Asymptotic enumeration of 2-covers and line graphs

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Abstract

In this paper we find asymptotic enumerations for the number of line graphs on n-labelled vertices and for different types of related combinatorial objects called 2-covers.

We find that the number of 2-covers, s_n , and proper 2-covers, t_n , on [n] both have asymptotic growth

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2}\log(2n/\log n)\right) = B_{2n} 2^{-n} \sqrt{\frac{\log n}{2n}},$$

where B_{2n} is the 2*n*th Bell number, while the number of restricted 2-covers, u_n , restricted, proper 2-covers on [n], v_n , and line graphs l_n , all have growth

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2}\log(2n/\log n)\right]^2\right).$$

In our proofs we use probabilistic arguments for the unrestricted types of 2-covers and and generating function methods for the restricted types of 2-covers and line graphs.

KEYWORDS: ASYMPTOTIC ENUMERATION, LINE GRAPHS, SET PARTITIONS

1 Introduction

A k-cover of $[n] := \{1, 2, ..., n\}$ is a multiset of subsets $\{S_1, S_2, ..., S_m\}$, $S_i \subseteq [n]$, (possibly with $S_i = S_j$ for some $i \neq j$), such that for each $d \in [n]$ the number of j such that $d \in S_j$ is exactly k. A k-cover is called *proper* if $S_i \neq S_j$ whenever $i \neq j$. A k-cover is called *restricted* if the intersection of any k of the S_i contains at most one element. These definitions have been taken from [4]. Note that for a proper k-cover $\{S_1, \ldots, S_m\}$ is a set.

The line graph L(G) of a simple graph G is the graph whose vertex set is the edge set of G and such that two vertices are adjacent in L(G) if and only if the corresponding edges of G are adjacent.

Let s_n be the number of 2-covers of [n]; let t_n be the number of proper 2-covers of [n]; let u_n be the number of restricted, proper 2-covers of [n]; let v_n be the number of restricted, proper 2-covers of [n]; and let l_n be the number of line graphs on n labelled vertices. Let B_n be the nth Bell number. Given sequences a_n and b_n , we write $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

Theorem 1 The number of 2-covers and the number of proper 2-covers have asymptotic growth

$$s_n \sim t_n \sim B_{2n} 2^{-n} \exp\left(-\frac{1}{2}\log(2n/\log n)\right) \tag{1}$$

while the number of restricted 2-covers, restricted, proper 2-covers and line graphs all have asymptotic growth

$$u_n \sim v_n \sim l_n \sim B_{2n} 2^{-n} n^{-1/2} \exp\left(-\left[\frac{1}{2}\log(2n/\log n)\right]^2\right).$$
 (2)

We make some initial observations regarding 2-covers, special graphs and orbits in Section 2. We use a probabilistic method to prove (1) in Section 3. A pair of technical lemmas are proven in Section 3.1, (1) is proven for s_n in Section 3.2 and it is proven for t_n in Section 3.3. We prove (2) in Section 4.

In both probabilistic and generating function proofs we will make use of Lambert's W-function W(t), which is a solution to

$$W(t)e^{W(t)} = t \tag{3}$$

and which has asymptotics (see (3.10) of [6])

$$W(t) = \log t - \log \log t + \frac{\log \log t}{\log t} + o\left(\frac{1}{\log t}\right) \quad \text{as} \quad t \to \infty.$$
 (4)

For each k-cover S_1, \ldots, S_m of [n] we can define an associated $m \times n$ incidence matrix M with entries given by

$$M_{i,j} = \begin{cases} 1 & \text{if } j \in S_i \\ 0 & \text{if } j \notin S_i \end{cases}$$

Note that M has exactly k ones in each column and that the rows are unordered. A k-cover is proper if and only if M has no repeated rows. A k-cover is restricted if and only if M has no repeated columns. Therefore, Theorem 1 is equivalent to the asymptotic enumeration of certain 0-1 matrices. The general methods of this paper were used for the asyptotic enumeration of other 0-1 matrices called incidence matrices in [2, 3].

2 2-covers, line graphs and orbits

In this section we establish correspondences between 2-covers, line graphs and orbits of certain permutation groups.

2.1 2-covers and graphs

We define a *special multigraph* to be a multigraph with no isolated vertices or loops. Our first result is

Proposition 1 There is a bijection between 2-covers on [n] and special multigraphs having unlabelled vertices and n labelled edges, such that

- proper 2-covers correspond to multigraphs having no connected component of size 2;
- restricted 2-covers correspond to simple graphs.

Proof Let S_1, \ldots, S_m be a 2-cover of [n]. Construct a graph G as follows:

- the vertex set is [m];
- for each $i \in [n]$, there is an edge e_i joining vertices j and k, where S_j and S_k are the two sets of the 2-cover containing i.

The graph G is a multigraph (that is, repeated edges are permitted), but it has no isolated vertices and no loops.

Conversely, given a multigraph without isolated vertices or loops, we can recover a 2-cover: number the edges e_1, \ldots, e_n , and let S_i be the set of indices j for which the *i*th vertex lies on edge e_j . Thus we have the first part of the proposition.

The second part comes from observing that a "repeated set" in a 2-cover corresponds to a pair of vertices lying on the same edges, while a pair of elements lying in two different sets correspond to a pair of edges incident to the same two vertices.

2.2 Generating function identities for 2-covers

Recall that s_n , t_n , u_n and v_n denote the numbers of 2-covers, proper 2-covers, restricted 2-covers, and restricted proper 2-covers respectively. Using Proposition 1 in this subsection we will find relationships between these quantities and derive corresponding generating function identities.

Proposition 2 Let S(n,k) denote the Stirling numbers of the second kind, that is, the number of set partitions of [n] into exactly k nonempty subsets. Then,

$$s_n = \sum_{k=1}^n S(n,k)u_k$$
$$t_n = \sum_{k=1}^n S(n,k)v_k$$
$$u_n = \sum_{k=0}^n \binom{n}{k}v_k$$

Proof We prove these for the corresponding special multigraphs.

Any special multigraph with edges e_1, \ldots, e_n can be described by giving a partition of [n] into, say, k parts, together with a special simple graph with k labelled edges; simply replace the *i*th edge of the simple graph by the *i*th set of edges of the partition (where the edges are ordered lexicographically, say). This is clearly a bijection. Moreover, the simple graph has no connected components of size 2 if and only if the same holds for the multigraph. This proves the first two equations.

Given a special simple graph, there is a distinguished subset of [n] (of size n - k, say) consisting of isolated edges; the remaining graph has no components of size 2. Again, the correspondence is bijective. So the third equation holds.

Proposition 2 can be reformulated in terms of exponential generating functions. Let $S(x) = \sum_{n\geq 0} s_n x^n/n!$, with similar definitions for the others. The proof of Proposition 3 is omitted.

Proposition 3

$$S(x) = U(e^{x} - 1)$$

$$T(x) = V(e^{x} - 1)$$

$$U(x) = V(x)e^{x}.$$

It follows from Proposition 3 that S(x) = T(x)B(x), where $B(x) = e^{e^x-1}$ is the exponential generating function for the Bell numbers. This is easily proved directly.

2.3 Unrestricted 2-covers and orbits

Recall the notation $F_n(G)$ for the number of orbits of the oligomorphic group G on ordered *n*-tuples of distinct elements, and $F_n^*(G)$ for the number of orbits on all *n*-tuples. Let $S_{\infty}^{\{2\}}$ denote the group induced by the infinite symmetric group on the set of all 2-element subsets of its domain.

Proposition 4 $F_n(S_{\infty}^{\{2\}}) = u_n \text{ and } F_n^*(S_{\infty}^{\{2\}}) = s_n.$

Proof Simply observe that an *n*-tuple of distinct 2-sets is the edge set of a special simple graph with n labelled edges, while an arbitrary *n*-tuple of 2-sets is the edge set of a special multigraph with n labelled edges.

We note that the relation

$$F^*(G) = \sum_{k=1}^n S(n,k)F_k(G)$$

gives an alternative proof of the first equation in Proposition 2. We do not know of a similar interpretation of the other two parameters.

2.4 Generating function identities for line graphs

Let $L(x) = \sum_{n \ge 0} l_n x^n / n!$. We now prove

Proposition 5

$$L(x) = e^{-x^3/3!} U(x) = e^{x-x^3/3!} V(x).$$

Proof According to Whitney's Theorem [5], an isomorphism between line graphs $L(G_1)$ and $L(G_2)$ of connected graphs is induced by an isomorphism from G_1 to G_2 , except in one case: the line graphs of the triangle K_3 and the star $K_{1,3}$ are isomorphic.

Now the connected components of line graphs which are triangles contribute a factor $e^{x^3/3!}$ to the exponential generating function L(x) for line graphs on [n]; that is, $L(x) = e^{x^3/3!}W'(x)$, where W'(x) is the e.g.f. for line graphs with no such components. Similarly, components which are triangles or stars contribute a factor $(e^{x^3/3!})^2$ to the e.g.f. for special simple graphs with *n* edges. Proposition 5 now follows by Whitney's Theorem and Proposition 3.

3 Unrestricted 2-covers: a probabilistic approach

In this section we prove (1) of Theorem 1 by using a probabilistic construction.

3.1 Technical results

We proceed with the following definitions and lemma. Let T_n be the set of proper 2-covers on [n]. Let S_n be the set of set partitions of [2n]. Let $E_{1,n} \subset S_n$ be the subset of set partitions of [2n] such that j and j + n are contained in different blocks for each $j \in [n]$. Define the function ψ from a subset \tilde{S} of [2n] to a subset of [n] by $\psi(\tilde{S}) = \{j : j \in \tilde{S} \text{ or } j + n \in \tilde{S}\}$. Let $E_{2,n} \subset S_n$ be the subset of set partitions of [2n] with blocks $\{\tilde{S}_1, \ldots, \tilde{S}_m\}$ such that $\psi(\tilde{S}_{i_1}) \neq \psi(\tilde{S}_{i_2})$ for each $i_1 \neq i_2$. Let $C_n = E_{1,n} \cap E_{2,n}$. Let ϕ be the function on S_n given by

$$\phi(\{\tilde{S}_1,\ldots,\tilde{S}_m\})=\{\psi(\tilde{S}_1),\ldots,\psi(\tilde{S}_m)\}.$$

Lemma 1 ϕ maps C_n onto T_n and $|\phi^{-1}(\mathbf{a})| = 2^n$ for all $\mathbf{a} \in T_n$.

Proof Fix $\{\tilde{S}_1, \ldots, \tilde{S}_m\} \in C_n$. Each $j \in [n]$ appears in exactly two blocks of $\phi(\{\tilde{S}_1, \ldots, \tilde{S}_m\})$ because of the definition of $E_{1,n}$ and the blocks of $\{\tilde{S}_1, \ldots, \tilde{S}_m\}$ are unique because of the definition of $E_{2,n}$ so $\phi(\{\tilde{S}_1, \ldots, \tilde{S}_m\}) \in T_n$.

Let $\mathbf{a} = \{S_1, \ldots, S_m\} \in T_n$. For each $j \in [n]$ there are two ways of assigning j, j + n to the appearances of j in \mathbf{a} (think of a fixed ordering of the blocks of \mathbf{a} to see this). The choices made for every $j \in [n]$ determine an *assignment*. Clearly, every element of $\phi^{-1}(\mathbf{a})$ must be of the form $\chi(\mathbf{a})$ for some assignment χ . There are 2^n assignments. We also write $\chi(S_i)$ for the block \tilde{S}_i corresponding to S_i in $\chi(\mathbf{a})$.

We claim that each assignment $\chi(\mathbf{a})$ gives a unique element of C_n . To see this, first note that j and j + n are clearly in different blocks of $\chi(\mathbf{a})$, so $\chi(\mathbf{a}) \in E_{1,n}$. Secondly, $\phi \circ \chi$ is the identity map on T_n . Therefore, $\chi(\mathbf{a}) \in E_{2,n}$ because \mathbf{a} is a proper 2-cover. Moreover, $\chi_1(\mathbf{a}_1) \neq \chi_2(\mathbf{a}_2)$ for all $\mathbf{a}_1, \mathbf{a}_2 \in T_n$ such that $\mathbf{a}_1 \neq \mathbf{a}_2$ and for all assignments χ_1 and χ_2 , which gives $\phi^{-1}(\mathbf{a}_1) \cap \phi^{-1}(\mathbf{a}_2) = \emptyset$.

We next prove that if χ_1 and χ_2 are two assignments such that $\chi_1(\mathbf{a}) = \chi_2(\mathbf{a})$, then $\chi_1 = \chi_2$. To see this, let

$$\mathcal{U} = \{ j \in [n] : \chi_1 \text{ and } \chi_2 \text{ differ for } j \}.$$

Without loss of generality, assume that $j \in S_1$ and $j \in S_2$. Then, either $j \in \chi_1(S_1)$ and $j \in \chi_2(S_2)$ or $j + n \in \chi_1(S_1)$ and $j + n \in \chi_2(S_2)$. It follows that $\chi_1(S_1) = \chi_2(S_2)$. Therefore, $\phi \circ \chi_1(S_1) = \phi \circ \chi_2(S_2)$ or $S_1 = S_2$ violating the assumption that **a** is proper. We conclude that $\mathcal{U} = \emptyset$ and that $\chi_1 = \chi_2$. This implies that $|\phi^{-1}(\mathbf{a})| = 2^n$.

Next we generalize Lemma 1 to (possibly) improper covers. Let U_n denote the set of 2-covers of [n].

Lemma 2 ϕ maps $E_{1,n}$ onto U_n . Let $\mathbf{a} = \{S_1, S_2, \ldots, S_m\}$ be a 2-cover of [n]. Let \mathcal{M} be the set of $i \in [m]$ such that there does not exist any $j \in [m] \setminus \{i\}$, $S_i = S_i$. Let

$$\rho = \frac{m - |\mathcal{M}|}{2}$$

be the number of pairs $\{i, j\}$ such that $S_i = S_j$. Then

$$\left.\phi^{-1}(\mathbf{a})\right| = 2^{n-\rho}.$$

Proof Clearly ϕ maps $E_{1,n}$ onto U_n . Let $\mathcal{N} = [n] \setminus \{\bigcup_{i \in \mathcal{M}} S_i\}$. Then $\{S_i : i \in \mathcal{M}\}$ is a proper cover of \mathcal{N} and Lemma 1 implies that

$$|\phi^{-1}(\{S_i : i \in \mathcal{N}\})| = 2^{|\mathcal{N}|}$$

For each pair S_{i_1} , S_{i_2} such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$, it must be true that $\phi^{-1}(S_i)$ consists of two sets \tilde{S}_1 and \tilde{S}_2 such that for each $j \in S_{i_1}$ either $j \in \tilde{S}_{i_1}$ and $j+n \in \tilde{S}_{i_2}$ or $j+n \in \tilde{S}_{i_1}$ and $j \in \tilde{S}_{i_2}$. The number of choosing unordered sets \tilde{S}_{i_1} , \tilde{S}_{i_2} is $2^{|S_{i_1}|-1}$. Therefore,

$$|\phi^{-1}(\mathbf{a})| = 2^{|\mathcal{N}|} \prod 2^{|S_{i_1}|-1} = 2^{n-\rho},$$

where the product is over pairs i_1, i_2 such that $i_1 \neq i_2$ and $S_{i_1} = S_{i_2}$.

3.2 Asymptotic enumeration of proper 2-covers

From Lemma 1 we conclude that $|C_n| = 2^n t_n$ so

$$t_n = 2^{-n} |C_n| = 2^{-n} \frac{|C_n|}{B_{2n}} B_{2n}$$
(5)

where B_{2n} is the 2*n*th Bell number.

We will now prove

Lemma 3

$$\frac{|E_{1,n}|}{B_{2n}} \sim \sqrt{\frac{\log n}{2n}} \tag{6}$$

and

$$\frac{|E_{2,n}|}{B_{2n}} = 1 - O\left(\frac{\log^2 n}{n}\right). \tag{7}$$

Proof To prove (6), choose an element of S_n uniformly at random and let X be the number of $j \in [n]$ for which j and j + n are in the same block. We have

$$\mathbb{P}(X=0) = \frac{|E_{1,n}|}{B_{2n}}.$$
(8)

We have $X = \sum_{j=1}^{n} I_j$ where I_j is the indicator random variable that j and j + n are in the same block. The *r*th falling moment of X_n is

$$\mathbb{E}(X)_r = \mathbb{E}X(X-1)\cdots(X-r+1)$$
$$= \sum \mathbb{E}(I_{j_1}I_{j_2}\cdots I_{j_r})$$

where the sum is over (j_1, \ldots, j_r) with no repetitions. To find $\mathbb{E}(I_{j_1}I_{j_2}\cdots I_{j_r})$ we take $[2n] \setminus \{j_1, j_2, \ldots, j_r\}$ and form a set partition. We then add j_k to the block containing $j_k + n$ for each $k \in [r]$. This process is uniquely reversible. Therefore,

$$\mathbb{E}(X)_r = \frac{(n)_r B_{2n-r}}{B_{2n}}.$$

We apply the formula in Corollary 13, page 18, of [1] to obtain

$$\mathbb{P}(X=0) = \sum_{r=0}^{\infty} (-1)^r \frac{\mathbb{E}(X)_r}{r!} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(n)_r B_{2n-r}}{B_{2n}}.$$
(9)

To analyze (9) we use the expansion of the Bell numbers [6, 8]

$$\log B_n = e^w (w^2 - w + 1) - \frac{1}{2} \log(1 + w) - 1 - \frac{w(2w^2 + 7w + 10)}{24(1 + w)^3} e^{-w} - \frac{w(2w^4 + 12w^3 + 29w^2 + 40w + 36)}{48(1 + w)^6} e^{-2w} + O(e^{-3w}) ,$$

where w = W(n) is given by (3), (4), from which we obtain (using Maple)

$$\log B_{n-r} - \log B_n = -rw + \frac{rw}{2n} \left(\frac{r}{w+1} + \frac{1}{(w+1)^2} \right) + O\left(\frac{r^3w}{n^2} \right).$$

In particular,

$$\frac{B_{n-1}}{B_n} \sim \frac{\log n}{n}$$

so there exists a constant C > 0 such that

$$\frac{B_{n-r}}{B_n} \le \frac{(C\log n)^r}{(n)_r}.$$
(10)

Moreover,

$$\log B_{2n-r} - \log B_{2n} = -rv + \frac{rv}{4n} \left(\frac{r}{v+1} + \frac{1}{(v+1)^2} \right) + O\left(\frac{r^3v}{n^2}\right)$$
$$= -r\log n + rc_n + r^2d_n + O\left(\frac{r^3\log n}{n^2}\right),$$

where v = W(2n) has the expansion

$$v = \log n - \log \log n + \log 2 + \frac{\log \log n}{\log n} - \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right),$$

where

$$c_n = \log n - v - \frac{rv}{4n(v+1)^2}$$

= $\log \log n - \log 2 - \frac{\log \log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right)$

and where

$$d_n = O\left(\frac{1}{n}\right).$$

Using (10) we estimate

$$\left| \sum_{r>\log^{3/2} n} (-1)^r \frac{\mathbb{E}(X)_r}{r!} \right| \leq \sum_{r>\log^{3/2} n} \frac{(n)_r B_{2n-r}}{r! B_n}$$
$$\leq \sum_{r>\log^{3/2} n} \frac{(C\log 2n)^r}{r!}$$
$$= (2n)^C \sum_{r>\log^{3/2} n} e^{-C\log 2n} \frac{(C\log 2n)^r}{r!}$$
$$= o(1). \tag{11}$$

For $r \leq \log^{3/2} n$, we have

$$\frac{B_{n-r}}{B_n} = n^{-r} \exp\left(rc_n + r^2d_n + O\left(\frac{\log^9 n}{n^2}\right)\right)$$

and

$$(n)_r = n^r \exp\left(O\left(\frac{r^2}{n}\right)\right),$$

hence

$$\mathbb{E}(X)_r = \exp\left(rc_n + r^2d_n + O\left(\frac{\log^9 n}{n^2}\right)\right).$$

Therefore,

$$\sum_{0 \le r \le \log^{3/2} n} (-1)^r \frac{\mathbb{E}(X)_r}{r!} = \sum_{0 \le r \le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n + r^2 d_n} \left(1 + O\left(\frac{\log^9 n}{n^2}\right) \right)$$
$$= \sum_{0 \le r \le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} \left(1 + d_n r^2 + O\left(\frac{\log^9 n}{n^2}\right) \right)$$
$$= \sum_{0 \le r \le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} + d_n \sum_{0 \le r \le \log^{3/2} n} \frac{(-1)^r r^2}{r!} e^{rc_n} + \left(\frac{\log^9 n}{n^2}\right) \sum_{0 \le r \le \log^{3/2} n} \frac{e^{rc_n}}{r!}.$$
(12)

We proceed to approximate the terms in (12). First, we find that

$$\sum_{0 \le r \le \log^{3/2} n} \frac{(-1)^r}{r!} e^{rc_n} = \exp\left(-e^{c_n}\right) + O\left(\sum_{\log^{3/2} n \le r \le n} \frac{e^{rc_n}}{r!}\right)$$
$$= \exp\left(-\frac{\log n}{2} \left[1 - \frac{\log \log n}{\log n} + \frac{\log 2}{\log n} + o\left(\frac{1}{\log n}\right)\right]\right) + o(n^{-1/2})$$
$$\sim \sqrt{\frac{\log n}{2n}}.$$
(13)

We estimate

$$d_{n} \left| \sum_{0 \leq r \leq \log^{3/2} n} \frac{(-1)^{r}}{r!} r^{2} e^{rc_{n}} \right|$$

$$= d_{n} \left| \sum_{2 \leq r \leq \log^{3/2} n} \frac{(-1)^{r}}{(r-2)!} e^{rc_{n}} + \sum_{1 \leq r \leq \log^{3/2} n} \frac{(-1)^{r}}{(r-1)!} e^{rc_{n}} \right|$$

$$= d_{n} \left| e^{2c_{n}} \sum_{2 \leq r \leq \log^{3/2} n} \frac{(-1)^{r}}{(r-2)!} e^{(r-2)c_{n}} + e^{c_{n}} \sum_{1 \leq r \leq \log^{3/2} n} \frac{(-1)^{r}}{(r-1)!} e^{(r-1)c_{n}} \right|$$

$$= d_{n} \left(\exp\left(-e^{c_{n}} + 2c_{n}\right) + \exp\left(-e^{c_{n}} + c_{n}\right) + O\left(e^{2c_{n}} \sum_{\log^{3/2} n \leq r \leq n} \frac{e^{rc_{n}}}{r!}\right)\right)$$

$$= o(n^{-1/2}).$$
(14)

Finally, we have

$$O\left(\frac{\log^9 n}{n^2}\right) \sum_{0 \le r \le \log^{3/2} n} \frac{e^{rc_n}}{r!} \le O\left(\frac{\log^9 n}{n^2}\right) e^{c_n} = o(n^{-1/2}).$$
(15)

Together, (8), (9), (11), (12), (13), (14) and (15) prove (6).

To show (7), let Y be the number of pairs S_i , S_j in an partition in S_n chosen uniformly at random for which $\psi(S_i) = \psi(S_j)$. For such S_i , S_j of size $|S_i| = |S_j| = k$, the probability that they are present in the random partition is B(2n - 2k)/B(2n). The total number of pairs S_i , S_j of size k is bounded by $\binom{n}{k}2^k$ (the number of ways of choosing a subset J of size k from [n] times a bound on the number of ways of choosing two subsets S_1 , S_2 of [2n] of size k such that either $j \in S_1$ and $j + n \in S_2$ or $j + n \in S_1$ and $j \in S_2$ for all $j \in J$.) Therefore, using (10) we get

$$1 - \frac{|E_{2,n}|}{B_{2n}} = \mathbb{P}(Y > 0)$$

$$\leq \mathbb{E}Y$$

$$\leq \sum_{k=1}^{n} \binom{n}{k} 2^k \frac{B_{2n-2k}}{B_{2n}}$$

$$\leq \sum_{k=1}^{n} \binom{n}{k} 2^k \frac{(C\log 2n)^{2k}}{(2n)_{2k}}$$

$$\leq \sum_{k=1}^{n} \frac{(n)_k (2C^2 \log^2 2n))^k}{(2n)_{2k}k!}$$

$$= O\left(\frac{\log^2 n}{n}\right).$$

Lemma 3 and (5) along with

$$\frac{|C_n|}{B_{2n}} \le \frac{|E_{1,n}|}{B_{2n}}$$

and

$$\frac{|C_n|}{B_{2n}} \ge \frac{|E_{1,n}| - (B_{2n} - |E_{2,n}|)}{B_{2n}}$$

prove (1) for t_n .

3.3 Asymptotic enumeration of 2-covers

In this subsection we prove (1) for s_n . Recall that U_n denotes the set of 2-covers of [n]. Each element of $E_{1,n}$ is mapped to a unique $\mathbf{a} \in U_n$ by ϕ . Given $\omega = \{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m\} \in \mathcal{S}_n$, let $Z(\omega)$ be the number of pairs $\{i_1, i_2\}$ such that $\psi(\tilde{S}_{i_1}) = \psi(\tilde{S}_{i_2})$. Note that in the case $\omega \in E_{1,n}$ we have $Z(\omega) = \rho$ with ρ defined with respect to $\mathbf{a} = \phi(\omega)$ in the statement of Lemma 2.

Define $D_{\rho,n}$ for $\rho \in \{0, 1, \ldots, n\}$ to be

$$D_{\rho,n} = \{ \omega \in E_{1,n} : Z(\omega) = \rho \}.$$

Note that $D_{0,n} = C_n$. By Lemma 2,

$$u_n = \sum_{\rho=0}^n |D_{\rho,n}| 2^{-n+\rho}$$

= $|C_n| 2^{-n} + \sum_{\rho=1}^n |D_{\rho,n}| 2^{\rho}$
= $B_{2n} 2^{-n} \left(\frac{|C_n|}{B_{2n}} + \sum_{\rho=1}^n \frac{|D_{\rho,n}|}{B_{2n}} 2^{\rho} \right)$

We have shown in the previous section that $C_n/B_{2n} \sim \sqrt{\log n/2n}$. Observe that $\sum_{\rho=1}^n |D_{\rho,n}| 2^{\rho}/B_{2n} \leq \sum_{\rho=1}^n \mathbb{P}(Z=\rho) 2^{\rho}$, where Z was defined in the last paragraph and ω is chosen uniformly at random from S_n . In light of these observations, to prove (1) for s_n it suffices to prove that

$$\sum_{\rho=1}^{n} \mathbb{P}(Z=\rho) 2^{\rho} = o\left(\sqrt{\frac{\log n}{2n}}\right).$$
(16)

The quantity $\mathbb{P}(Z \ge \rho)$ is equal to the probability that the randomly chosen element of \mathcal{S}_n contains at least ρ disjoint pairs of equal sets, therefore,

$$\mathbb{P}(Z \ge \rho) \le \sum_{s_1=1}^n \sum_{s_2=1}^n \cdots \sum_{s_\rho=1}^n \binom{n}{s_1, s_2, \dots, s_\rho, n-\sum s_i} \frac{B_{2n-2\sum s_i}}{B_{2n}}$$

Let σ be defined by $\sigma = \sum_{i=1}^{\rho} s_i$. We can assume $\sigma \leq n$. From (10) we have

$$\mathbb{P}(Z \ge \rho) \le \sum_{s_1=1}^n \sum_{s_2=1}^n \dots \sum_{s_\rho=1}^n \left(\binom{n}{s_1, s_2, \dots, s_\rho, n-\sigma} \right) \frac{(C \log n)^{2\sigma}}{(2n)_{2\sigma}} \\ = \sum_{s_1=1}^n \sum_{s_2=1}^n \dots \sum_{s_\rho=1}^n \frac{(n)_\sigma}{\prod_i s_i!} \frac{(C \log n)^{2\sigma}}{(2n)_{2\sigma}}.$$

Observing that

$$\frac{(n)_{\sigma}}{(2n)_{2\sigma}} = \frac{(n)_{\sigma}}{(2n)_{\sigma}(2n-\sigma)_{\sigma}} \le \frac{1}{(2n)_{\sigma}} \le n^{-\sigma},$$

we have

$$\mathbb{P}(Z \ge \rho) \le \sum_{\sigma=\rho}^{n} \sum_{\substack{s_1,\dots,s_{\rho}:\\\sum_i s_i = \sigma}} \frac{1}{\prod_i s_i!} \left(\frac{C^2 \log^2 n}{n}\right)^{\sigma}$$
$$= \sum_{\sigma=\rho}^{n} \frac{\rho^{\sigma}}{\sigma!} \left(\frac{C^2 \log^2 n}{n}\right)^{\sigma}$$

Therefore,

$$\sum_{\rho=1}^{n} \mathbb{P}(Z=\rho) 2^{\rho} \leq \sum_{\rho=1}^{n} \mathbb{P}(Z\geq\rho) 2^{\rho}$$

$$\leq \sum_{\rho=1}^{n} \sum_{\sigma=\rho}^{n} \frac{2^{\rho} \rho^{\sigma}}{\sigma!} \left(\frac{C^{2} \log^{2} n}{n}\right)^{\sigma}$$

$$= \sum_{\sigma=1}^{n} \sum_{\rho=1}^{\sigma} \frac{2^{\rho} \rho^{\sigma}}{\sigma!} \left(\frac{C^{2} \log^{2} n}{n}\right)^{\sigma}$$

$$\leq \sum_{\sigma=1}^{n} \sum_{\rho=1}^{\sigma} \frac{\rho^{\sigma}}{\sigma!} \left(\frac{2C^{2} \log^{2} n}{n}\right)^{\sigma}$$

$$\leq \sum_{\sigma=1}^{n} \frac{(\sigma+1)^{\sigma}}{\sigma!} \left(\frac{2C^{2} \log^{2} n}{n}\right)^{\sigma}$$

$$= O\left(\frac{\log^{2} n}{n}\right)$$

$$= o\left(\sqrt{\frac{\log n}{2n}}\right).$$

The last estimate proves (16).

4 Restricted 2-covers and line graphs: an analytic approach

Our proof of (2) will use generating function analysis. Let $a_{n,m}$ be the number of restricted, proper 2-covers on [n] with m blocks. The generating function for restricted, proper 2-covers

$$A(x,y) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n} \frac{a_{n,m}}{n!} x^n y^m$$

equals

$$A(x,y) = \exp\left(-y - \frac{xy^2}{2}\right) \sum_{m \ge 0} \frac{y^m}{m!} (1+x)^{\binom{m}{2}};$$
(17)

see page 203 of [4]. Therefore,

$$V(x) = A(x,1) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{-x/2}$$
(18)

and

$$v_n = n! e^{-1} \sum_{m=0}^{\infty} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{1}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k}.$$
 (19)

Note that for $m \ge 2$,

$$\left| \sum_{k=0}^{n} \frac{n!}{k!} \left(-\frac{1}{2} \right)^{k} m^{-2n} \binom{\binom{m}{2}}{n-k} \right| \leq \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2} \right)^{k} m^{-2n} \binom{m}{2}^{n-k}$$
$$\leq 2^{-n} \sum_{k=0}^{n} \binom{n}{k} m^{-2k}$$
$$\leq 2^{-n} \left(\frac{1+m^{-2}}{2} \right)^{n} = O(2^{-n}). \quad (20)$$

We will make use of the asymptotic analysis of the Bell numbers in Example 5.4 of [7], which uses the identity

$$B_n = e^{-1} \sum_{m=0}^{\infty} \frac{m^n}{m!}.$$

Let m_0 be the nearest integer to $\frac{2n}{W(2n)}$, where W is defined by (3). (The choice of m_0 is slightly different here than in [7], but the analysis giving (21) and (22) below remains valid.) In [7] it is proved that

$$\sum_{\substack{1 \le m \le n \\ |m - m_0| > \sqrt{n} \log n}} \frac{m^{2n}}{m!} = O\left(\frac{m_0^{2n}}{m_0!} \sqrt{n} \exp\left(-(\log n)^3\right)\right)$$
(21)

and that

$$\sum_{\substack{1 \le m \le n \\ |m-m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!} = \frac{m_0^{2n+1}}{m_0!} \sqrt{\frac{2\pi}{2n+m_0}} \left(1 + O\left((\log n)^6 n^{-1/2}\right)\right)$$
(22)

$$\sim eB_{2n}.$$
 (23)

It follows from (20) and (21) that

$$\sum_{\substack{1 \le m \le n \\ |m-m_0| > \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{n!}{k!} \left(-\frac{1}{2} \right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} = O\left(\frac{m_0^{2n}}{m_0!} \sqrt{n} 2^{-n} \exp\left(-(\log n)^3\right)\right) = O\left(B_{2n} 2^{-n} \exp\left(-\frac{(\log n)^3}{2}\right)\right) = O\left(B_{2n} 2^{-n} \exp\left(-\frac{(\log n)^3}{2}\right)\right)$$

We have

$$\sum_{\substack{1 \le m \le n \\ |m-m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=0}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k} = \sum_{\substack{1 \le m \le n \\ |m-m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n} + \Delta,$$
(25)

where

$$\Delta := \sum_{\substack{1 \le m \le n \\ |m - m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=1}^n \frac{n!}{k!} \left(-\frac{1}{2}\right)^k m^{-2n} \binom{\binom{m}{2}}{n-k}$$

is bounded by

$$\begin{aligned} |\Delta| &\leq \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} \sum_{k=1}^n \frac{n!}{k!} m^{-2n} \binom{\binom{m}{2}}{n} \left(\frac{n}{\binom{m}{2}-n}\right)^k \\ &= O\left(\frac{\log^2 n}{n}\right) \sum_{\substack{1 \leq m \leq n \\ |m-m_0| \leq \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n}. \end{aligned}$$

One may show that uniformly for m in the range $|m - m_0| \le \sqrt{n} \log n$

$$m^{-2n} \binom{\binom{m}{2}}{n} n! = 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) \left(1 + O\left(n^{-1/2} \log^6 n\right)\right),$$

hence,

$$|\Delta| = O\left(\frac{\log^2 n}{n}\right) 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) B_{2n}.$$
 (26)

The main term of (25) is

$$\sum_{\substack{1 \le m \le n \\ |m-m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!} m^{-2n} n! \binom{\binom{m}{2}}{n} = 2^{-n} \exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right) (1+o(1)) \sum_{\substack{1 \le m \le n \\ |m-m_0| \le \sqrt{n} \log n}} \frac{m^{2n}}{m!}$$

$$= eB_{2n}2^{-n}\exp\left(-\frac{n}{m_0} - \frac{n^2}{m_0^2}\right)(1+o(1))$$

$$= eB_{2n}\frac{1}{2^n\sqrt{n}}e^{-\left(\frac{1}{2}\log(2n/\log n)\right)^2}(1+o(1))$$
(27)

where we have used the asymptotic expansion (4) and the definition of m_0 at the last step. Now (19), (24), (26) and (27) prove (2) for v_n .

In the previous argument the result would have been the same if the $e^{-x/2}$ in (18) were replaced by 1 because in the Taylor expansion of $e^{-x/2}$ the constant term 1 corresponds to the main term of (25) and the higher order terms contribute to Δ , which is negligable. The argument for restricted partitions and line graphs are similar, starting from the identities obtained from Proposition 17 and (18)

$$U(x) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{x/2}.$$

and

$$L(x) = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} (1+x)^{\binom{m}{2}} e^{x/2 - x^3/6}.$$

In each case only the contribution of the constant term of the Taylor expansion of the exponential is 1 and the remaining terms contribute to a quantity like Δ which is asymptotically insignificant.

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