Measure preserving homomorphisms and independent sets in tensor graph powers

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Abstract

In this note, we study the behavior of independent sets of maximum probability measure in tensor graph powers. To do this, we introduce an upper bound using measure preserving homomorphisms. This work extends some previous results about independence ratios of tensor graph powers.

1 Introduction

The graphs in this note can have infinite number of vertices. A *homomorphism* from a graph H to a graph G is a map h from the vertices of H to the vertices of G such that $h(u)h(v)$ is an edge in G for every edge $uv \in E(H)$. For every graph G, we assume that there is a *probability measure* μ_G on the vertices of G. A homomorphism $h : V(H) \to V(G)$ is *measure preserving*, if h is measurable and for every measurable $S \subseteq V(G)$, $\mu_H(h^{-1}(S)) = \mu_G(S)$. By $H \to G$, we mean that there exists a measure preserving homomorphism from H to G.

Definition 1 *Let G be a graph with the probability measure μ_G on its vertices. We call G* vertex transitive*, if*

1. *there exists a set S of measure preserving homomorphisms $\phi : V(G) \to V(G)$;*
2. *there exists a probability measure ν on S, such that for almost every $v \in V(G)$, $\phi(v)$ has the same distribution as μ_G when ϕ is chosen according to ν.*

Note that for a finite graph with the uniform measure, this definition coincides with the known definition of vertex transitivity of finite graphs (take S to be the group of automorphisms of G with the uniform measure).

The *tensor* product of two graphs, G and H, has vertex set $V(G) \times V(H)$, where (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. The

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measure on the new vertex set is the product measure. The characteristics of tensor products of graphs have been studied extensively (for example see [4, 6]).

Let G^n be the tensor product of n copies of G. For a graph G, define $\overline{\alpha}(G) := \sup_I \mu_G(I)$, where I is a measurable independent set. It is easy to see that if $H \to G$, then $\overline{\alpha}(H) \geq \overline{\alpha}(G)$ and $H^n \to G^n$. Since $G^{i+1} \to G^i$, this in particular implies that $\overline{\alpha}(G^n)$ is a nondecreasing sequence, and $\lim_{n\to\infty} \overline{\alpha}(G^n)$ exists. For a finite vertex transitive graph H with the uniform measure, it is known that $\overline{\alpha}(H^n) = \overline{\alpha}(H)$ (see [2]). Now we prove an infinite version of this fact:

Lemma 1 *Let H be a (possibly infinite) vertex transitive graph. Then for any positive integer n,*

$$\overline{\alpha}(H^n) = \overline{\alpha}(H).$$

Proof. Since $\overline{\alpha}(H^n) \geq \overline{\alpha}(H)$, it is enough to prove that $\overline{\alpha}(H^n) \leq \overline{\alpha}(H)$. According to Definition 1, there exists a probability measure ν on a set S that together they satisfy Definition 1 (property 1 and 2). Consider an arbitrary measurable independent set $I \subseteq H^n$ and for a vertex $w \in H^n$ denote by $[w \in I]$ the function that is 1 if $w \in I$ and 0 otherwise. Note that

$$\mu_{H^n}(I) = \Pr_{v_i \in V(H)}[(v_1, \ldots, v_n) \in I] = \Pr_{\phi_i \in S, v \in V(H)}[(\phi_1(v), \ldots, \phi_n(v)) \in I]$$

Thus, there exists a choice of $\overline{\phi}_1, \ldots, \overline{\phi}_n$ such that

$$\begin{aligned}\mu_{H^n}(I) &\leq \Pr_{v \in V(H)}[(\overline{\phi}_1(v), \ldots, \overline{\phi}_n(v)) \in I] \\ &= \mu(\{v : (\overline{\phi}_1(v), \ldots, \overline{\phi}_n(v)) \in I, v \in V(H)\}).\end{aligned}$$

But $\{v : (\overline{\phi}_1(v), \ldots, \overline{\phi}_n(v)) \in I\}$ is an independent set in H because I is an independent set and $\{\overline{\phi}_i\}$ are homomorphisms. Thus we obtain that $\mu_{H^n}(I) \leq \overline{\alpha}(H)$ which completes the proof. ■

We call a vertex transitive graph H, a *descriptor* of G, if $H \to G$. Thus, for a descriptor H, we have

$$\overline{\alpha}(H) = \lim_{n\to\infty} \overline{\alpha}(H^n) \geq \lim_{n\to\infty} \overline{\alpha}(G^n).$$

Now, define $\mathrm{u}(G)$ as below:

$$\mathrm{u}(G) = \inf_{\textit{descriptor}\, H} \overline{\alpha}(H).$$

Trivially, we have

$$\lim_{n\to\infty} \overline{\alpha}(G^n) \leq \mathrm{u}(G). \tag{1}$$

This raises the following question:

Question 1 *Does every finite graph G satisfy* $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$*?*

This question is inspired by the work of Dinur and Friedgut [5], in which measure preserving homomorphisms are used to give a new proof for an Erdös-Ko-Rado-type theorem. We study the behavior of $\lim \overline{\alpha}(G^n)$ for graphs with probability measures. This is closely related and can be considered as the generalizations of some results in [4] and [3].

2 The results

The following lemma is the generalization of a result of [4] to graphs with probability measures.

Lemma 2 *For every finite graph G, if* $\lim \overline{\alpha}(G^n) > \frac{1}{2}$*, then* $\lim \overline{\alpha}(G^n) = 1$.

Proof. If $\lim \overline{\alpha}(G^n) > \frac{1}{2}$, then there exists a positive integer i such that $\overline{\alpha}(G^i) > \frac{1}{2}$. By letting $H = G^i$, trivially $\lim \overline{\alpha}(H^n) = \lim \overline{\alpha}(G^n)$. Let I be an independent set of measure $\frac{1}{2} + \epsilon$ of H. Define $J \subseteq V(H^n)$ as the set of vertices with strictly more than half coordinates in I. Clearly, J is an independent set of H^n. To prove that $\overline{\alpha}(H^n) = 1$, it suffices to prove that as n goes to infinity a random vertex which is taken from H^n with respect to μ_{H^n} is in J almost surely. Let X_i be an indicator random variable, such that $X_i = 1$ if the ith coordinate of the random vertex belongs to I and $X_i = 0$ otherwise. As a result, we have $E[X_i] = \overline{\alpha}(H)$ and the mean and variance of X_i is finite. Thus, by applying the weak law of large numbers for the random variable $X = \frac{1}{n}\sum_{i=1}^{n} X_i$, we obtain $\lim_{n\to\infty} P(|X - \overline{\alpha}(H)| < \epsilon') = 1$ for every positive real ϵ'. Therefore, X is greater than $\frac{1}{2}$ almost surely as desired. ∎

Now, we characterize the graphs for which $\lim \overline{\alpha}(G^n) = 1$ and by using this, we present some classes of graphs satisfying $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$.

Lemma 3 *For every finite graph G, if* $\mathrm{u}(G) = 1$ *then there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$, where $N(I)$ is the set of the vertices in $V(G)$ that are adjacent to at least one vertex in I.*

Proof. Suppose that every independent set $I \subseteq V(G)$ satisfies $\mu_G(I) \leq \mu_G(N(I))$. We claim that for all $Q \subseteq V(G)$, we have $\mu_G(Q) \leq \mu_G(N(Q))$. Suppose that for a $Q \subseteq V(G)$, we have $\mu_G(Q) > \mu_G(N(Q))$. Let I be the set of all vertices of Q without any neighbor in Q. Clearly, I is an independent set and since $\mu_G(Q) > \mu_G(N(Q))$, I is nonempty. Let $Q' = Q \backslash I$. Hence, $Q' \subseteq N(Q)$ and $N(I) \subseteq N(Q)\backslash Q'$. Therefore,

$$\mu_G(N(I)) \leq \mu_G(N(Q)) - \mu_G(Q') < \mu_G(Q) - \mu_G(Q') = \mu_G(I),$$

a contradiction.

Now let $G' = G \times K_2$, where $K_2 = uv$ has the uniform measure. It is clear that $X = \{(z, u) \in V(G') : z \in V(G)\}$ and $Y = V(G') - X$ is a bipartition of G'. Consider a flow network with vertices $V(G') \cup \{s, t\}$ and nonnegative capacities $c(s, x) = \mu_{G'}(x)$, and $c(y, t) = \mu_{G'}(y)$, for $x \in X$ and $y \in Y$, and $c(x, y) = \infty$ if $xy \in E(G')$. All the other capacities are 0. Let (S, T) be a minimum cut of this network with capacity $c(S, T)$. By the structure of the flow network, we have $c(S, T) \leq \frac{1}{2}$. Now, let $X_1 = S \cap X$, $Y_1 = S \cap Y$, $X_2 = T \cap X$, and $Y_2 = T \cap Y$. Since $c(x, y) = \infty$ if $xy \in E(G')$, there is not any edge between X_1 and Y_2. Therefore, $X_1 \cup Y_2$ is an independent set in G'. Since for all $Q \subseteq V(G)$, $\mu_G(Q) \leq \mu_G(N(Q))$, we have $\mu_{G'}(X_1) \leq \mu_{G'}(N(X_1))$ and $\mu_{G'}(Y_2) \leq \mu_{G'}(N(Y_2))$, which yields $\mu_{G'}(X_1) + \mu_{G'}(Y_2) \leq \mu_{G'}(N(X_1)) + \mu_{G'}(N(Y_2))$. Thus, we obtain $\mu_{G'}(X_1) + \mu_{G'}(Y_2) \leq \frac{1}{2}$. Therefore, we have $\mu_{G'}(X_2) + \mu_{G'}(Y_1) \geq \frac{1}{2}$ and because $c(S,T) = \mu_{G'}(X_2) + \mu_{G'}(Y_1)$, we obtain $c(S,T) = \frac{1}{2}$. Thus by the max-flow min-cut theorem, the value of a maximum flow f must be equal to $\frac{1}{2}$.

Now by using the maximum flow f, we construct a descriptor graph H for G' together with the measure preserving homomorphism $h : H \to G'$ as follows. The vertices of H are the elements of the interval $[0, 1)$ endowed with the (uniform) Lebesgue measure, and $E(H) = \{\{a, a + \frac{1}{2}\} : a \in [0, \frac{1}{2})\}$. It is easy to see that H is vertex transitive. Now we have to specify h. For $xy \in E(G')$, let f_{xy} denote the amount of the flow that passes through this edge. Since the value of f is equal to $\frac{1}{2}$, we have $\sum_{xy\in E(G')} f_{xy} = \frac{1}{2}$. So it is possible to partition the interval $[0, \frac{1}{2})$ into *disjoint* intervals in the following way: $[0, \frac{1}{2}) = \bigcup_{xy\in E(G')}[a_{xy}, a_{xy} + f_{xy})$, where $a_{xy} \geq 0$. Now h is defined as for every $z \in V(G') = X \cup Y$:

$$h^{-1}(z) = \begin{cases} \bigcup_{y:\, zy\in E(G')}[a_{zy}, a_{zy} + f_{zy}) & \text{if } z \in X \\ \bigcup_{x:\, xz\in E(G')}[\frac{1}{2} + a_{xz}, \frac{1}{2} + a_{xz} + f_{xz}) & \text{if } z \in Y \end{cases}$$

It is not hard to see that h is a measure preserving homomorphism from H to G'. Since $G' \to G$, H is a descriptor of G. Hence, we have $\mathrm{u}(G) \leq \frac{1}{2}$. ■

Lemma 4 *For every finite graph G, if there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$, then $\lim \overline{\alpha}(G^n) = 1$.*

Proof. Let $U = V(G)\backslash(I \cup N(I))$. Let $m_n = \overline{\alpha}(G^n)$. Trivially, $\mu_G(I) + \mu_G(N(I)) + \mu_G(U) = 1$, $m_1 \geq \mu_G(I)$ and $\mu_G(U) < 1$. Consider the union of the vertices with first coordinate in I and the vertices with first coordinate in U and last $n-1$ coordinates in the maximum measure independent set of G^{n-1}. It can be seen that this is an independent set and we have $m_n \geq \mu_G(I) + \mu_G(U)m_{n-1}$. By applying this inequality repeatedly, we obtain:

$$\begin{aligned} m_n &\geq \mu_G(I) + \mu_G(I)\mu_G(U) + ... + \mu_G(U)^{n-1}.m_1 \\ &\geq \mu_G(I) + \mu_G(I)\mu_G(U) + ... + \mu_G(I)\mu_G(U)^{n-1} = \frac{\mu_G(I) - \mu_G(I)\mu_G(U)^n}{1 - \mu_G(U)} \end{aligned}$$

Thus, we have $\lim_{n\to\infty} m_n \geq \frac{\mu_G(I)}{1-\mu_G(U)} = \frac{\mu_G(I)}{\mu_G(I)+\mu_G(N(I))} > \frac{1}{2}$, and by Lemma 2, we have $\lim \overline{\alpha}(G^n) = 1$. ■

Theorem 1 *For every finite graph G, the followings are equivalent:*

(i) $\lim \overline{\alpha}(G^n) = 1$*;*

(ii) $\mathrm{u}(G) = 1$*;*

(iii) there exists an independent set $I \subseteq V(G)$ such that $\mu_G(I) > \mu_G(N(I))$.

Proof. (i) implies (ii) by the inequality (1), (ii) implies (iii) by Lemma 3, and (iii) implies (i) by Lemma 4. ■

Corollary 1 *For every finite graph G, if $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$ then $\lim \overline{\alpha}(G^n) = \mathrm{u}(G)$.*

Remark 1 *It is not hard to see that for graphs with rational measures, Theorem 1(i) directly yields Theorem 1(iii). Noga Alon showed us that a density argument can be used to generalize this to graphs with real measures [1]. But, since we are mainly interested in* $\mathrm{u}(G)$*, we do not state his proof here.*

Corollary 1 presents a family of graphs for which equality holds in Question 1. In the next proposition, we show that this family contains bipartite graphs. Trivially, finite vertex transitive graphs are another family of graphs for which equality holds in Question 1.

Proposition 1 *For a finite bipartite graph* G*, we have* $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$.

Proof. Let X and Y be a bipartition of G. The set of the vertices of G^n whose first coordinates are in X and the set of the vertices of G^n whose first coordinates are in Y is a bipartition of G^n. Thus, for the bipartite graph G^n, $\overline{\alpha}(G^n) \geq \frac{1}{2}$. Therefore, by Lemma 2, we obtain $\lim \overline{\alpha}(G^n) \in \{\frac{1}{2}, 1\}$. ■

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