}_{C}$, of $G$ is canonical will follow from the uniqueness of the sets of corner vertices in Proposition 3.

Clearly, the sets of graphs corresponding to the cases (i), (ii) and (iii) of Proposition 3 are mutually disjoint. Suppose $G$ contains a subdivision $T K_{5}$ or $T M$ and all the augmented side components of $T K_{5}$ or $T M$, respectively, in $G$ are planar graphs as in Proposition 3(i, iii). Then $G=K_{5} \uparrow\left(N_{1}, N_{2}, \ldots, N_{10}\right)$ or $G=M \uparrow\left(N_{1}, N_{2}, \ldots, N_{19}\right)$, respectively, $K_{5}, M \in \mathcal{T}_{C}$ and all the $N_{j}$ 's are in $\mathcal{N}_{P}$. The uniqueness of the decomposition
in cases (i) and (iii) of Proposition 3 can be proved by analogy with Theorem 3 in [9]: the set of corners of the $K_{5}$-subdivision in Proposition 3(i) and the set of corners of the $M$-graph subdivision in Proposition 3(iii) are uniquely defined. This covers toroidal cores $K_{5}$ and the $M$-graph.

Suppose $S$ is the unique cylindrical side component of $T K_{5}$ in $G$ as in Proposition 3(ii). Notice that $G \backslash \operatorname{Int}(S)$ itself is a cylindrical network of the form $K_{5} \backslash e \uparrow\left(N_{1}, N_{2}, \ldots, N_{9}\right)$, where $e=a b$ and $N_{j} \in \mathcal{N}_{P}, j=1,2, \ldots, 9$. By Proposition 4, the block-cutvertex decomposition of $S$ forms a path of blocks $S_{1}, S_{2}, \ldots, S_{k}, k \geq 1$, as in Figure 7, and at least one of the blocks $S_{1}, S_{2}, \ldots, S_{k}, k \geq 1$, is a cylindrical network. In this path we can regroup maximal series of consecutive strongly planar networks into single strongly planar networks so that at most one strongly planar network $N^{\prime}$ is separating two cylindrical networks in the resulting path, and the poles of the strongly planar network $N^{\prime}$ are uniquely defined by maximality. By Proposition 4, the cylindrical networks in the path are of the form $K_{5} \backslash e \uparrow\left(N_{1}, N_{2}, \ldots, N_{9}\right)$, where $N_{j} \in \mathcal{N}_{P}, j=1,2, \ldots, 9$, and the corners $a^{\prime}$ and $b^{\prime}, e=a^{\prime} b^{\prime}$, are uniquely defined with respect to the corresponding $K_{5}$-subdivision $T K_{5}^{\prime}$ in $G$. Therefore the unique set of corners completely defines a toroidal core $M^{*}$ or a circular crown $H$ having $k$ edges and the set of corresponding strongly planar networks $N_{1}, N_{2}, \ldots, N_{k}$, such that $G=M^{*} \uparrow\left(N_{1}, N_{2}, \ldots, N_{18}\right)$ or $G=H \uparrow\left(N_{1}, N_{2}, \ldots, N_{k}\right)$, respectively.

Theorems 1 and 2 imply that a projective-planar graph with no $K_{3,3}$-subdivisions is toroidal. However an arbitrary projective-planar graph can be non-toroidal. The characterizations of Theorems 1 and 2 can be used to detect projective-planar or toroidal graphs with no $K_{3,3}$-subdivisions in linear time. The implementation of this algorithm can be derived from [8] by using a breadth-first or depth-first search technique for the decomposition and by doing a linear-time planarity testing. The linear-time complexity follows from the linear-time complexity of the decomposition and from the fact that each vertex of the initial graph can appear in at most 7 different components.

A corollary to Euler's formula for the plane says that a planar graph with $n \geq 3$ vertices can have at most $3 n-6$ edges (see, for example, [5] and [6]). Let us state this for 2 -connected planar graphs with $n$ vertices and $m$ edges as follows:

$$
m \leq \begin{cases}3 n-5 & \text { if } n=2  \tag{1}\\ 3 n-6 & \text { if } n \geq 3\end{cases}
$$

In fact, $m=3 n-5=1$ if $n=2$. The generalized Euler formula (see, for example, [15]) implies that a toroidal graph $G$ with $n$ vertices can have up to $3 n$ edges. An arbitrary graph $G$ without a $K_{3,3}$-subdivision is known to have at most $3 n-5$ edges (see [1). The following proposition shows that toroidal graphs with no $K_{3,3}$-subdivisions satisfy a stronger relation, which is analogous to planar graphs.

Proposition 5 The number $m$ of edges of a non-planar $K_{3,3}$-subdivision-free toroidal $n$-vertex graph $G$ satisfies $m \leq 3 n-5$ if $n=5$ or 8 , and

$$
\begin{equation*}
m \leq 3 n-6, \text { if } n \geq 6 \text { and } n \neq 8 \tag{2}
\end{equation*}
$$

Proof. Clearly, toroidal graphs satisfying Theorem 2 also satisfy Proposition 3. By Proposition $3(i, i i)$, each side component $S_{i}$ of $T K_{5}$ in $G, i=1,2, \ldots, 10$, satisfies the condition (11) with $n=n_{i}$, the number of vertices, and $m=m_{i}$, the number of edges of $S_{i}$, $i=1,2, \ldots, 10$. Since each corner of $T K_{5}$ is in precisely 4 side components, we have $\sum_{i=1}^{10} n_{i}=n+15$ and we obtain, by summing these 10 inequalities,

$$
m=\sum_{i=1}^{10} m_{i} \leq \begin{cases}3 \sum_{i=1}^{10} n_{i}-50=3(n+15)-50=3 n-5 & \text { if } n=5 \\ 3 \sum_{i=1}^{10} n_{i}-51=3(n+15)-51=3 n-6 & \text { if } n \geq 6\end{cases}
$$

since $n=5$ iff $n_{i}=2, i=1,2, \ldots, 10$, and $n \geq 6$ if and only if at least one $n_{j} \geq 3$, $j=1,2, \ldots, 10$.

Similarly, by Proposition 3(iii), each side component $S_{i}$ of $T M$ in $G, i=1,2, \ldots, 19$, satisfies the condition (11) with $n=n_{i}$, the number of vertices, and $m=m_{i}$, the number of edges of $S_{i}, i=1,2, \ldots, 19$. Since 2 vertices of $T M$ are in precisely 7 side components, 6 vertices of $T M$ are in precisely 4 side components, and all the other vertices of $G$ are in a unique side component, we have $\sum_{i=1}^{19} n_{i}=n+30$ and we obtain, by summing these 19 inequalities,

$$
m=\sum_{i=1}^{19} m_{i} \leq \begin{cases}3 \sum_{i=1}^{19} n_{i}-95=3(n+30)-95=3 n-5 & \text { if } n=8 \\ 3 \sum_{i=1}^{19} n_{i}-96=3(n+30)-96=3 n-6 & \text { if } n \geq 9\end{cases}
$$

since $n=8$ iff $n_{i}=2, i=1,2, \ldots, 19$, and $n \geq 9$ if and only if at least one $n_{j} \geq 3$, $j=1,2, \ldots, 19$.

An analogous result for the projective-planar graphs can be found in 9. Also note that Corollary 8.3.5 of [6] implies that graphs with no $K_{5}$-minors can have at most $3 n-6$ edges.

## 5 Counting labelled $K_{3,3}$-subdivision-free toroidal graphs

Now let us consider the question of the labelled enumeration of toroidal graphs with no $K_{3,3}$-subdivisions according to the numbers of vertices and edges. First, we review some basic notions and terminology of labelled enumeration together with the counting methods and technique used in [17, 9]. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). For example, see [2], 11], 14], or 18].

By a labelled graph, we mean a simple graph $G=(V, E)$ where the set of vertices $V=V(G)$ is itself the set of labels and the labelling function is the identity function. $V$ is called the underlying set of $G$. An edge $e$ of $G$ then consists of an unordered pair $e=u v$ of elements of $V$ and $E=E(G)$ denotes the set of edges of $G$. If $W$ is another
set and $\sigma: V \stackrel{\sim}{\rightarrow} W$ is a bijection, then any graph $G=(V, E)$ on $V$, can be transformed into a graph $G^{\prime}=\sigma(G)=(W, \sigma(E))$, where $\sigma(E)=\{\sigma(e)=\sigma(u) \sigma(v) \mid e \in E\}$. We say that $G^{\prime}$ is obtained from $G$ by vertex relabelling and that $\sigma$ is a graph isomorphism $G \xrightarrow{\sim} G^{\prime}$. An unlabelled graph is then seen as an isomorphism class $\gamma$ of labelled graphs. We write $\gamma=\gamma(G)$ if $\gamma$ is the isomorphism class of $G$. By the number of ways to label an unlabelled graph $\gamma(G)$, where $G=(V, E)$, we mean the number of distinct graphs $G^{\prime}$ on the underlying set $V$ which are isomorphic to $G$. Recall that this number is given by $n!/|\operatorname{Aut}(G)|$, where $n=|V|$ and $\operatorname{Aut}(G)$ denotes the automorphism group of $G$.

A species of graphs is a class of labelled graphs which is closed under vertex relabellings. Thus any class $\mathcal{G}$ of unlabelled graphs gives rise to a species, also denoted by $\mathcal{G}$, by taking the set union of the isomorphism classes in $\mathcal{G}$. For any species $\mathcal{G}$ of graphs, we introduce its (exponential) generating function $\mathcal{G}(x, y)$ as the formal power series

$$
\begin{equation*}
\mathcal{G}(x, y)=\sum_{n \geq 0} g_{n}(y) \frac{x^{n}}{n!}, \quad \text { with } \quad g_{n}(y)=\sum_{m \geq 0} g_{n, m} y^{m} \tag{3}
\end{equation*}
$$

where $g_{n, m}$ is the number of graphs in $\mathcal{G}$ with $m$ edges over a given set of vertices $V_{n}$ of size $n$. Here $y$ is a formal variable which acts as an edge counter. For example, for the species $\mathcal{G}=K=\left\{K_{n}\right\}_{n \geq 0}$ of complete graphs, we have

$$
\begin{equation*}
K(x, y)=\sum_{n \geq 0} y^{\binom{n}{2}} x^{n} / n! \tag{4}
\end{equation*}
$$

while for the species $\mathcal{G}=\mathcal{G}_{a}$ of all simple graphs, we have $\mathcal{G}_{a}(x, y)=K(x, 1+y)$.
A species of graphs is molecular if it contains only one isomorphism class. For a molecular species $\gamma=\gamma(G)$, where $G$ has $n$ vertices and $m$ edges, we have $\gamma(x, y)=$ $\frac{y^{m} n!}{|\operatorname{Aut}(G)|} x^{n} / n!=y^{m} x^{n} /|\operatorname{Aut}(G)|$. For example,

$$
\begin{equation*}
K_{5}(x, y)=\frac{x^{5} y^{10}}{5!} \tag{5}
\end{equation*}
$$

Also, for the graphs $M$ and $M^{*}$ described in Section 2, we have

$$
\begin{equation*}
M(x, y)=280 \frac{x^{8} y^{19}}{8!}, \quad M^{*}(x, y)=280 \frac{x^{8} y^{18}}{8!} \tag{6}
\end{equation*}
$$

since $|\operatorname{Aut}(M)|=\left|\operatorname{Aut}\left(M^{*}\right)\right|=144$.
For the enumeration of networks, we consider that the poles $a$ and $b$ are not labelled, or, in other words, that only the internal vertices form the underlying set. Hence the generating function of a class (or species) $\mathcal{N}$ of networks is defined by

$$
\begin{equation*}
\mathcal{N}(x, y)=\sum_{n \geq 0} \nu_{n}(y) \frac{x^{n}}{n!}, \quad \text { with } \quad \nu_{n}(y)=\sum_{m \geq 0} \nu_{n, m} y^{m} \tag{7}
\end{equation*}
$$

where $\nu_{n, m}$ is the number of networks in $\mathcal{N}$ with $m$ edges and a given set of internal vertices $V_{n}$ of size $n$. For example, we have

$$
\begin{equation*}
\left(K_{5} \backslash e\right)(x, y)=\frac{x^{3} y^{9}}{3!} \tag{8}
\end{equation*}
$$

A species $\mathcal{N}$ of networks is called symmetric if for any $\mathcal{N}$-network $N$ (i.e. $N$ in $\mathcal{N}$ ), the opposite network $\tau \cdot N$, obtained by interchanging the poles $a$ and $b$, is also in $\mathcal{N}$. Examples of symmetric species of networks are the classes $\mathcal{N}_{P}$, of strongly planar networks, and $\mathcal{R}$, of series-parallel networks (see [17, 9]).

Lemma 1 (T. Walsh [17, [9]) Let $\mathcal{G}$ be a species of graphs and $\mathcal{N}$ be a symmetric species of networks such that the composition $\mathcal{G} \uparrow \mathcal{N}$ is canonical. Then the following generating function identity holds:

$$
\begin{equation*}
(\mathcal{G} \uparrow \mathcal{N})(x, y)=\mathcal{G}(x, \mathcal{N}(x, y)) \tag{9}
\end{equation*}
$$

By Theorem 2 and Lemma 1, we have the following proposition.
Proposition 6 The generating function $\mathcal{T}(x, y)$ of labelled non-planar $K_{3,3}$-subdivisionfree toroidal graphs is given by

$$
\begin{equation*}
\mathcal{T}(x, y)=\left(\mathcal{T}_{C} \uparrow \mathcal{N}_{P}\right)(x, y)=\mathcal{T}_{C}\left(x, \mathcal{N}_{P}(x, y)\right) \tag{10}
\end{equation*}
$$

where $\mathcal{T}_{C}$ denotes the class of toroidal cores (see Definition 3).
Let $P$ denote the species of 2-connected planar graphs. Then the generating function of $\mathcal{N}_{P}$, the associated class of strongly planar networks, is given by

$$
\begin{equation*}
\mathcal{N}_{P}(x, y)=(1+y) \frac{2}{x^{2}} \frac{\partial}{\partial y} P(x, y)-1 \tag{11}
\end{equation*}
$$

(see [17, [9]). Methods for computing the generating function $P(x, y)$ of labelled 2connected planar graphs are described in [3] and [4]. Formula (11) can then be used to compute $\mathcal{N}_{P}(x, y)$. Therefore there remains only to compute the generating function $\mathcal{T}_{C}(x, y)$ for toroidal cores. Recall that $\mathcal{T}_{C}=K_{5}+M+M^{*}+C C$, where $C C$ denotes the class of circular crowns. Circular crowns can be enumerated as follows using matching polynomials.

Proposition 7 The mixed generating series $C C(x, y)$ of circular crowns is given by

$$
\begin{equation*}
C C(x, y)=-\frac{12 x^{4} y^{9}+12 x^{5} y^{10}+x^{8} y^{18}+72 \ln \left(1-\frac{x^{4} y^{9}}{6}-\frac{x^{5} y^{10}}{6}\right)}{144} \tag{12}
\end{equation*}
$$

Proof. Recall that a matching $\mu$ of a finite graph $G$ is a set of disjoint edges of $G$. We define the matching polynomial of $G$ as

$$
\begin{equation*}
M_{G}(y)=\sum_{\mu \in \mathcal{M}(G)} y^{|\mu|} \tag{13}
\end{equation*}
$$

where $\mathcal{M}(G)$ denotes the set of matchings of $G$. In particular, the matching polynomials $U_{n}(y)$ and $T_{n}(y)$ for paths and cycles of size $n$ are well known (see [10]). They are closely related to the Chebyshev polynomials. To be precise, let $P_{n}$ denote the path graph ( $V, E$ )
with $V=[n]=\{1,2, \ldots, n\}$ and $E=\{\{i, i+1\} \mid i=1,2, \ldots, n-1\}$ and $C_{n}$ denote the cycle graph with $V=[n]$ and $E=\{\{i, i+1(\bmod n)\} \mid i=1,2, \ldots, n\}$. Then we have

$$
\begin{equation*}
U_{n}(y)=\sum_{\mu \in \mathcal{M}\left(P_{n}\right)} y^{|\mu|}, \quad T_{n}(y)=\sum_{\mu \in \mathcal{M}\left(C_{n}\right)} y^{|\mu|} \tag{14}
\end{equation*}
$$

The dichotomy caused by the membership of the edge $\{n-1, n\}$ in the matchings of the path $P_{n}$ leads to the recurrence relation

$$
\begin{equation*}
U_{n}(y)=y U_{n-2}(y)+U_{n-1}(y) \tag{15}
\end{equation*}
$$

for $n \geq 2$, with $U_{0}(y)=U_{1}(y)=1$. It follows that the ordinary generating function of the matching polynomials $U_{n}(y)$ is rational. In fact, it is easily seen that

$$
\begin{equation*}
\sum_{n \geq 0} U_{n}(y) x^{n}=\frac{1}{1-x-y x^{2}} \tag{16}
\end{equation*}
$$

Now, the dichotomy caused by the membership of the edge $\{1, n\}$ in the matchings of the cycle $C_{n}$ leads to the relation

$$
\begin{equation*}
T_{n}(y)=y U_{n-2}(y)+U_{n}(y) \tag{17}
\end{equation*}
$$

for $n \geq 3$. It is then a simple matter, using (16) and (17) to compute their ordinary generating function, denoted by $G(x, y)$. We find

$$
\begin{equation*}
G(x, y)=\sum_{n \geq 3} T_{n}(y) x^{n}=\frac{x^{3}\left(1+3 y+y x+2 y^{2} x\right)}{1-x-y x^{2}} \tag{18}
\end{equation*}
$$

In fact, we also need to consider the homogeneous matchings polynomials

$$
\begin{equation*}
T_{n}(y, z)=z^{n} T_{n}\left(\frac{y}{z}\right)=\sum_{\mu \in \mathcal{M}\left(C_{n}\right)} y^{|\mu|} z^{n-|\mu|}, \tag{19}
\end{equation*}
$$

where the variable $z$ marks the edges which are not selected in the matchings, whose generating function $G(x, y, z)=\sum_{n \geq 3} T_{n}(y, z) x^{n}$ is given by

$$
\begin{equation*}
G(x, y, z)=G\left(x z, \frac{y}{z}\right)=\frac{x^{3} z^{2}\left(z+3 y+x y z+2 x y^{2}\right)}{1-x z-x^{2} y z} . \tag{20}
\end{equation*}
$$

We now introduce the species $B C$ of pairs $(c, \mu)$, where $c$ is a cycle of length $n \geq 3$ and $\mu$ is a matching of $c$, with weight $y^{|\mu|} z^{n-|\mu|}$. Since there are $\frac{(n-1)!}{2}$ non-oriented cycles on a set of size $n \geq 3$, and all these cycles admit the same homogeneous matching polynomial
$T_{n}(y, z)$, the mixed generating function of labelled $B C$-structures is

$$
\begin{array}{r}
B C(x, y, z)=\sum_{n \geq 3} \frac{(n-1)!}{2} T_{n}(y, z) \frac{x^{n}}{n!} \\
=\frac{1}{2} \sum_{n \geq 3} T_{n}(y, z) \frac{x^{n}}{n} \\
=\frac{1}{2} \int_{0}^{x} \frac{1}{t} G(t, y, z) d t \\
=-\frac{2 x z+2 x^{2} z y+x^{2} z^{2}+2 \ln \left(1-x z-x^{2} y z\right)}{4} \tag{21}
\end{array}
$$

Notice that in a circular crown, the unsubstituted edges are not adjacent, by definition, and hence form a matching of the underlying cycle, while the substituted edges are replaced by $K_{5} \backslash e$-networks. We can thus write

$$
\begin{equation*}
C C=B C \uparrow_{z}\left(K_{5} \backslash e\right), \tag{22}
\end{equation*}
$$

where the notation $\uparrow_{z}$ means that only the edges marked by $z$ are replaced by $K_{5} \backslash e$ networks. Moreover the decomposition (22) is canonical and we have

$$
\begin{equation*}
C C(x, y)=B C\left(x, y,\left(K_{5} \backslash e\right)(x, y)\right), \tag{23}
\end{equation*}
$$

which implies (12) using (8).
A substitution of the generating function $\mathcal{N}_{P}(x, y)$ of (11) counting the strongly planar networks for the variable $y$ in (6), (5), and (12) gives the generating function for labelled 2-connected non-planar toroidal graphs with no $K_{3,3}$-subdivision, i.e.

$$
\begin{equation*}
\mathcal{T}(x, y)=K_{5}\left(x, \mathcal{N}_{P}(x, y)\right)+M\left(x, \mathcal{N}_{P}(x, y)\right)+M^{*}\left(x, \mathcal{N}_{P}(x, y)\right)+C C\left(x, \mathcal{N}_{P}(x, y)\right) . \tag{24}
\end{equation*}
$$

Notice that the term $K_{5}\left(x, \mathcal{N}_{P}(x, y)\right)$ in (24) also enumerates non-planar 2-connected $K_{3,3}$-subdivision-free projective-planar graphs and that corresponding tables are given in [9]. Here we present the computational results just for labelled graphs in $\mathcal{T}$ that are not projective-planar. Numerical results are presented in Tables 1 and 2, where $\mathcal{T}(x, y)-K_{5}\left(x, \mathcal{N}_{P}(x, y)\right)=\sum_{n \geq 8} \sum_{m} t_{n, m} x^{n} y^{m} / n!$ and $t_{n}=\sum_{m} t_{n, m}$ count labelled non-projective-planar graphs in $\mathcal{T}$.

The homeomorphically irreducible non-projective-planar graphs in $\mathcal{T}$, i.e. the graphs having no vertex of degree two, can be counted by using several methods described in detail in Section 4 of [9]. We used the approach of Proposition 8 of [9] to obtain the numerical data presented in Tables 3 and 4 for labelled homeomorphically irreducible graphs in $\mathcal{T}$ that are not projective-planar.

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Table 1: The number of labelled non-planar non-projective-planar toroidal 2-connected graphs without a $K_{3,3^{-}}$-subdivision (having $n$ vertices and $m$ edges).

| $n$ | $t_{n}$ |
| ---: | ---: |
| 8 | 560 |
| 9 | 191520 |
| 10 | 42058800 |
| 11 | 7864256400 |
| 12 | 1407126890400 |
| 13 | 257752421166240 |
| 14 | 50607986220311520 |
| 15 | 10995419195575214400 |
| 16 | 2692773804667509763200 |
| 17 | 747221542837742897724800 |
| 18 | 233698171655650029030743040 |
| 19 | 81472765051132560093387934080 |
| 20 | 31268587126068905034073041062400 |

Table 2: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs (having $n$ vertices).

| $n$ | $m$ | $t_{n, m}$ | $n$ | $m$ | $t_{n, m}$ | $n$ | $m$ | $t_{n, m}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 18 | 280 | 14 | 26 | 6054048000 | 16 | 29 | 5811886080000 |
| 8 | 19 | 280 | 14 | 27 | 285751065600 | 16 | 30 | 621544891968000 |
| 9 | 19 | 5040 | 14 | 28 | 3361812854400 | 16 | 31 | 11935943091072000 |
| 10 | 20 | 25200 | 14 | 29 | 17840270448000 | 16 | 32 | 101350194001056000 |
| 10 | 22 | 226800 | 14 | 30 | 55133382704400 | 16 | 33 | 499371733276416000 |
| 10 | 23 | 466200 | 14 | 31 | 108994658572800 | 16 | 34 | 1611221546830896000 |
| 10 | 24 | 239400 | 14 | 32 | 141179453415000 | 16 | 35 | 3605404135132800000 |
| 11 | 23 | 10256400 | 14 | 33 | 118498240060200 | 16 | 36 | 5738963267481444000 |
| 11 | 24 | 30492000 | 14 | 34 | 61801664324400 | 16 | 37 | 6540526990277280000 |
| 11 | 25 | 43520400 | 14 | 35 | 18158435895600 | 16 | 38 | 5293490794557966000 |
| 11 | 26 | 31185000 | 14 | 36 | 2294786894400 | 16 | 39 | 2967845927880834000 |
| 11 | 27 | 7900200 | 15 | 28 | 1961511552000 | 16 | 40 | 1095216458944608000 |
| 12 | 24 | 189604800 | 15 | 29 | 57537672192000 | 16 | 41 | 239190441890400000 |
| 12 | 25 | 1079416800 | 15 | 30 | 557188343712000 | 16 | 42 | 23417178744960000 |
| 12 | 26 | 3044487600 | 15 | 31 | 2827950253128000 | 17 | 31 | 3903916528512000 |
| 12 | 27 | 5080614000 | 15 | 32 | 8936155496268000 | 17 | 32 | 174648084811200000 |
| 12 | 28 | 4776294600 | 15 | 33 | 18886100303070000 | 17 | 33 | 2606052624215040000 |
| 12 | 29 | 2261536200 | 15 | 34 | 27395286118200000 | 17 | 34 | 20178959825344320000 |
| 12 | 30 | 410810400 | 15 | 35 | 27296971027326000 | 17 | 35 | 97287841256493888000 |
| 13 | 25 | 1686484800 | 15 | 36 | 18324093378591000 | 17 | 36 | 319780940570307216000 |
| 13 | 26 | 22875652800 | 15 | 37 | 7906712877063000 | 17 | 37 | 751384930811218704000 |
| 13 | 27 | 126680954400 | 15 | 38 | 1978851858984000 | 17 | 38 | 1292496613555066920000 |
| 13 | 28 | 382608626400 | 15 | 39 | 218263565520000 | 17 | 39 | 1642597679422623924000 |
| 13 | 29 | 700723623600 |  |  |  | 17 | 40 | 1539140405659676820000 |
| 13 | 30 | 788388400800 |  |  |  | 17 | 41 | 1049167407329489448000 |
| 13 | 31 | 525156231600 |  |  |  | 17 | 42 | 505608857591934096000 |
| 13 | 32 | 188324136000 |  |  |  | 17 | 43 | 163183484418946992000 |
| 13 | 33 | 27935107200 |  |  |  | 17 | 44 | 31635477128166912000 |
|  |  |  |  |  |  | 17 | 45 | 2784602773016064000 |

Table 3: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs with no vertex of degree 2 (having $n$ vertices and $m$ edges).

| $n$ | $t_{n}$ |
| ---: | ---: |
| 8 | 560 |
| 9 | 5040 |
| 10 | 957600 |
| 11 | 123354000 |
| 12 | 16842764400 |
| 13 | 2764379217600 |
| 14 | 527554510282800 |
| 15 | 114387072405606000 |
| 16 | 27728561968887780000 |
| 17 | 7418031804967840056000 |
| 18 | 2167306256125914230527200 |
| 19 | 685709965521372865035362400 |
| 20 | 233306923207078035272369412000 |

Table 4: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs with no vertex of degree 2 (having $n$ vertices).

