# On Whitney numbers of the Order Ideals of Generalized Fences and Crowns 

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#### Abstract

We solve some recurrences given by E. Munarini and N. Zagaglia Salvi proving explicit closed formulas for Whitney numbers of the distributive lattices of order ideals of the fence poset and crown poset. Moreover, we get explicit closed formulas for Whitney numbers of lattices of order ideals of fences with higher asymmetric peaks.


## 1 Introduction and Preliminaries

In [10] authors consider the distributive lattices of all order ideals of the fence poset and crown poset ordered by inclusion, and they are able to prove

[^0]recursive formulas for their Whitney numbers. In this paper, using purely combinatorial methods, we solve these recursions giving explicit closed formulas for the corresponding rank polynomials. Moreover, in $\S 3$ we consider a more general class of fence posets, namely fences with higher asymmetric peaks, and we get explicit closed formulas for Whitney numbers of lattices of their order ideals.

For others combinatorial results about lattices of order ideals of finite posets and their Whitney numbers, we remind to [3, 7, 14, 15].

In the sequel we collect some definitions, notations and results that will be used in the following. For $x \in \mathbb{R}$ we let $\lfloor x\rfloor=\max \{n \in \mathbb{Z}: n \leq x\}$ and $\lceil x\rceil=\min \{n \in \mathbb{Z}: n \geq x\} ;$ for any $n, m \in \mathbb{N}, n \leq m$, we let $[n, m]=\{t \in$ $\mathbb{N}: n \leq t \leq m\}$, and $[n]=[1, n]$, therefore $[0]=\emptyset$. For any complex number $a$, we define the rising factorial as $(a)_{0}=1$ and $(a)_{m}=\prod_{j=0}^{m-1}(a+j)$ for any $m \in \mathbb{N} \backslash\{0\}$. The cardinality of a set $\mathcal{X}$ will be denoted by $\# \mathcal{X}$.

We follow [1, 6, 13] for combinatorics notations and terminology. We recall that a ranked poset is a poset $P$ with a function $\rho: P \longrightarrow \mathbb{N}$, called rank, such that $\rho(y)=\rho(z)+1$ whenever $z$ is covered by $y$ in $P$ and $\min \{\rho(z): z \in P\}=0$. The rank polynomial of a ranked finite poset $P$ is the polynomial

$$
\sum_{z \in P} X^{\rho(z)}=\sum_{j \geq 0} \omega_{j} X^{j},
$$

where $\omega_{j}=\#\{z \in P: \rho(z)=j\}$ are called Whitney numbers of $P$.
An order ideal of a poset $P$ is a subset $I \subset P$ such that if $y \in I$ and $z \leq y$, then $z \in I$; it is well known that the set of all order ideals of $P$ ordered by inclusion is closed under unions and intersections, and hence forms a distributive lattice: we denote it by $\mathcal{J}(P)$, viz. $\mathcal{J}(P)=\{I \subset$ $P: I$ is an order ideal\}. It is not hard to see that its rank function is the cardinality of order ideals.

Given a finite poset $(P, \leq)$, we denote with $W_{P}(k)$ the $k$-th Whitney numbers of the ranked poset of all order ideals of $P$, i.e. $W_{P}(k)=\#\{I \in$ $\mathcal{J}(P): \rho(I)=j\}$, where $\rho$ is the rank function of $\mathcal{J}(P)$, and the rank polynomial of $\mathcal{J}(P)$ is denoted by $\mathcal{R}_{P}(X)$, i. e. $\mathcal{R}_{P}(X)=\sum_{k \geq 0} W_{P}(k) X^{k}$.

We denote by $\mathcal{Z}_{n}$ the fence poset of order $n$, viz. the poset $\left\{z_{1}, \ldots, z_{n}\right\}$ in
which $z_{2 j-1} \triangleleft z_{2 j} \triangleright z_{2 j+1}$, for all $j \geq 1$, are the cover relations, by $\mathcal{I}_{n}(k)$ the set of order ideals of $\mathcal{Z}_{n}$ with cardinality $k$, and by $f_{n, k}$ the Whitney numbers of the poset of all order ideals of a fence of order $n$, viz. $f_{n, k}=\# \mathcal{I}_{n}(k)=W_{\mathcal{Z}_{n}}(k)$. We denote by $\mathcal{Y}_{n}$ the crown poset of order $2 n$, viz. the poset $\left\{\zeta_{0}, \ldots, \zeta_{2 n-1}\right\}$ in which the cover relations are the following: for any $h \in\{0, \ldots n-1\}$ and $k \in[n], \zeta_{2 h} \triangleleft \zeta_{2 k-1}$ if and only if $|2 h-2 k+1| \equiv 1(\bmod 2 n)$, therefore

$$
\rho\left(\zeta_{j}\right)=\left\{\begin{array}{lll}
0 & \text { if } j \equiv 0 & (\bmod 2) \\
1 & \text { if } j \equiv 1 & (\bmod 2)
\end{array}\right.
$$

We also denote by $\mathcal{O}_{n}(k)$ the set of order ideals of $\mathcal{Y}_{n}$ with cardinality $k$, and by $c_{n, k}$ the Whitney numbers of the poset of all order ideals of a crown of order $2 n$, viz. $c_{n, k}=\# \mathcal{O}_{n}(k)=W_{\mathcal{Y}_{n}}(k)$.

Finally we recall, gluing together, Propositions 1,3 and 5 of [10], which give recursions for the sequences $f_{n, k}$ and $c_{n, k}$.

Proposition 1.1. For any integer $n$ the recurrence identity

$$
\begin{cases}f_{2 n, k} & =f_{2 n-1, k}+f_{2 n-2, k-2} \\ f_{2 n+1, k} & =f_{2 n, k-1}+f_{2 n-1, k}\end{cases}
$$

holds, where

$$
\begin{cases}f_{n, k}=0 & \text { if } k \notin[0, n] \text { or } n<0 \\ f_{n, 0}=1 & \text { for all } n \in \mathbb{N}\end{cases}
$$

are the initial values.
Moreover, with the same initial values the formula

$$
f_{n+4, k+2}=f_{n+2, k+2}+f_{n+2, k+1}+f_{n+2, k}-f_{n, k}
$$

holds, for all $0 \leq k \leq n \in \mathbb{N}$.
Furthermore,

$$
\begin{aligned}
c_{n+2, k+3} & =f_{2 n+3, k+3}+f_{2 n+1,2 n+1-k} \\
c_{n+2, k+2} & =c_{2 n+1, k}+f_{2 n+3, k+2}-f_{2 n-1, k} \\
c_{n+2, k+2} & =f_{2 n+4, k+2}-f_{2 n, k}
\end{aligned}
$$

hold, for all $n \in \mathbb{N}$ and all $0 \leq k \leq 2 n$.

## 2 Closed Formulas for Whitney Numbers

We need the following Proposition, whose proof can be found in [13].
Proposition 2.1. For all non-negative integers $k \leq n$,

$$
\begin{aligned}
\#\left\{\overline{\mathbf{x}}=\left(x_{1}, \ldots, x_{k}\right) \in(\mathbb{N} \backslash\{0\})^{k}: \sum_{j=1}^{k} x_{j}=n\right\} & =\binom{n-1}{k-1} \\
\#\left\{\overline{\mathbf{x}}=\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{N}^{k}: \sum_{j=1}^{k} x_{j}=n\right\} & =\binom{n+k-1}{k-1}, \\
\#\left\{\overline{\mathbf{x}}=\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{N}^{k}: \sum_{j=1}^{k} x_{j} \leq n\right\} & =\binom{n+k}{k},
\end{aligned}
$$

hold.
Theorem 2.2. For all $k, v \in \mathbb{N}$ such that $k \leq 2 v+1$,

$$
f_{2 v+1, k}=\binom{v+1}{k}+\sum_{j \geq 1} \frac{(k-2 j+1)_{j-1}(v+j-k+2)_{k-2 j}}{j!(k-2 j-1)!}
$$

holds.

Proof. For all integers $0 \leq k \leq n$, we can write

$$
f_{n, k}=\# \mathcal{I}_{n}(k)=\sum_{j \geq 0} \mathcal{A}(n, k, j)=\sum_{j=0}^{\min \left\{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{k}{2}\right\rceil-1\right\}} \mathcal{A}(n, k, j),
$$

where

$$
\mathcal{A}(n, k, j)=\#\left\{\mathcal{J} \in \mathcal{I}_{n}(k): \#\{x \in \mathcal{J}: \rho(x)=1\}=j\right\}
$$

thus we have that $f_{n, 0}=1, f_{n, 1}=\#\left\{x \in \mathcal{Z}_{n}: \rho(x)=0\right\}=\left\lfloor\frac{n+1}{2}\right\rfloor=\left\lceil\frac{n}{2}\right\rceil$, and $\mathcal{A}(n, k, 0)=\binom{f_{n, 1}}{k}$.

Consider a fence $\mathcal{Z}_{n}$ with odd cardinality, i.e. $n=2 v+1$ for some $v \in \mathbb{N}$, and write it as the poset $\left\{z_{1}, \ldots, z_{2 v+1}\right\}$ in which $z_{2 \alpha-1} \triangleleft z_{2 \alpha} \triangleright z_{2 \alpha+1}$, for all $\alpha \geq 1$, are the cover relations.
For any given $\mathcal{J} \in \mathcal{I}_{2 v+1}(k)$ (with $k \geq 2$ ) such that $\#\{x \in \mathcal{J}: \rho(x)=1\}=$
$j \geq 1$ we can split the set $\{x \in \mathcal{J}: \rho(x)=1\}$ in $r$ separated non-empty blocks $\mathcal{X}_{1}, \ldots, \mathcal{X}_{r}$, such that $\sum_{t=1}^{r} \# \mathcal{X}_{t}=j ; z_{2 a}, z_{2 b} \in \mathcal{J}$ with $1 \leq a<b \leq v$ are in the same block if and only if $z_{2 c} \in \mathcal{J}$ for all $c$ such that $1 \leq a<c<$ $b \leq v$. Each $\mathcal{X}_{t}$ determines $2 \# \mathcal{X}_{t}+1$ elements in $\mathcal{J}$, so this decomposition fix $\sum_{t=1}^{r}\left(2 \# \mathcal{X}_{t}+1\right)=2 j+r$ elements of $\mathcal{J}(j$ of these have rank 1 , and the others $j+r$ have rank 0 ), and obviously the others can be chosen in $\binom{v+1-(j+r)}{k-(2 j+r)}$ ways between the remainder elements with rank 0 .
Moreover, the number of such decompositions $\mathcal{X}_{1}, \ldots, \mathcal{X}_{r}$ is $\# \mathcal{C}(j, r)$ times the the total numbers of shifts of all blocks $\mathcal{X}_{1}, \ldots, \mathcal{X}_{r}$, which can be evaluated in the following way: at least one element of rank 1 has to be into the slot between the blocks $\mathcal{X}_{t}$ and $\mathcal{X}_{t+1}$, for any $t \in[r-1]$, and the others $v-(j+r-1)$ elements can be freely distributed into the $r+1$ slots, viz. before $\mathcal{X}_{1}$, between $\mathcal{X}_{t}$ and $\mathcal{X}_{t+1}$, for any $t \in[r-1]$, and after $\mathcal{X}_{r}$, thus from Proposition $2.1\binom{v-j+1}{r}$ is the searched value.
Therefore if we define $\mathcal{C}(\mu, \nu)=\left\{\overline{\mathbf{x}}=\left(x_{1}, \ldots, x_{\nu}\right) \in(\mathbb{N} \backslash\{0\})^{\nu}: \sum_{j=1}^{\nu} x_{j}=\right.$ $\mu\}$ for any $1 \leq \nu \leq \mu$, from Proposition 2.1] we have $\# \mathcal{C}(\mu, \nu)=\binom{\mu-1}{\nu-1}$, hence for any $j \geq 1$

$$
\begin{aligned}
\mathcal{A}(2 v+1, k, j) & =\sum_{r=1}^{j} \sum_{\substack{\overline{\mathbf{x}} \in \mathcal{C}(j, r) \\
2 j \leq r \leq k \\
j+r-1 \leq v}}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)} \\
& =\sum_{r=1}^{j} \sum_{\overline{\mathbf{x}} \in \mathcal{C}(j, r)}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)} \\
& =\sum_{r=1}^{j}\binom{j-1}{r-1}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)} .
\end{aligned}
$$

Therefore we have

$$
\begin{gathered}
f_{2 v+1, k}=\binom{v+1}{k}+\sum_{j=1}^{\min \left\{v,\left\lceil\frac{k}{2}\right\rceil-1\right\}} \sum_{r=1}^{j}\binom{j-1}{r-1}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)} \\
=\binom{v+1}{k}+\sum_{j \geq 1} \sum_{r \geq 1}\binom{j-1}{j-r}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)}
\end{gathered}
$$

Writing the sum over $r$ in hypergeometric notation and applying Chu-

Vandermonde summation, see (4, 5, 4, we get

$$
\begin{aligned}
& \sum_{r \geq 1}\binom{j-1}{j-r}\binom{v-j+1}{r}\binom{v+1-(j+r)}{k-(2 j+r)} \\
& =\frac{{ }_{2} F_{1}\left[\begin{array}{c}
1-j, 1+2 j-k \\
2
\end{array} ; 1\right](v+j-k+2)_{k-2 j}}{(k-2 j-1)!} \\
& =\frac{(k-2 j+1)_{j-1}(v+j-k+2)_{k-2 j}}{j!(k-2 j-1)!}
\end{aligned}
$$

and the desired result follows.
Corollary 2.3. For any $v \in \mathbb{N}$ and all $0 \leq k \leq 2 v+1$, the sequence $f_{2 v+1, k}$ is increasing in $v$, viz. $f_{2(v+1)+1, k}>f_{2 v+1, k}$.

Definition 2.4. Let $\left(P_{1}, \leq_{1}\right),\left(P_{2}, \leq_{2}\right)$ be finite posets with cover relations $\triangleleft_{1}$ and $\triangleleft_{2}$, respectively, and let $x_{1} \in P_{1}, x_{2} \in P_{2}$ be minimal elements.
We consider a new element $\widetilde{x}$ which does not belong to $P_{1} \biguplus P_{2}$ and we define a new poset $\left(P_{1}\left(x_{1}\right) \circledast P_{2}\left(x_{2}\right), \leq\right)$ with cover relations $\triangleleft$, where

$$
P_{1}\left(x_{1}\right) \circledast P_{2}\left(x_{2}\right)=P_{1} \biguplus P_{2} \biguplus\{\widetilde{x}\}
$$

and for any $x, y \in P_{1}\left(x_{1}\right) \circledast P_{2}\left(x_{2}\right)$ we have $x \triangleleft y$ if and only if one of the following conditions holds:

- $x, y \in P_{1}$ and $x \triangleleft_{1} y$ in $P_{1}$,
- $x, y \in P_{2}$ and $x \triangleleft_{2} y$ in $P_{2}$,
- $x_{1} \triangleleft \widetilde{x}$,
- $x_{2} \triangleleft \widetilde{x}$.

Theorem 2.5. Let $\left(P_{1}, \leq_{1}\right),\left(P_{2}, \leq_{2}\right)$ be finite posets, $x_{1} \in P_{1}, x_{2} \in P_{2}$ be minimal elements, and $\widehat{P}=P_{1}\left(x_{1}\right) \circledast P_{2}\left(x_{2}\right)$; then

$$
\mathcal{R}_{\hat{P}}(X)=\mathcal{R}_{P_{1}}(X) \mathcal{R}_{P_{2}}(X)+X^{3} \mathcal{R}_{P_{1} \backslash\left\{x_{1}\right\}}(X) \mathcal{R}_{P_{2} \backslash\left\{x_{2}\right\}}(X)
$$

holds.

Proof. Let us write $\mathcal{J}(\widehat{P})=\biguplus_{k=0}^{\# \widehat{P}} \mathcal{J}_{k}$, where $\mathcal{J}_{k}=\{I \in \mathcal{J}(\widehat{P}): \rho(I)=$ $\# I=k\}$, thus

$$
W_{\widehat{P}}(k)=\# \mathcal{J}_{k}=\#\left\{I \in \mathcal{J}_{k}: \widetilde{x} \notin I\right\}+\#\left\{I \in \mathcal{J}_{k}: \widetilde{x} \in I\right\} .
$$

It is not hard to see that

$$
\#\left\{I \in \mathcal{J}_{k}: \widetilde{x} \notin I\right\}=\sum_{j=0}^{k} W_{P_{1}}(j) \cdot W_{P_{2}}(k-j)
$$

and

$$
\begin{aligned}
\#\left\{I \in \mathcal{J}_{k}: \widetilde{x} \in I\right\} & =\#\left\{I \in \mathcal{J}_{k}: x_{1}, x_{2}, \widetilde{x} \in I\right\} \\
& =\sum_{j=0}^{k-3} W_{P_{1} \backslash\left\{x_{1}\right\}}(j) \cdot W_{P_{2} \backslash\left\{x_{2}\right\}}(k-3-j),
\end{aligned}
$$

and the desired result follows.
Theorem 2.6. For all $k, v \in \mathbb{N}$ such that $k \leq 2 v$,

$$
\begin{aligned}
f_{2 v, k} & =\sum_{j \geq 0} \sum_{r \geq 0}\binom{j}{r}\binom{v-j}{r}\binom{v-(j+r)}{k-(2 j+r)} \\
& =\sum_{j \geq 0} \frac{(k-2 j+1)_{j}(v+j-k+1)_{k-2 j}}{j!(k-2 j)!}
\end{aligned}
$$

holds.

Proof. For any $n \in \mathbb{N} \backslash\{0\}$ write the fence poset $\mathcal{Z}_{n}$ as the poset $\left\{z_{1}, \ldots, z_{n}\right\}$ in which $z_{2 j-1} \triangleleft z_{2 j} \triangleright z_{2 j+1}$, for all $j \geq 1$, are the cover relations, so $\rho\left(z_{j}\right)=0$ if and only if $j \equiv 1(\bmod 2)$ and $\rho\left(z_{j}\right)=1$ if and only if $j \equiv 0(\bmod 2)$. If we consider $P_{1}=\mathcal{Z}_{2 v+1}=\left\{a_{1}, \ldots, a_{2 v+1}\right\}, P_{2}=\mathcal{Z}_{1}=\left\{b_{1}\right\}$, we have that $P_{1}\left(a_{2 v+1}\right) \circledast P_{2}\left(b_{1}\right) \simeq \mathcal{Z}_{2 v+3}$, and the desired result follows applying Theorems 2.5and 2.2, and Chu-Vandermonde summation for hypergeometric series as in the proof of Theorem 2.2

Corollary 2.7. For any $v \in \mathbb{N}$ and all $0 \leq k \leq 2 v$, the sequence $f_{2 v, k}$ is increasing in $v$, viz. $f_{2(v+1), k}>f_{2 v, k}$.

From Proposition 1.1 and Theorem [2.6 we immediately get the following result.

Theorem 2.8. For all $k, n \in \mathbb{N}$ such that $k \leq 2 n$,

$$
\begin{aligned}
c_{n, k} & =\sum_{j \geq 0} \frac{(k-2 j+1)_{j-2}(n+j-k+1)_{k-2 j-2}}{j!(k-2 j)!} \\
& \cdot\left[(k-j-1)_{2}(n-j-1)_{2}-\left((k-2 j-1)_{2}\right)^{2}\right]
\end{aligned}
$$

holds.

Therefore Theorems [2.2, 2.6 and 2.8 give the solution of the recursive identities in Proposition 1.1 .

## 3 Generalized Fences with higher asymmetric peaks

Now we define an asymmetric peak poset with two positive integers parameters $\mu, \nu$.

Definition 3.1. Let $\mu, \nu \in \mathbb{N} \backslash\{0\}$; we define the poset asymmetric peak $\left(\mathrm{AP}_{\mu, \nu}, \leq\right)$ in the following way: $\mathrm{AP}_{\mu, \nu}=\left\{a_{j}: j \in[\mu]\right\} \biguplus\left\{b_{j}: j \in[\nu]\right\} \biguplus\{\omega\}$, and the cover relations are

- $a_{j} \triangleleft a_{j+1}$ for all $j \in[\mu-1]$,
- $b_{j} \triangleleft b_{j+1}$ for all $j \in[\nu-1]$,
- $a_{\mu} \triangleleft \omega$,
- $b_{\nu} \triangleleft \omega$.

Proposition 3.2. Let $\mu, \nu \in \mathbb{N} \backslash\{0\}$; then

$$
W_{\mathrm{AP}_{\mu, \nu}}(k)= \begin{cases}1 & \text { if } k=0 \text { or } k=\mu+\nu+1 \\ k+1 & \text { if } k \leq \min \{\mu, \nu\} \\ \min \{\mu, \nu\}+1 & \text { if } \min \{\mu, \nu\} \leq k \leq \max \{\mu, \nu\} \\ 1+\mu+\nu-k & \text { if } \max \{\mu, \nu\} \leq k\end{cases}
$$

holds, for any $k=0, \ldots, \# \mathrm{AP}_{\mu, \nu}=\mu+\nu+1$.

Proof. The result is clear is $k=0$ or $k=\mu+\nu+1$.
We consider the case $\mu \leq \nu$, the case $\mu \geq \nu$ is completely symmetric.
If $k \in[\mu+\nu]$ then any $I \in \mathcal{J}\left(\mathrm{AP}_{\mu, \nu}\right)$ with $\rho(I)=\# I=k$ has the shape $I=\left\{a_{j}: j \in[r]\right\} \biguplus\left\{b_{j}: j \in[t]\right\}$ with

$$
\begin{equation*}
r+t=k, \tag{1}
\end{equation*}
$$

so $W_{\mathrm{AP}_{\mu, \nu}}(k)$ equals the number of solutions of (11) with the constraints

$$
\begin{aligned}
0 \leq r \leq k \text { and } 0 \leq t \leq k & \text { if } k \leq \mu, \\
0 \leq r \leq \mu \text { and } k-\mu \leq t \leq k & \text { if } \mu \leq k \leq \nu, \\
k-\nu \leq r \leq \mu \text { and } k-\mu \leq t \leq \nu & \text { if } \nu \leq k .
\end{aligned}
$$

The desired result follows.

Results proved in $\S 2$ allows to get explicit closed formulas for Whitney numbers of lattices of order ideals of "fences with higher asymmetric peaks", i.e. the alternate composition of fences and asymmetric peaks by the operator $\circledast$, see Definition [2.4.

For example, we can consider a fence with one higher asymmetric peak, which can be formally defined as the following poset $(F A P(w, x, y, z), \leq)$ where with $w, x, y, z \in \mathbb{N}$ and $w \equiv 1(\bmod 2)$ :

$$
\operatorname{FAP}(w, x, y, z)=\left\{a_{1}, \ldots, a_{w}, b_{1}, \ldots, b_{x}, \omega, c_{1}, \ldots, c_{y}, d_{1}, \ldots, d_{z}\right\}
$$

where the cover relations are

- $a_{2 j-1} \triangleleft a_{2 j} \triangleright a_{2 j+1}$, for all $j \geq 1$,
- $a_{w} \triangleleft b_{1}$,
- $b_{j} \triangleleft b_{j+1}$ for all $j \in[x-1]$,
- $c_{j} \triangleleft c_{j+1}$ for all $j \in[y-1]$,
- $b_{x} \triangleleft \omega$,
- $c_{y} \triangleleft \omega$,
- $d_{1} \triangleleft c_{1}$,
- $d_{2 j-1} \triangleleft d_{2 j} \triangleright d_{2 j+1}$, for all $j \geq 1$.

In Figure 1 the Hasse diagram of $\operatorname{FAP}(7,10,6,7)$ is depicted.


Figure 1: FAP (7, 10, 6, 7)

Inside FAP $(w, x, y, z)$ consider the subposets

$$
\begin{aligned}
& \mathcal{P}_{1}=\left\{a_{1}, \ldots, a_{w-2}\right\} \simeq \mathcal{Z}_{w-2}, \\
& \mathcal{P}_{2}=\left\{a_{w}, b_{1}, \ldots, b_{x}, \omega, c_{1}, \ldots, c_{y}, d_{1}\right\} \simeq \mathrm{AP}_{x+1, y+1}, \\
& \mathcal{P}_{3}=\left\{d_{3}, \ldots, d_{z}\right\} \simeq \mathcal{Z}_{z-2}
\end{aligned}
$$

therefore $\operatorname{FAP}(w, x, y, z)=\mathcal{P}_{1} \biguplus\left\{a_{w-1}\right\} \biguplus \mathcal{P}_{2} \biguplus\left\{d_{2}\right\} \biguplus \mathcal{P}_{3}$.
We have that

$$
\begin{aligned}
\operatorname{FAP}(w, x, y, z) & \simeq\left(\mathcal{P}_{1}\left(a_{w-2}\right) \circledast \mathcal{P}_{2}\left(a_{w}\right)\right)\left(d_{1}\right) \circledast \mathcal{P}_{3}\left(d_{3}\right) \\
& \simeq \mathcal{P}_{1}\left(a_{w-2}\right) \circledast\left(\mathcal{P}_{2}\left(d_{1}\right) \circledast \mathcal{P}_{3}\left(d_{3}\right)\right)\left(a_{w}\right),
\end{aligned}
$$

therefore from Theorems [2.2, 2.5, [2.6, 2.8 and Proposition 3.2 we get an explicit closed formulas for the rank polynomial of the distributive lattice of all order ideals of the poset $\operatorname{FAP}(w, x, y, z)$.

We remark that the same construction can be iterated, so for any nonnegative integer $k$ we can recursively have a formula for the rank polynomial of the lattice of all order ideals of a fence with $k$ higher asymmetric peaks.

## 4 Open problems and Conjectures

In [10] using recursive formulas stated in Proposition 1.1 it is proved that sequences $f_{n, k}$ and $c_{n, k}$ are indeed unimodal; for definitions and comprehensive surveys about unimodal and (strong) $\log$-concave sequences we refer to [2, 8, 11, 12, 16] (and the references therein).

We feel that the following stronger statement is true.
Conjecture 4.1. For any $3 \neq n \in \mathbb{N} \backslash\{0\}$ and all $0 \leq k \leq n$, the sequence $f_{n, k}$ is log-concave in $k$, viz. $f_{n, k}^{2} \geq f_{n, k-1} f_{n, k+1}$ for any $k \in[n-1]$.

Moreover, for any $4 \leq n \in \mathbb{N}$ and all $0 \leq k \leq 2 n$, the sequence $c_{n, k}$ is strong $\log$-concave in $k$, viz. $c_{n, k}^{2}>c_{n, k-1} c_{n, k+1}$ for any $k \in[2 n-1]$.

Using a computer, Conjecture 4.1 has been verified for distributive lattices of order ideals of fences and crowns, for all fences and crowns with cardinality less or equal than 90.

Moreover, we note that it would be of very great interest to study the following much more general problem.

Open Problem 4.2. Characterize finite posets for which the distributive lattice of order ideals is rank (strong) log-concave or just rank unimodal.

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