On The Signed Edge Domination Number of Graphs *†

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Abstract

Let $\gamma'_s(G)$ be the signed edge domination number of G. In 2006, Xu conjectured that: for any 2-connected graph G of order $n(n \geq 2)$, $\gamma'_s(G) \geq 1$. In this article we show that this conjecture is not true. More precisely, we show that for any positive integer m, there exists an m-connected graph G such that $\gamma'_s(G) \leq -\frac{m}{6}|V(G)|$. Also for every two natural numbers m and n, we determine $\gamma'_s(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph with part sizes m and n.

Introduction

In this paper all of graphs that we consider are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G denotes the number of vertices of G. For any $v \in V(G)$, $d(v)$ is the degree of v and $E(v)$ is the set of all edges incident with v. If $e = uv \in E(G)$, then we put $N[e] = \{u'v' \in E(G)|u' = u \text{ or } v' = v\}$. Let G be a graph and $f : E(G) \longrightarrow \{-1, 1\}$ be a function. For every vertex v, we define $s_v = \sum_{e \in E(v)} f(e)$. We denote the complete bipartite graph with two parts of sizes m and n, by $K_{m,n}$. Also we denote the cycle of order n, by C_n. In [4] the signed edge domination function of graphs was introduced as follows:

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Let $G = (V(G), E(G))$ be a non-empty graph. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge domination function* (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$, for every $e \in E(G)$. The *signed edge domination number* of G is defined as,

$$\gamma'_s(G) = min\{ \sum_{e \in E(G)} f(e) \mid f \text{ is an SEDF of } G\}.$$

Several papers have been published on lower bounds and upper bounds of the signed edge domination number of graphs, for instance, see [2], [3], [4], [5], [6]. In [2], Xu posed the following conjecture:

For any 2-connected graph G of order $n(n \geq 2)$, $\gamma'_s(G) \geq 1$.

In the first section we give some counterexamples to this conjecture by showing that for any natural number m, there exists an m-connected graph G such that $\gamma'_s(G) \leq -\frac{m}{6}|V(G)|$. For any natural number k, let $g(k) = min\{\gamma'_s(G) \,|\, |V(G)| = k\}$. In [2] the following problem was posed:

Determine the exact value of $g(k)$ for every positive integer k. In Section 1, it is shown that for every natural number k, $k \geq 12$, $g(k) \leq \frac{-(k-8)^2}{72}$.

1. Counterexamples to a Conjecture

In this section we present some counterexamples to a conjecture that appeared in [2]. We start this section by the following simple lemma and leave the proof to the reader.

Lemma 1. *Let $f : E(G) \longrightarrow \{-1, 1\}$ be a function. Then f is an* SEDF *of G, if and only if for any edge $e = uv$, $s_u + s_v - f(e) \geq 1$. Moreover, if f is an* SEDF, *then $s_u + s_v \geq 0$.*

An $L_{(m,n)}$-graph G is a graph of order $(n + 1)(mn + m + 1)$, whose vertices can be partitioned into $n + 1$ subsets $V_1, \ldots, V_{n+1}$ such that:

(i) The induced subgraph on V_1 is the complete graph K_{mn+m+1}.

(ii) The induced subgraph on V_i, $2 \le i \le n+1$ is the complement of K_{mn+m+1}.

(iii) For every i, $2 \le i \le n+1$, all edges between V_1 and V_i form m disjoint matchings of size $mn+m+1$.

(iv) There is no edge between V_i and V_j for any i, j, $2 \le i < j \le n+1$.

It is well-known that for any natural number r, the edge chromatic number of $K_{r,r}$ is r, see Theorem 6 of [1, p.93]. Thus for every pair of natural numbers m and n, there is an $L_{(m,n)}$-graph.

Theorem 1. *Let m and n be two natural numbers. Then for every $L_{(m,n)}$-graph G, we have,*

$$\gamma'_s(G) \le \frac{(mn+m+1)(m-mn)}{2}.$$

Proof. To prove the inequality we provide an SEDF for G, say f, such that,

$$\sum_{e\in E(G)} f(e) = \frac{(mn+m+1)(m-mn)}{2}.$$

Define $f(e) = 1$, if both end points of e are contained in V_1, and $f(e) = -1$, otherwise. We find,

$$\begin{aligned}\sum_{e\in E(G)} f(e) &= \frac{(mn+m+1)(mn+m)}{2} - (mn+m+1)mn \\ &= \frac{(mn+m+1)(m-mn)}{2}.\end{aligned}$$

It can be easily verified that for every $v \in V_1$, $s_v = m$, and for every $v \in V(G) \setminus V_1$, $s_v = -m$. Now, Lemma 1 yields that f is an SEDF for G. $\square$

Example 1. Consider the $L_{(2,1)}$-graph G shown in Figure 1. The graph clearly has perfect matching; and by applying Lemma 1 to the edges of this matching we may conclude that for every SEDF f of this graph, $\sum_{e\in E(G)} f(e) = \frac{1}{2}\sum_{v\in V(G)} s_v \ge 0$, hence $\gamma'_s(G) \ge 0$. But

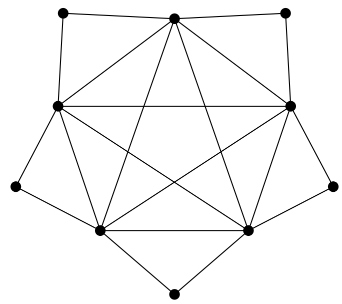

Figure 1: A 2-connected $L_{(2,1)}$-graph with $\gamma'_s < 1$.

it follows from Theorem 1 that $\gamma'_s(G) \leq 0$. Consequently, $\gamma'_s(G) = 0$ and the bound in Theorem 1 is sharp for this graph.

In [2], Xu conjectured that for any 2-connected graph G of order $n(n \geq 2)$, $\gamma'_s(G) \geq 1$. The next theorem shows that conjecture fails.

Theorem 2. *For any natural number m, there exists an m-connected graph G such that $\gamma'_s(G) \leq -\frac{m}{6}|V(G)|$.*

Proof. First we claim that for each pair of natural numbers m and n, every $L_{(m,n)}$-graph is an m-connected graph. To see this we note that if one omits at most $m-1$ vertices of an $L_{(m,n)}$-graph, then some vertices of V_1 remain (because $|V_1| = mn + m + 1$) and since the degree of each vertex of V_i, $2 \leq i \leq n+1$ is m, the claim is proved.

Now, for any natural number m, consider an $L_{(m,2)}$-graph G. By Theorem 1, the following inequality holds:

$$\gamma'_s(G) \leq \frac{1}{2}(2m + m + 1)(m - 2m) = -\frac{m}{6}|V(G)|.$$

$\square$

Remark 1. If we repeat the previous proof for an $L_{(m,n)}$-graph instead of an $L_{(m,2)}$-graph, then we find $\gamma'_s(G) \leq \frac{-m(n-1)}{2(n+1)}|V(G)|$. Hence for large enough n, $\gamma'_s(G) \leq \frac{-m+1}{2}|V(G)|$.

Lemma 2. *Let G be a graph with an* SEDF*. If G contains C_n as subgraph, then*

$$\sum_{v \in V(C_n)} s_v \geq 0.$$

Proof. Let $V(C_n) = \{v_1, \ldots, v_n\}$. Clearly, we have,

$$\sum_{i=1}^{n} s_{v_i} = \frac{1}{2} \sum_{i=1}^{n} (s_{v_i} + s_{v_{i+1}}),$$

where indices are modulo n. Thus by Lemma 1, the proof is complete. $\square$

Theorem 3. *For every graph G of order n, $\gamma'_s(G) \geq \frac{-n^2}{16}$.*

Proof. An elementary graph is a graph in which each component is a 1-regular graph or a 2-regular graph. Let H be an elementary subgraph of G with maximum number of vertices. With no loss of generality we may assume that H has no even cycle, since one can replace an even cycle of size $2k$ by k vertex-disjoint edges. Suppose α is the number of vertices of G which are not covered by H. We claim that for every vertex v which is not covered by H, $d(v) \leq \frac{n-\alpha}{2}$.

To see this, we note that v is adjacent to none of the other $\alpha - 1$ vertices which are not covered by H, because otherwise we could find an elementary subgraph H' which covers more vertices of G, a contradiction. Also, v is adjacent to none of the vertices of an odd cycle of H, because if v is adjacent to a vertex u of an odd cycle C, we can decompose the set $E(C) \bigcup \{uv\}$ into vertex-disjoint edges which cover $V(C) \bigcup \{v\}$, obtaining an elementary subgraph H' which covers more vertices, a contradiction. If v is adjacent to both end points of an edge in the matching part of H, then we can add an odd cycle of length 3 to H, obtaining a bigger elementary subgraph, a contradiction. Thus the degree of v does not exceed the number of the edges in the matching part of H, so, $d(v) \leq \frac{n-\alpha}{2}$.

By Lemmas 1 and 2, $\sum_{v \in V(H)} s_v \geq 0$. Therefore we have,

$$\sum_{e\in E(G)} f(e) = \frac{1}{2}(\sum_{v\in V(H)} s_v + \sum_{v\in V(G)\setminus V(H)} s_v) \geq \frac{1}{2}\sum_{v\in V(G)\setminus V(H)} s_v$$
$$\geq \frac{-1}{2}\sum_{v\in V(G)\setminus V(H)} d(v) \geq \frac{-\alpha(n-\alpha)}{4} \geq \frac{-n^2}{16}.$$

□

Corollary 1. *If G has a spanning elementary subgraph, then $\gamma'_s(G) \geq 0$.*

Proof. In the proof of the previous theorem replace α by 0. □

In [2] the following problem has been posed:

Determine the exact value of $g(k)$ for every positive integer k. In the next theorem we find a lower and an upper bound for $g(k)$, $k \geq 12$.

Theorem 4. *For every natural number k, $k \geq 12$, $-\frac{k^2}{16} \leq g(k) \leq -\frac{(k-8)^2}{72}$.*

Proof. The lower bound is an immediate consequence of Theorem 3. First we obtain the upper bound for $k = 9m+3$. In the proof of the Theorem 1, we constructed a graph G of order $(n+1)(mn+m+1)$ vertices for which,

$$\gamma'_s(G) \leq \frac{(mn+m+1)(m-mn)}{2}.$$

Assume that $n = 2$. We have,

$$g(9m+3) \leq \frac{-m}{6}(9m+3).$$

Since $k \geq 12$, for $k = 9m+3$ we find,

$$g(k) \leq \frac{-\left(\frac{k-3}{9}\right)}{6}k \leq \frac{-k^2}{72}.$$

Now, for every k, we may write $k = 9m + 3 + r$, where $0 \le r < 9$. By adding r isolated vertices to the constructed graph for $9m + 3$, and using the previous inequality for $g(9m + 3)$, we have the following:

$$g(k) \le \frac{-(k-r)^2}{72} \le \frac{-(k-8)^2}{72},$$

and the proof is complete. $\square$

2. Signed Edge Domination of Complete Bipartite Graphs

In this section we want to obtain the signed edge domination number of complete bipartite graphs.

Theorem 5. *Let m and n be two natural numbers where $m \le n$. Then the following hold:*

(i) If m and n are even, then $\gamma_s'(K_{m,n}) = \min(2m, n)$,

(ii) If m and n are odd, then $\gamma_s'(K_{m,n}) = \min(2m-1, n)$,

(iii) If m is even and n is odd, then $\gamma_s'(K_{m,n}) = \min(3m, \max(2m, n+1))$,

(iv) If m is odd and n is even, then $\gamma_s'(K_{m,n}) = \min(3m-1, \max(2m, n))$.

Proof. Let (X, Y) be two parts of the complete bipartite graph $K_{m,n}$ and $X = \{u_1, \ldots, u_m\}$ and $Y = \{v_1, \ldots, v_n\}$. We note that if f is an SEDF for $K_{m,n}$, then we have,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u = \sum_{v \in Y} s_v.$$

(i) First we show that $\gamma_s'(K_{m,n}) \ge \min(2m, n)$. It suffices to show that if f is an SEDF such that $\sum_{e \in E(K_{m,n})} f(e) < 2m$, then $\sum_{e \in E(K_{m,n})} f(e) \ge n$. Since $\sum_{e \in E(K_{m,n})} f(e) < 2m$, there exists a vertex $u \in X$ such that $s_u < 2$. But s_u is even and so $s_u \le 0$. If $s_u = 0$, then u is incident with $n/2$ edges with value 1 and $n/2$ edges with value -1. If $f(uv) = 1$, for some $v \in Y$, then by Lemma 1, $s_v \geq 2$. If $f(uv) = -1$, for some $v \in Y$, then we find $s_v \geq 0$. Thus we have $\sum_{e \in E(K_{m,n})} f(e) = \sum_{v \in Y} s_v \geq 2\left(\frac{n}{2}\right) = n$. If $s_u < 0$, then $s_u \leq -2$. Now, for each $v \in Y$, by Lemma 1, $s_v \geq 2$. Therefore we have the following:

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{v \in Y} s_v \geq 2n > n.$$

Hence $\gamma'_s(K_{m,n}) \geq \min(2m, n)$.

We now show that there exist two SEDF, say f and g, such that $\sum_{e \in E(K_{m,n})} f(e) = 2m$ and $\sum_{e \in E(K_{m,n})} g(e) = n$. Let f be define as follows:

$$f(u_i v_j) = \begin{cases} 1 & \text{if } i + j \text{ is odd} \\ 1 & \text{if } i = j \\ -1 & \text{otherwise.} \end{cases}$$

It is clear that for every u_i, $s_{u_i} = 2$. Also one can see that $s_{v_i} \geq 0$, for $i = 1, \ldots, n$. Now, by Lemma 1, we see that f is an SEDF. Therefore,

$$\gamma'_s(K_{m,n}) \leq \sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u = 2m,$$

as required.

Define g as follows:

$$g(u_i v_j) = \begin{cases} 1 & \text{if } i + j \text{ is odd} \\ 1 & \text{if } i \text{ is even and } i = j \text{ modulo } m \\ -1 & \text{otherwise.} \end{cases}$$

We note that if i is even, then $s_{v_i} = 2$; and if i is odd, then $s_{v_i} = 0$. Also, if i is even, then $s_{u_i} \geq 2$; and if i is odd, then $s_{u_i} = 0$. Now, Lemma 1 implies that g is an SEDF. Therefore,

$$\gamma'_s(K_{m,n}) \leq \sum_{e \in E(K_{m,n})} g(e) = \sum_{i=1}^{n} s_{v_i} = \frac{2n}{2} = n,$$

as required.

(ii) First we show that $\gamma_s'(K_{m,n}) \geq \min(2m-1, n)$. It is enough to show that if f is an SEDF with $\sum_{e\in E(K_{m,n})} f(e) < n$, then $\sum_{e\in E(K_{m,n})} f(e) \geq 2m-1$. Since $\sum_{e\in E(K_{m,n})} f(e) < n$, there exists a vertex $v \in Y$ such that $s_v < 1$. But s_v is odd and so $s_v \leq -1$. If $s_v = -1$, then v is incident with $\frac{m-1}{2}$ edges with value 1 and $\frac{m+1}{2}$ edges with value -1. If $f(uv) = 1$, for some $u \in X$, then by Lemma 1, $s_u \geq 3$. If $f(uv) = -1$, for some $u \in X$, then similarly we have $s_u \geq 1$. Thus we have the following:

$$\sum_{e\in E(K_{m,n})} f(e) = \sum_{u\in X} s_u \geq 3\left(\frac{m-1}{2}\right) + \frac{m+1}{2} = 2m-1.$$

If $s_v < -1$, then $s_v \leq -3$. Now, by Lemma 1, $s_u \geq 3$ for each $u \in X$. Therefore we find that,

$$\sum_{e\in E(K_{m,n})} f(e) = \sum_{u\in X} s_u \geq 3m > 2m-1.$$

Hence $\gamma_s'(K_{m,n}) \geq \min(2m-1, n)$. We now show that there are two SEDF f and g such that $\sum_{e\in E(K_{m,n})} f(e) = 2m-1$ and $\sum_{e\in E(K_{m,n})} g(e) = n$.

Define f and g as follows,

$$f(u_iv_j) = \begin{cases} 1 & \text{if } i+j \text{ is odd} \\ 1 & \text{if } i = j \\ -1 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $s_{u_i} = 3$, if i is even; and $s_{u_i} = 1$, if i is odd. Also, we have,

$$s_{v_j} = \begin{cases} 3 & \text{if } j \text{ is even and } j \leq m \\ 1 & \text{if } j \text{ is odd and } j \leq m \\ 1 & \text{if } j \text{ is even and } j > m \\ -1 & \text{if } j \text{ is odd and } j > m. \end{cases}$$

Consequently, f is an SEDF, by lemma 1. Therefore,

$$\gamma_s'(K_{m,n}) \leq \sum_{e\in E(K_{m,n})} f(e) = \sum_{u\in X} s_u = 3\left(\frac{m-1}{2}\right) + \frac{m+1}{2} = 2m-1,$$

as required.

Define g as follows:

$$g(u_iv_j) = \begin{cases} 1 & \text{if } i+j \text{ is odd} \\ 1 & \text{if } j \text{ is odd and } i = j \text{ modulo } (m+1) \\ -1 & \text{otherwise.} \end{cases}$$

It is not hard to see that for any $u \in X$, $s_u \geq 1$ and for any $v \in Y, s_v = 1$. Therefore g is an SEDF and $\gamma'_s(K_{m,n}) \leq \sum_{e \in E(K_{m,n})} g(e) = \sum_{v \in Y} s_v = n$.

(iii) Three cases may be considered:

Case 1. $n+1 \leq 2m$. We claim that $\gamma'_s(K_{m,n}) = 2m$. First we show that $\gamma'_s(K_{m,n}) \geq 2m$. By contradiction suppose that there exists an SEDF, say f, such that $\sum_{e \in E(K_{m,n})} f(e) < 2m$. Since $m \leq n$, we find that $\sum_{e \in E(K_{m,n})} f(e) < 2n$. Thus there exists a vertex $v \in Y$ such that $s_v < 2$. On the other hand since s_v is even, $s_v \leq 0$. If $s_v = 0$, then v is incident with $m/2$ edges with value 1 and $m/2$ edges with value -1. If $f(uv) = 1$, for some $u \in X$, then by Lemma 1, we have, $s_u \geq 2$. Since s_u is odd we find $s_u \geq 3$. If $f(uv) = -1$, for some $u \in X$, then by a similar argument one can see that $s_u \geq 1$. Thus,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u \geq 3m/2 + m/2 = 2m,$$

a contradiction. Hence $\gamma'_s(K_{m,n}) \geq 2m$.

If $s_v < 0$, then $s_v \leq -2$. By Lemma 1, for every $u \in X$, $s_u \geq 2$. Hence we obtain that,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u \geq 2m,$$

a contradiction.

We now define an SEDF, say f, such that $\sum_{e \in E(K_{m,n})} f(e) = 2m$. Let $X_1 = \{u_1, \ldots, u_{\frac{m}{2}}\}$,$X_2 = X - X_1, Y_1 = \{v_1, \ldots, v_{\frac{n+1}{2}}\}$ and $Y_2 = Y - Y_1$.

Now, define f as follows:

$$f(e) = \begin{cases} 1 & \text{if } e \text{ meets } X_1 \text{ and } Y_2 \\ 1 & \text{if } e \text{ meets } X_2 \text{ and } Y_1 \\ 1 & \text{if } e = u_i v_i,\ 1 \leq i \leq m/2 \\ 1 & \text{if } e = u_i v_j,\ 1 \leq i \leq m/2 \text{ and } j = (i + m/2) \text{ modulo } (n+1)/2 \\ -1 & \text{otherwise.} \end{cases}$$

For each $u \in X_1$, we have $s_u = 3$. For every $u \in X_2$, we have $s_u = 1$. Also for each $v \in Y_1$, we have $s_v \geq 2$. For each $v \in Y_2$, $s_v = 0$. By Lemma 1, it is not hard to see that f is an SEDF. Also we have,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u = \frac{3m}{2} + \frac{m}{2} = 2m.$$

Case 2. $2m < n + 1 \leq 3m$. We claim that $\gamma'_s(K_{m,n}) = n + 1$. First we show that $\gamma'_s(K_{m,n}) \geq n + 1$. By contradiction assume that there exists an SEDF, f, such that $\sum_{e \in E(K_{m,n})} f(e) < n + 1$. Since $n + 1 \leq 3m$, we have $\sum_{e \in E(K_{m,n})} f(e) < 3m$. Therefore there exists a vertex $u \in X$ such that $s_u < 3$. Since s_u is odd, $s_u \leq 1$. If $s_u = 1$, then u is incident with $\frac{n+1}{2}$ edges with value 1 and $\frac{n-1}{2}$ edges with value -1. If $f(uv) = 1$, for some $v \in Y$, then by Lemma 1, $s_v \geq 1$ and since s_v is even, we have $s_v \geq 2$. If $f(uv) = -1$, for some $v \in Y$, then one can see that $s_v \geq 0$. Hence,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{v \in Y} s_v \geq 2\left(\frac{n+1}{2}\right) = n + 1,$$

which is a contradiction.

If $s_u < 1$, then $s_u \leq -1$. By Lemma 1, $s_v \geq 1$, for each $v \in Y$. Thus, $\sum_{e \in E(K_{m,n})} f(e) = \sum_{v \in Y} s_v \geq n$. Since the number of edges is even, $\sum_{e \in E(K_{m,n})} f(e)$ is also even. Now, since n is odd, $\sum_{e \in E(K_{m,n})} f(e) \geq n + 1$, a contradiction. Hence $\gamma'_s(K_{m,n}) \geq n + 1$.

We now define an SEDF, say f, such that $\sum_{e \in E(K_{m,n})} f(e) = n + 1$. Let $X_1 = \{u_1, \ldots, u_{\frac{m}{2}}\}$,$X_2 = X - X_1$,$Y_1 = \{v_1, \ldots, v_{\frac{n+1}{2}}\}$ and $Y_2 = Y - Y_1$. Let us define,

$$f(e) = \begin{cases} 1 & \text{if } e \text{ meets } X_1 \text{ and } Y_2 \\ 1 & \text{if } e \text{ meets } X_2 \text{ and } Y_1 \\ 1 & \text{if } e = u_i v_j \text{ and } i = j \text{ modulo } \frac{m}{2},\ 1 \le i \le \frac{m}{2}, 1 \le j \le \frac{n+1}{2} \\ -1 & \text{otherwise.} \end{cases}$$

It is straightforward to see that for each vertex $u \in X_1$, $s_u \geq 3$ and for each vertex $u \in X_2$, $s_u = 1$. Also, for each $v \in Y_1, s_v = 2$ and for each $v \in Y_2$, $s_v = 0$. Thus we have,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{v \in Y} s_v = \frac{2(n+1)}{2} = n + 1.$$

By Lemma 1, it can be easily seen that f is an SEDF.

Case 3. $3m < n+1$. We claim that $\gamma'_s(K_{m,n}) = 3m$. First we prove that $\gamma'_s(K_{m,n}) \geq 3m$. By contradiction assume that there exists an SEDF f such that $\gamma'_s(K_{m,n}) < 3m$. Hence there exists a vertex $u \in X$ such that $s_u < 3$. By a similar method as we saw in the proof of Case 2, we conclude that $\sum_{e \in E(K_{m,n})} f(e) \geq n + 1$, which contradicts the inequality $3m < n + 1$. Hence $\gamma'_s(K_{m,n}) \geq 3m$.

We now define an SEDF, say f, such that $\sum_{e \in E(K_{m,n})} f(e) = 3m$. Consider a partition of X such as X_1 and X_2, each of them containing $m/2$ vertices. Also suppose that Y_1, Y_2 and Y_3 is a partition of Y such that $|Y_1| = |Y_2| = \frac{n-3}{2}$ and $|Y_3| = 3$. We define f as follows:

$$f(e) = \begin{cases} -1 & \text{if } e \text{ meets } X_1 \text{ and } Y_1 \\ -1 & \text{if } e \text{ meets } X_2 \text{ and } Y_2 \\ 1 & \text{otherwise.} \end{cases}$$

Now, it can be easily seen that for any $u \in X$, $s_u = 3$ and for any $v \in Y, s_v \geq 0$. By Lemma 1, f is an SEDF. Also we have,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u = 3m.$$

(iv) Three cases may be considered:

Case 1. $n \le 2m$. We claim that $\gamma_s'(K_{m,n}) = 2m$. First we show that $\gamma_s'(K_{m,n}) \ge 2m$. By contradiction suppose that f is an SEDF such that $\sum_{e\in E(K_{m,n})} f(e) < 2m$. Thus, there exists a vertex $u \in X$ such that $s_u < 2$. Since s_u is even, $s_u \le 0$. If $s_u = 0$, then $\frac{n}{2}$ edges incident with u have value 1 and other $\frac{n}{2}$ edges have value -1. If $f(uv) = 1$, for some $v \in Y$, then by Lemma 1, $s_v \ge 2$ and since s_v is odd, we have $s_v \ge 3$. If $f(uv) = -1$, then we have $s_v \ge 1$. Therefore,

$$\sum_{e\in E(K_{m,n})} f(e) = \sum_{v\in Y} s_v \ge 3n/2 + n/2 = 2n > 2m,$$

a contradiction.

Now, assume that $s_u < 0$. Thus $s_u \le -2$. By Lemma 1, $s_v \ge 2$, for any $v \in Y$. Therefore,

$$\sum_{e\in E(K_{m,n})} f(e) = \sum_{v\in Y} s_v \ge 2n > 2m,$$

a contradiction. Hence $\gamma_s'(K_{m,n}) \ge 2m$.

We now define an SEDF, say f, such that $\sum_{e\in E(K_{m,n})} f(e) = 2m$. We know that all edges of $K_{m,n}$ can be decomposed into $K_{m,m}$ and $K_{n-m,m}$. Note that m and $n-m$ are odd and $n - m \le m$. By Part (ii) there exists an SEDF, g_1, for $K_{m,m}$ such that $\sum_{e\in E(K_{m,m})} g_1(e) = m$ and for each vertex x, $s_x = 1$. Also there exists an SEDF, say g_2, for $K_{n-m,m}$ such that $\sum_{e\in E(K_{n-m,m})} g_2(e) = m$ and for every vertex $u \in X$, $s_u = 1$ and for other vertex v, $s_v \ge 1$. Now, define an SEDF, say f, for $K_{m,n}$ such that for each $e \in E(K_{m,m})$, $f(e) = g_1(e)$ and for every $e \in E(K_{n-m,m})$, $f(e) = g_2(e)$. Now, for every $u \in X$, we have $s_u = 2$ and for each $v \in Y$, we have $s_v \ge 1$. By Lemma 1, f is an SEDF and moreover we find,

$$\sum_{e\in E(K_{m,n})} f(e) = \sum_{e\in E(K_{m,m})} g_1(e) + \sum_{e\in E(K_{n-m,m})} g_2(e) = m + m = 2m.$$

Case 2. $2m < n \le 3m-1$. We claim that $\gamma_s'(K_{m,n}) = n$. First we show that $\gamma_s'(K_{m,n}) \ge n$. By contradiction assume that f is an SEDF and $\sum_{e\in E(K_{m,n})} f(e) < n$. This implies that there exists a vertex $v \in Y$ such that $s_v < 1$. Since s_v is odd, we have $s_v \le -1$. If $s_v = -1$, then v is incident with $\frac{m-1}{2}$ edges with value 1 and $\frac{m+1}{2}$ edges with value -1. If

$f(uv)=1$, for some $u \in X$, then by Lemma 1 , $s_u \geq 3$. Now, since s_u is even, $s_u \geq 4$. If $f(uv)=-1$, then we conclude that $s_u \geq 2$. Thus,

$$\sum_{e \in E(K_{m,n})} f(e)=\sum_{u \in X} s_u \geq \frac{4(m-1)}{2}+\frac{2(m+1)}{2}=3m-1 \geq n,$$

a contradiction.

If $s_v<-1$, then $s_v \leq -3$. By Lemma 1, for every $u \in X$, $s_u \geq 3$. Hence we obtain,

$$\sum_{e \in E(K_{m,n})} f(e)=\sum_{u \in X} s_u \geq 3m>n,$$

a contradiction. Hence $\gamma_s'(K_{m,n}) \geq n$.

By a similar argument as we did in the Case 1, we may find an SEDF, say f, for $K_{m,n}$ such that $\sum_{e \in E(K_{m,n})} f(e)=m+(n-m)=n$, as desired.

Case 3. $3m-1<n$. We claim that $\gamma_s'(K_{m,n})=3m-1$. First we show that $\gamma_s'(K_{m,n}) \geq 3m-1$. By contradiction assume that f is an SEDF such that $\sum_{e \in E(K_{m,n})} f(e)<3m-1$. Since $3m-1<n$, there exists a vertex $v \in Y$ such that $s_v<1$. Now, by a similar argument as we did in Case 2, one can see that $\sum_{e \in E(K_{m,n})} f(e) \geq 3m-1$, a contradiction.

We now define an SEDF, say f, such that $\sum_{e \in E(K_{m,n})} f(e)=3m-1$. Consider a partition of X into two subsets X_1 and X_2 such that $|X_1|=\frac{m+1}{2}$ and $|X_2|=\frac{m-1}{2}$. Also consider a partition of Y such as Y_1, Y_2 and Y_3 such that $|Y_1|=\frac{3m+3}{2}$, $|Y_2|=\frac{n}{2}-2$, $|Y_3|=\frac{n-(3m-1)}{2}$. Let $X_1=\{u_1,\ldots,u_{\frac{m+1}{2}}\}, Y_1=\{v_1,\ldots,v_{\frac{3m+3}{2}}\}$. Define f as follows:

$$f(e)=\begin{cases} 1 & \text{if e meets } X_1 \text{ and } Y_2 \\ 1 & \text{if e meets } X_2 \text{ and } Y_1 \\ 1 & \text{if e meets } X_2 \text{ and } Y_3 \\ 1 & e=u_iv_j,\ 1 \leq i \leq \frac{m+1}{2} \text{ and } j \in \{3i-2,3i-1,3i\} \\ -1 & \text{otherwise.} \end{cases}$$

One can easily see that for any $u \in X_1$, $s_u=2$, and for any $u \in X_2$, $s_u=4$. Also we have,

$$s_v = \begin{cases} 1 & v \in Y_1 \cup Y_2 \\ -1 & v \in Y_3. \end{cases}$$

Now, Lemma 1 implies that f is an SEDF.

Also, we have,

$$\sum_{e \in E(K_{m,n})} f(e) = \sum_{u \in X} s_u = \frac{2(m+1)}{2} + \frac{4(m-1)}{2} = 3m - 1.$$

□

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References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, North-Holland, 1976.

[2] B. Xu, Two classes of edge domination in graphs, Disc. Appl. Math. 154 (2006), No. 10, 1541-1546.

[3] B. Xu, On edge domination numbers of graphs, Disc. Math. 294 (2005), No. 3, 311-316.

[4] B. Xu, On signed edge domination numbers of graphs, Disc. Math. 239 (2001) 179-189.

[5] B. Zelinka, On signed edge domination numbers of trees, Math. Bohem. 127 (2002), no. 1, 49-55.

[6] Z. Zhang, B. Xu, Y. Li, L. Liu, A note on the lower bounds of signed edge domination number of a graph, Discrete Math. 195 (1999), No. 1-3, 295-298

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