A REALIZATION OF GRAPH-ASSOCIAHEDRA

SATYAN L. DEVADOSS

ABSTRACT. Given any finite graph G, we offer a simple realization of the graph-associahedron $\mathcal{P}G$ using integer coordinates.

1. INTRODUCTION

Given a finite graph G, the graph-associahedron $\mathcal{P}G$ is a simple, convex polytope whose face poset is based on the connected subgraphs of G. This polytope was has been studied in [3], and has appeared in combinatorial [1, 8] and geometric contexts [4, 11]. In particular, it appears as tilings of minimal blow-ups of certain Coxeter complexes, which themselves are natural generalizations of the Deligne-Knudsen-Mumford compactification $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of the real moduli space of curves [5].

For special examples of graphs, their graph-associahedra become well-known, sometimes classical, polytopes. For instance, when G is a set of vertices, $\mathcal{P}G$ is the simplex. Moreover, when G is a path, a cycle, or a complete graph, $\mathcal{P}G$ results in the associahedron, cyclohedron, and permutohedron, respectively. Loday [7] provided a formula for the coordinates of the vertices of the associahedron which contains the classical realization of the permutohedron. Recently, Hohlweg and Lange [6] offer different realizations of the associahedron and cyclohedron. We offer a realization of graph-associahedra for any graph.

2. Convex Hull

2.1. We begin with definitions; the reader is encouraged to see [3, Section 1] for details.

Definition. Let G be a finite graph. A *tube* is a proper nonempty set of nodes of G whose induced graph is a proper, connected subgraph of G. There are three ways that two tubes u_1 and u_2 may interact on the graph.

- (1) Tubes are *nested* if $u_1 \subset u_2$.
- (2) Tubes intersect if $u_1 \cap u_2 \neq \emptyset$ and $u_1 \not\subset u_2$ and $u_2 \not\subset u_1$.
- (3) Tubes are *adjacent* if $u_1 \cap u_2 = \emptyset$ and $u_1 \cup u_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible. A *k*-tubing is a tubing with k tubes.

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Remark. When G is a disconnected graph with connected components G_1, \ldots, G_k , we place an additional restriction. Let u_i be the tube of G whose induced graph is G_i . Then any tubing of G cannot contain all of the tubes $\{u_1, \ldots, u_k\}$. Thus, for a graph G with n nodes, a tubing of G can at most contain n-1 tubes. Figure 1 shows examples of (a) valid tubings and (b) invalid tubings.



FIGURE 1. (a) Valid tubings and (b) invalid tubings.

Definition. For a graph G, the graph-associahedron $\mathcal{P}G$ is a simple, convex polytope whose face poset is isomorphic to set of tubings of G, ordered such that $U \prec U'$ if U is obtained from U' by adding tubes.

2.2. Let G be a graph with n nodes and let M_G be the collection of maximal (n-1)-tubings of G. For each such tubing U in M_G , define a map f_U from the nodes of G to the integers as follows: If a node v of G is a tube of U, then $f_U(v) = 0$. Otherwise, let t(v) be the smallest tube containing v, and let all other nodes of G satisfy the recursive condition

(2.1)
$$\sum_{x \in t(v)} f_U(x) = 3^{|t(v)|-2}.$$

Figure 2 gives some examples of integer values of nodes associated to tubings.



FIGURE 2. Integer values of nodes associated to tubings.

Let G be a graph with an ordering v_1, v_2, \ldots, v_n of its nodes. Define $c: M_G \to \mathbb{R}^n$ where

$$c(U) = (f_U(v_1), f_U(v_2), \dots, f_U(v_n)).$$

Theorem 1. If G is a graph with n nodes, the convex hull of the points $c(M_G)$ in \mathbb{R}^n yields the graph-associahedron $\mathcal{P}G$.

The proof of this is given at the end of the paper.

3. Examples

3.1. Simplex. Let G be the graph with n (disjoint) nodes. The set M_G of maximal tubings has n elements, each corresponding to choosing n-1 out of the n possible nodes. An element of M_G will be assigned a point in \mathbb{R}^n consisting of zeros for all coordinates except one with value 3^{n-2} . Due to Theorem 1, $\mathcal{P}G$ is the convex hull of the n vertices in \mathbb{R}^n yielding the (n-1)-simplex. Figure 3 shows this when n = 3, resulting in the 2-simplex in \mathbb{R}^3 .



FIGURE 3. The maximal tubings of G and its convex hull, resulting in the simplex.

3.2. **Permutohedron.** Let G be the complete graph on n nodes. Each maximal tubing of G can be seen as a sequential nesting of all n nodes. In other words, they are in bijection with permutations on n letters. The elements of M_G will be assigned coordinate values based on all permutations of $\{0, 1, \ldots, 3^{n-2} - 3^{n-3}\}$. Theorem 1 shows $\mathcal{P}G$ as the convex hull of the n! vertices in \mathbb{R}^n , resulting in the permutohedron. Figure 4 shows this when n = 3, resulting in the hexagon, the two-dimensional permutohedron.



FIGURE 4. The maximal tubings of G and its convex hull, resulting in the permutohedron.

3.3. Associahedron. Let G be an n-path. The number of such maximal tubings is in bijection with the Catalan number c_n . Due to Theorem 1, the convex hull of these vertices in \mathbb{R}^n yields the (n-1) dimension associahedron. Stasheff originally defined the associahedron for use in homotopy theory in connection with associativity properties of H-spaces [9]. Figure 5 shows this when n = 3, resulting in the pentagon, the two-dimensional associahedron.



FIGURE 5. The maximal tubings of G and its convex hull, resulting in the associahedron.

3.4. Cyclohedron. Let G be an n-cycle. In this case, the number of maximal tubings is the type B Catalan number $\binom{2n-2}{n-1}$. Theorem 1 shows $\mathcal{P}G$ as the cyclohedron, a polytope originally manifested in the work of Bott and Taubes in relation to knot and link invariants [2]. Figure 4 shows this when n = 3, since both the permutohedron and cyclohedron are identical in dimension two.

4. Constructing the Graph-Associahedron

4.1. For a graph G with n nodes $v_1, \ldots v_n$, let Δ be the (n-1)-simplex in which each facet (codimension 1 face) corresponds to a particular node of G. Thus, each proper subset of nodes of G corresponds to a unique face of Δ , defined by the intersection of the faces associated to those nodes. The following construction of the graph-associahedron is based on truncations of a simplex.

Theorem 2. [3, Section 2] For a given graph G, truncating faces of Δ which correspond to 1-tubings in increasing order of dimension results in $\mathcal{P}G$.

Indeed, truncations should not only be in increasing order of dimension (certain vertices of Δ are truncated first, and then the edges, and so forth), but they should also not form "deep cuts". Consider Figure 6 as an example. Part (a) shows a 3-simplex with two vertices marked for truncation; part (b) shows appropriate truncations of the vertices, with (c) and (d) showing inappropriate cuts which are too deep.



FIGURE 6. Iterated truncations of the 3-simplex based on an underlying graph.

Remark. In order to recover Loday's elegant construction of the classical permutohedron as part of the associahedron, we simply use the following recursive definition of f_U :

$$\sum_{x \in t(v)} f_U(x) = \binom{|t(v)|+1}{2}.$$

Although this works for the associahedron, it fails for graph-associahedra in general. The reason for this is that the cuts needed to construct the polytopes are too deep.

Figure 7 shows a tetrahedron truncated according to a graph, resulting in $\mathcal{P}G$. Note that its facets are labeled with 1-tubings. One can verify that the edges correspond to all possible 2-tubings and the vertices to 3-tubings.



FIGURE 7. Iterated truncations of the 3-simplex based on an underlying graph.

4.2. We are now in position to prove Theorem 1. This is influenced by the work of Stasheff and Schnider [10, Appendix B].

Proof of Theorem 1. Consider the affine hyperplane H defined by

$$(4.1) \qquad \qquad \sum x_i = 3^{n-2}.$$

The intersection of the quadrant $\{(x_1, \ldots, x_n) \mid x_i \geq 0\}$ with H yields a standard (n-1)simplex Δ . Let G_u be the set of all 1-tubings of G, where G_u^i be the set of 1-tubings containing i nodes. The faces of Δ which need to truncated correspond to the 1-tubings G_u^i , where $i \geq 2$. Let $u = \{v_{i_1}, \ldots, v_{i_k}\}$ be a 1-tubing in G_u^k ; note that this corresponds to a n-1-k face of Δ , seen as the intersection of the hyperplane

$$\sum_{v_i \in u} x_i = 0$$

of \mathbb{R}^n with Δ . Truncate this face with the hyperplane

(4.2)
$$\sum_{v_i \in u} x_i = 3^{k-2}.$$

We claim that this collection of hyperplanes, one for each element of G_u^i , results in $\mathcal{P}G$.

By Theorem 2 above, the appropriate faces of Δ have been truncated, one for each 1tubing. However, we need to show the any two cuts of a given dimension are not deep; that is, their corresponding hyperplanes must not intersect in H. This is done by induction. Two vertices of Δ which are truncated correspond to 1-tubings in G_u^{n-1} , say $u = \{v_1, \ldots, v_{n-2}, v_{n-1}\}$ and $u' = \{v_1, \ldots, v_{n-2}, v_n\}$. These hyperplanes cannot intersect in H since Eqs. (4.1) and (4.2) show

$$\sum x_i = 3^{n-2} > 3^{n-3} + 3^{n-3} = \sum_{v_i \in u} x_i + \sum_{v_i \in u'} x_i.$$

In general, let u_* be in G_u^{n-1-k} , a k-dimensional face of Δ that is truncated. Let u and u' be two (k+1)-dimensional faces of Δ in G_u^{n-2-k} which are incident to u_* . The cuts u and u' will not be deep with respect to u_* . To see this, notice that the nodes of u and u' are contained in u_* . Thus, in H the hyperplanes of u and u' cannot intersect in u_* since

$$\sum_{v_i \in u_*} x_i = 3^{n-3-k} > 3^{n-2-k} + 3^{n-2-k} = \sum_{v_i \in u} x_i + \sum_{v_i \in u'} x_i$$

Recall that each vertex of $\mathcal{P}G$ corresponds to a (n-1)-tubing T of G. This, in turn, corresponds to the intersection of the n-1 hyperplanes of (4.2) for each 1-tubing of T. In particular, a tube containing one node assigns the value 0 to that node; these are incident to the original facets of Δ . Thus Eq. (2.1) is satisfied inductively.

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References

- F. Ardila, V. Reiner, L. Williams. Bergman complexes, Coxeter arrangements, and graph associahedra, Seminaire Lotharingien de Combinatoire 54A(2006).
- 2. R. Bott and C. Taubes. On the self-linking of knots, J. Math. Phys. 35 (1994) 5247-5287.
- M. Carr and S. L. Devadoss. Coxeter Complexes and graph-associahedra, *Topology and its Appl.* 153 (2006) 2155-2168.
- M. Davis, T. Januszkiewicz, R. Scott. Fundamental groups of blow-ups, Advances in Math. 177 (2003) 115-179.
- S. Devadoss. Tessellations of moduli spaces and the mosaic operad, *Contemp. Math.* 239 (1999) 91-114.
- C. Hohlweg and C. Lange. Realizations of the associahedron and cyclohedron, preprint math.CO/0510614.
- 7. J.-L. Loday. Realization of the Stasheff polytope, Archiv der Mathematik 83 (2004) 267-278.
- 8. A. Postnikov. Permutohedra, associahedra, and beyond, preprint math.CO/0601339.
- 9. J. D. Stasheff. Homotopy associativity of H-spaces, Trans. Amer. Math. Soc. 108 (1963) 275-292.
- J. D. Stasheff (Appendix B coauthored with S. Shnider). From operads to "physically" inspired theories, Contemp. Math. 202 (1997) 53-81.
- V. Toledano-Laredo. Quasi-Coxeter algebras, Dynkin diagram cohomology and quantum Weyl groups, preprint math.QA/0506529.
 - S. DEVADOSS: WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267 E-mail address: satyan.devadoss@williams.edu