# Ring graphs and complete intersection toric ideals 

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#### Abstract

We study the family of graphs whose number of primitive cycles equals its cycle rank. It is shown that this family is precisely the family of ring graphs. Then we study the complete intersection property of toric ideals of bipartite graphs and oriented graphs. An interesting application is that complete intersection toric ideals of bipartite graphs correspond to ring graphs and that these ideals are minimally generated by Gröbner bases. We prove that any graph can be oriented such that its toric ideal is a complete intersection with a universal Gröbner basis determined by the cycles. It turns out that bipartite ring graphs are exactly the bipartite graphs that have complete intersection toric ideals for any orientation.


## 1 Introduction

Let $G$ be a graph (no loops or multiple edges) with $n$ vertices and $q$ edges, and let $\operatorname{frank}(G)$ be the number of primitive cycles of $G$, i.e., cycles without chords. The number $\operatorname{frank}(G)$ is called the free rank of $G$ and the number $\operatorname{rank}(G)=q-n+r$ is called the cycle rank of $G$, where $r$ is the number of connected components of $G$. The cycle rank of $G$ can be expressed as the dimension of the cycle space of $G$. These two numbers satisfy $\operatorname{rank}(G) \leq \operatorname{frank}(G)$, as is seen in Proposition 2.2. The aim of this paper is to study and classify the family of graphs where the equality occurs. It will turn out that this family is precisely the family of ring graphs. The precise definition of a ring graph can be found in Section2. Roughly speaking ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices.

The contents of this paper are as follows. Before stating our main results, recall that a graph $G$ has the primitive cycle property ( PCP ) if any two primitive

[^0]cycles intersect in at most one edge. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. As usual we denote the complete graph on $n$ vertices by $\mathcal{K}_{n}$. In Section 2, which is the core of the paper, we prove the following implications for any graph $G$ :
\[

$$
\begin{array}{cc}
\text { outerplanar } \Rightarrow & \Leftrightarrow \begin{array}{c}
\text { PCP }+ \text { contains no } \\
\text { subdivision of } \mathcal{K}_{4} \\
\text { as a subgraph }
\end{array} \Rightarrow \text { planar }
\end{array}
$$ \Rightarrow frank \quad $$
\begin{gathered}
\text { and }
\end{gathered}
$$
\]

These purely graph theoretical results are applied in Sections 3 and 4, where graphs with complete intersection toric ideals are studied, both in the oriented and unoriented case. For bipartite graphs the equality $\operatorname{rank}(G)=\operatorname{frank}(G)$ is related to these special types of toric ideals as we explain below.

Let $R=k\left[x_{1}, \ldots, x_{n}\right]$ be a polynomial ring over a field $k$ and let $G$ be a graph with vertex set $V(G)=\left\{x_{1}, \ldots, x_{n}\right\}$ and edge set $E(G)=\left\{t_{1}, \ldots, t_{q}\right\}$. The edge subring of $G$ is the $k$-subalgebra of $R$ :

$$
k[G]=k\left[\left\{x_{i} x_{j} \mid x_{i} \text { is adjacent to } x_{j}\right\}\right] \subset R .
$$

There is an epimorphism of $k$-algebras

$$
\varphi: k\left[t_{1}, \ldots, t_{q}\right] \longrightarrow k[G], \quad\{x, y\} \longmapsto x y,
$$

where $k\left[t_{1}, \ldots, t_{q}\right]$ is a polynomial ring. The kernel of $\varphi$, denoted by $P(G)$, is called the toric ideal of $G$. Toric ideals of graphs are studied in Section 3. The height of $P(G)$ is equal to $g=q-\operatorname{rank}\left(A_{G}\right)$, where $A_{G}$ is the incidence matrix of $G$. By a result of Krull [2] the ideal $P(G)$ cannot be generated by less than $g$ polynomials. The toric ideal of $G$ is called a complete intersection if it can be generated by $g$ polynomials. The complete intersection property of $P(G)$ was first studied in [6, 19], and later in [8, 13].

An interesting result of Simis [19] shows that if $G$ is a bipartite graph, then $\operatorname{rank}(G)=\operatorname{frank}(G)$ if and only if $P(G)$ is a complete intersection. Thus by describing the graphs where equality occurs, we are in particular describing the toric ideals of bipartite graphs that are complete intersections (see Corollary 3.4). We prove that complete intersection toric ideals of 2-connected bipartite graphs are minimally generated by Gröbner bases (see Corollary 3.7).

In Section 4 we introduce and study toric ideals of oriented graphs and their Gröbner bases. To the best of our knowledge these toric ideals have not been studied much except for the case of acyclic tournaments [12]. Oriented graphs share some properties with bipartite graphs. For instance in both cases their incidence matrices are totally unimodular. This is a key fact to understand the Gröbner bases of toric ideals of oriented graphs (see Lemma 4.1). We prove that the toric ideal of any oriented graph is completely determined by its primitive
cycles and has a universal Gröbner basis determined by the cycles (see Proposition 4.3 and Corollary 4.5). It is shown that toric ideals of oriented ring graphs are complete intersections for any orientation. As an interesting consequence of the results of Section 22 we obtain that for bipartite graphs this property characterizes ring graphs (see Corollary 4.9). One of our main results shows that any graph has an acyclic orientation such that the corresponding toric ideal is a complete intersection (see Theorem 4.16).

The paper is essentially self contained. For unexplained terminology and notation on graph theory we refer to [5, 10]. Our main references for edge subrings are [21, 22].

## 2 Ring graphs

Let $G$ be a graph with $n$ vertices and $q$ edges. We denote the vertex set and edge set of $G$ by $V(G)=\left\{x_{1}, \ldots, x_{n}\right\}$ and $E(G)=\left\{t_{1}, \ldots, t_{q}\right\}$ respectively. Recall that a 0 -chain (resp. 1-chain) of $G$ is a formal linear combination $\sum a_{i} x_{i}$ (resp. $\sum b_{i} t_{i}$ ) of vertices (resp. edges), where $a_{i} \in \mathbb{Z}_{2}$ (resp. $b_{i} \in \mathbb{Z}_{2}$ ). The boundary operator is the linear map $\partial: C_{1} \rightarrow C_{0}$ defined by

$$
\partial(\{x, y\})=x+y,
$$

where $C_{i}$ is the $\mathbb{Z}_{2}$-vector space of $i$-chains. A cycle vector is a 1 -chain of the form $t_{1}+\cdots+t_{r}$ where $t_{1}, \ldots, t_{r}$ are the edges of a cycle of $G$. The cycle space $\mathcal{Z}(G)$ of $G$ over $\mathbb{Z}_{2}$ is equal to $\operatorname{ker}(\partial)$. The vectors in $\mathcal{Z}(G)$ can be regarded as a set of edge-disjoint cycles. A cycle basis for $G$ is a basis for $\mathcal{Z}(G)$ which consists entirely of cycle vectors, such a basis can be constructed as follows:

Remark 2.1 [10, pp. 38-39] If $G$ is connected, then $G$ has a spanning tree $T$. The subgraph of $G$ consisting of $T$ and any edge in $G$ not in $T$ has exactly one cycle, the collection of all cycle vectors of cycles obtained in this way form a cycle basis for $G$. Hence $\operatorname{dim}_{\mathbb{Z}_{2}} \mathcal{Z}(G)=q-n+r$ if $G$ is a graph with $r$ connected components.

Let $c$ be a cycle of $G$. A chord of $c$ is any edge of $G$ joining two non adjacent vertices of $c$. A cycle without chords is called primitive. The number $\operatorname{dim}_{\mathbb{Z}_{2}} \mathcal{Z}(G)$ is called the cycle rank of $G$ and is denoted by $\operatorname{rank}(G)$. The number of primitive cycles of a graph $G$, denoted by $\operatorname{frank}(G)$, is called the free rank of $G$.

Proposition 2.2 If $G$ is a graph, then $\mathcal{Z}(G)$ is generated by cycle vectors of primitive cycles. In particular $\operatorname{rank}(G) \leq \operatorname{frank}(G)$.

Proof. Let $\mathbf{c}_{1}, \ldots, \mathbf{c}_{r}$ be a cycle basis for the cycle space of $G$ and let $c_{1}, \ldots, c_{r}$ be the corresponding cycles of $G$. It suffices to notice that if some $c_{j}$ has a chord,
we can write $\mathbf{c}_{j}=\mathbf{c}_{j}^{\prime}+\mathbf{c}_{j}^{\prime \prime}$, where $\mathbf{c}_{j}^{\prime}$ and $\mathbf{c}_{j}^{\prime \prime}$ are cycle vectors of cycles of length smaller than that of $c_{j}$.

Corollary 2.3 Let $G$ be a graph. Then the following are equivalent:
(a) $\operatorname{rank}(G)=\operatorname{frank}(G)$.
(b) The set of cycle vectors of primitive cycles is a basis for $\mathcal{Z}(G)$.
(c) The set of cycle vectors of primitive cycles is linearly independent.

Proof. (a) $\Rightarrow$ (b): By Proposition 2.2 there is a basis $\mathcal{B}$ of $\mathcal{Z}(G)$ consisting of cycle vectors of primitive cycles. By hypothesis $\operatorname{rank}(G)=\operatorname{frank}(G)$. Thus $\mathcal{B}$ is the set of all cycle vectors of primitive cycles and $\mathcal{B}$ is a basis. That (b) implies (c) and (c) implies (a) are also very easy to prove.

Let $G$ be a graph. A vertex $v$ (resp. an edge $e$ ) of $G$ is called a cutvertex (resp. bridge) if the number of connected components of $G \backslash\{v\}$ (resp. $G \backslash\{e\}$ ) is larger than that of $G$. A maximal connected subgraph of $G$ without cutvertices is called a block. A graph $G$ is 2-connected if $|V(G)|>2$ and $G$ has no cutvertices. Thus a block of $G$ is either a maximal 2-connected subgraph, a bridge or an isolated vertex. By their maximality, different blocks of $G$ intersect in at most one vertex, which is then a cutvertex of $G$. Therefore every edge of $G$ lies in a unique block, and $G$ is the union of its blocks.

Lemma 2.4 Let $G$ be a graph and let $G_{1}, \ldots, G_{r}$ be its blocks. Then $\operatorname{rank}(G)=$ $\operatorname{frank}(G)$ if and only if $\operatorname{rank}\left(G_{i}\right)=\operatorname{frank}\left(G_{i}\right)$ for all $i$.

Proof. $\Rightarrow)$ Let $G_{i}$ be any block of $G$. We may assume $\left|V\left(G_{i}\right)\right|>2$, otherwise $\operatorname{rank}\left(G_{i}\right)=\operatorname{frank}\left(G_{i}\right)=0$. If $c$ is a primitive cycle of $G_{i}$, then by the maximality condition of a block one has that $c$ is also a primitive cycle of $G$. Thus by Corollary 2.3 the set of cycle vectors of primitive cycles of $G_{i}$ is linearly independent and $\operatorname{rank}\left(G_{i}\right)=\operatorname{frank}\left(G_{i}\right)$.
$\Leftarrow)$ Let $\mathcal{B}_{i}$ and $\mathcal{B}$ be the set of cycle vector of primitive cycles of $G_{i}$ and $G$ respectively. As $\cup_{i=1}^{r} \mathcal{B}_{i}$ is linearly independent, by Corollary 2.3 it suffices to prove that $\cup_{i=1}^{r} \mathcal{B}_{i}=\mathcal{B}$. In the first part of the proof we have already observed that $\cup_{i=1}^{r} \mathcal{B}_{i} \subset \mathcal{B}$. To prove the equality take any cycle vector $\mathbf{c}$ of a primitive cycle $c$ of $G$. Since $c$ is a 2-connected subgraph, it must be contained in some block of $G$, i.e., in some $G_{i}$. Thus $c$ is a primitive cycle of $G_{i}$, so $\mathbf{c}$ is in $\mathcal{B}_{i}$.

Definition 2.5 Given a graph $H$, we call a path $\mathcal{P}$ an $H$-path if $\mathcal{P}$ is non-trivial and meets $H$ exactly in its ends.

In order to describe, in graph theoretical terms, the family of graphs satisfying the equality $\operatorname{rank}(G)=\operatorname{frank}(G)$ we need to introduce another notion.

Definition 2.6 A graph $G$ is a ring graph if each block of $G$ which is not a bridge or a vertex can be constructed from a cycle by successively adding $H$-paths of length at least 2 that meet graphs $H$ already constructed in two adjacent vertices.

Families of ring graphs include forests and cycles. These graphs are planar by construction.

Remark 2.7 Let $G$ be a 2-connected ring graph and let $c$ be a fixed primitive cycle of $G$, then $G$ can be constructed from $c$ by successively adding $H$-paths of length at least 2 that meet graphs $H$ already constructed in two adjacent vertices.

A graph $H$ is called a subdivision of a graph $G$ if $H=G$ or $H$ arises from $G$ by replacing edges by paths.

Lemma 2.8 [1, Lemma 7.78, p. 387] Let $G$ be a graph with vertex set $V$. If $G$ is 2 -connected and $\operatorname{deg}(v) \geq 3$ for all $v \in V$, then $G$ contains a subdivision of $\mathcal{K}_{4}$ as a subgraph.

Lemma 2.9 Let $G$ be a graph. If $\operatorname{rank}(G)=\operatorname{frank}(G)$ and $x, y$ are two non adjacent vertices of $G$, then there are at most two vertex disjoint paths joining $x$ and $y$.

Proof. Assume that there are three vertex disjoint paths joining $x$ and $y$ :

$$
\begin{aligned}
& \mathcal{P}_{1}=\left\{x, x_{1}, \ldots, x_{r}, y\right\}, \quad \mathcal{P}_{2}=\left\{x, z_{1}, \ldots, z_{t}, y\right\}, \\
& \mathcal{P}_{3}=\left\{x, y_{1}, \ldots, y_{s}, y\right\},
\end{aligned}
$$

where $r, s, t$ are greater or equal than 1 . We may assume that the sum of the lengths of the $\mathcal{P}_{i}$ 's is minimal. Consider the cycles

$$
\begin{aligned}
& c_{1}=\left\{x, x_{1}, \ldots, x_{r}, y, z_{t}, \ldots, z_{1}, x\right\}, \\
& c_{2}=\left\{x, z_{1}, \ldots, z_{t}, y, y_{s}, \ldots, y_{1}, x\right\}, \\
& c_{3}=\left\{x, x_{1}, \ldots, x_{r}, y, y_{s}, \ldots, y_{1}, x\right\} .
\end{aligned}
$$

Thus we are in the following situation:


Observe that, by the choice of the $\mathcal{P}_{i}$ 's, a chord of the cycle $c_{1}$ (resp. $c_{2}, c_{3}$ ) must join $x_{i}$ and $z_{j}$ (resp. $z_{i}$ and $y_{j}, x_{i}$ and $y_{j}$ ) for some $i, j$. If $c_{1}$ is not primitive, we can write

$$
\mathbf{c}_{1}=\mathbf{a}_{1}+\cdots+\mathbf{a}_{n_{1}}
$$

for some distinct cycle vectors $\mathbf{a}_{1}, \ldots, \mathbf{a}_{n_{1}}$ of primitive cycles $\mathrm{a}_{1}, \ldots, \mathrm{a}_{n_{1}}$ such that each cycle $\mathrm{a}_{i}$ contains at least one edge of the form $\left\{x_{j}, z_{k}\right\}$. Similarly if $c_{2}$ (resp. $c_{3}$ ) is not primitive we can write:

$$
\mathbf{c}_{2}=\mathbf{b}_{1}+\cdots+\mathbf{b}_{n_{2}} \quad\left(\text { resp. } \quad \mathbf{c}_{3}=\mathbf{d}_{1}+\cdots+\mathbf{d}_{n_{3}}\right)
$$

for some distinct cycle vectors $\mathbf{b}_{1}, \ldots, \mathbf{b}_{n_{2}}$ (resp. $\mathbf{d}_{1}, \ldots, \mathbf{d}_{n_{3}}$ ) of primitive cycles such that each cycle $\mathrm{b}_{i}$ (resp. $\mathrm{d}_{i}$ ) contains at least one edge of the form $\left\{z_{j}, y_{k}\right\}$ (resp. $\left\{x_{j}, y_{k}\right\}$ ). Therefore we can write

$$
\mathbf{c}_{1}=\sum_{i=1}^{n_{1}} \mathbf{a}_{i}, \quad \mathbf{c}_{2}=\sum_{i=1}^{n_{2}} \mathbf{b}_{i}, \quad \mathbf{c}_{3}=\sum_{i=1}^{n_{3}} \mathbf{d}_{i}
$$

where $\mathbf{a}_{1}, \ldots, \mathbf{a}_{n_{1}}, \mathbf{b}_{1}, \ldots, \mathbf{b}_{n_{2}}, \mathbf{d}_{1}, \ldots, \mathbf{d}_{n_{3}}$ are distinct cycle vectors of primitive cycles of $G$. Thus from the equality $\mathbf{c}_{3}=\mathbf{c}_{1}+\mathbf{c}_{2}$ we get a non trivial linear relation of the set $\mathcal{B}$ of cycle vectors of primitive cycles, i.e., $\mathcal{B}$ is linearly dependent, a contradiction to Corollary 2.3.

Lemma 2.10 Let $G$ be a graph. If $\operatorname{rank}(G)=\operatorname{frank}(G)$, then $G$ has the primitive cycle property.

Proof. Let $c_{1}, c_{2}$ be two distinct primitive cycles. Assume that $c_{1}$ and $c_{2}$ intersect in at least two edges. Thus $c_{1}$ and $c_{2}$ must intersect in at least two non adjacent vertices $u, v$. The cycle $c_{2}$ can be written as:

$$
c_{2}=\left\{u=u_{0}, u_{1}, \ldots, u_{s}, v=u_{s+1}, v_{1}, \ldots, v_{m}, u\right\} .
$$

At least one of the paths $\mathcal{P}_{1}=\left\{u, u_{1}, \ldots, u_{s}, v\right\}, \mathcal{P}_{2}=\left\{v, v_{1}, \ldots, v_{m}, u\right\}$ that form the cycle $c_{2}$ must contain a vertex not in $c_{1}$, otherwise $c_{1}=c_{2}$. Assume that the path $\mathcal{P}_{1}$ has this property. Hence there is $u_{k} \notin c_{1}$ such that $u_{i} \in c_{1}$ for $i<k$, and there is $u_{\ell} \in c_{1}$, with $k<\ell$, such that $u_{i} \notin c_{1}$ for $k \leq i<\ell$. Hence there are two non adjacent vertices $x=u_{k-1}, y=u_{\ell}$ in $c_{1}$ and a path $\mathcal{P}=\left\{x, u_{k}, \ldots, u_{\ell-1}, y\right\}$ of length at least two that intersect $c_{1}$ in exactly the vertices $x, y$ :


This contradicts Lemma 2.9.

Lemma 2.11 Let $G$ be a graph. If $G$ satisfies PCP and $G$ does not contain a subdivision of $\mathcal{K}_{4}$ as a subgraph, then for any two non adjacent vertices $x, y$ of $G$ there are at most two vertex disjoint paths joining $x$ and $y$.

Proof. Assume that there are three vertex disjoint paths joining $x$ and $y$ :

$$
\mathcal{P}_{1}=\left\{x, x_{1}, \ldots, x_{r}, y\right\}, \mathcal{P}_{2}=\left\{x, z_{1}, \ldots, z_{t}, y\right\}, \mathcal{P}_{3}=\left\{x, y_{1}, \ldots, y_{s}, y\right\}
$$

where $r, s, t$ are greater or equal than 1 . We may assume that the sum of the lengths of the $\mathcal{P}_{i}$ 's is minimal. Consider the cycles

$$
\begin{aligned}
c_{1} & =\left\{x, x_{1}, \ldots, x_{r}, y, z_{t}, \ldots, z_{1}, x\right\}, \quad c_{2}=\left\{x, z_{1}, \ldots, z_{t}, y, y_{s}, \ldots, y_{1}, x\right\} \\
c_{3} & =\left\{x, x_{1}, \ldots, x_{r}, y, y_{s}, \ldots, y_{1}, x\right\}
\end{aligned}
$$

Thus we are in the following situation:


Observe that, by the choice of the $\mathcal{P}_{i}$ 's, a chord of the cycle $c_{1}$ (resp. $c_{2}, c_{3}$ ) must join $x_{i}$ and $z_{j}$ (resp. $z_{i}$ and $y_{j}, x_{i}$ and $y_{j}$ ) for some $i, j$. Notice that the cycles $c_{1}$ and $c_{3}$ are primitive. Indeed if $c_{1}$ or $c_{3}$ have a chord, then one of the following

is a subgraph of $G$, which is impossible because both subgraphs are subdivisions of $\mathcal{K}_{4}$. Since $c_{1}$ and $c_{3}$ are primitive and have at least two edges in common we obtain that $G$ does not satisfy PCP, a contradiction.

Lemma 2.12 Let $G$ be a graph. If $\operatorname{rank}(G)=\operatorname{frank}(G)$, then $G$ does not contain a subdivision of $\mathcal{K}_{4}$ as a subgraph.

Proof. Assume there is a subgraph $H \subset G$ which is a subdivision of $\mathcal{K}_{4}$. If $\mathcal{K}_{4}$ is a subgraph of $G$, then $G$ has four distinct triangles whose cycle vectors are linearly dependent, a contradiction to Corollary 2.3. If $\mathcal{K}_{4}$ is not a subgraph of $G$, then $H$ is a strict subdivision of $\mathcal{K}_{4}$, i.e., $H$ has more than four vertices. It follows that there are two vertices $x, y$ in $V(H)$ which are non adjacent in $G$. Notice that $x, y$ can be chosen in $\mathcal{K}_{4}$ before subdivision. Therefore there are at least three non adjacent paths joining $x$ and $y$, a contradiction to Lemma 2.9,

The main result of this section is:
Theorem 2.13 Let $G$ be a graph. Then the following conditions are equivalent:
(a) $G$ is a ring graph.
(b) $\operatorname{rank}(G)=\operatorname{frank}(G)$.
(c) $G$ satisfies PCP and $G$ does not contain a subdivision of $\mathcal{K}_{4}$ as a subgraph.

Proof. $(\mathrm{a}) \Rightarrow(\mathrm{b})$ : By induction on the number of vertices it is not hard to see that any ring graph $G$ satisfies the equality $\operatorname{rank}(G)=\operatorname{frank}(G)$.
(b) $\Rightarrow$ (c): It follows at once from Lemmas 2.10 and 2.12,
(c) $\Rightarrow$ (a): Let $G_{1}, \ldots, G_{r}$ be the blocks of $G$. The proof is by induction on the number of vertices of $G$. If each $G_{i}$ is either a bridge or an isolated vertex, then $G$ is a forest and consequently a ring graph. Hence by Lemma 2.4 we may assume that $G$ is 2 -connected and that $G$ is not a cycle. We claim that $G$ has at least one vertex of degree 2. If $\operatorname{deg}(v) \geq 3$ for all $v \in V(G)$, then by Lemma 2.8 there is a subgraph $H \subset G$ which is a subdivision of $\mathcal{K}_{4}$, which is impossible. Let $v_{0} \in V(G)$ be a vertex of degree 2 as claimed. By the primitive cycle property there is a unique primitive cycle $c=\left\{v_{0}, v_{1}, \ldots, v_{s}=v_{0}\right\}$ of $G$ containing $v_{0}$. The graph $H=G \backslash\left\{v_{0}\right\}$ satisfies PCP and does not has a subdivision of $\mathcal{K}_{4}$ as a subgraph. Consequently $H$ is a ring graph. Thus we may assume that $c$ is not a triangle, otherwise $G$ is a ring graph because it can be obtained by adding the $H$-path $\left\{v_{2}, v_{0}, v_{1}\right\}$ to $H$.

Next we claim that if $1 \leq i<j<k \leq s-1$, then $v_{i}$ and $v_{k}$ cannot be in the same connected component of $H \backslash\left\{v_{j}\right\}$. Otherwise there is a path of $H \backslash\left\{v_{j}\right\}$ than joins $v_{i}$ with $v_{k}$. It follows that there is a path $\mathcal{P}$ of $H \backslash\left\{v_{j}\right\}$ with at least three vertices that joins a vertex of $\left\{v_{j+1}, \ldots, v_{s-1}\right\}$ with a vertex of $\left\{v_{1}, \ldots, v_{j-1}\right\}$ and such that $\mathcal{P}$ intersects $c$ exactly in its ends, but this contradicts Lemma 2.11. This proves the claim. In particular $v_{i}$ is a cutvertex of $H$ for $i=2, \ldots, s-2$ and $v_{i-1}$, $v_{i+1}$ are in different connected components of $H \backslash\left\{v_{i}\right\}$. For each $1 \leq i \leq s-2$ there is a block $K_{i}$ of $H$ such that $\left\{v_{i}, v_{i+1}\right\}$ is an edge of $K_{i}$. Notice that if $1 \leq i<j<k \leq s-1$, then $v_{i}, v_{j}, v_{k}$ cannot lie in some $K_{\ell}$. Indeed if the three vertices lie in some $K_{\ell}$, then there is a path $\mathcal{P}^{\prime}$ in $K_{\ell} \backslash\left\{v_{j}\right\}$ that joins $v_{i}$ and $v_{k}$. Since $\mathcal{P}^{\prime}$ is also a path in $H \backslash\left\{v_{j}\right\}$, we get that $v_{i}$ and $v_{k}$ are in the same connected
component of $H \backslash\left\{v_{j}\right\}$, but this contradicts the last claim. In particular $V\left(K_{\ell}\right)$ intersects the cycle $c$ in exactly the vertices $v_{\ell}, v_{\ell+1}$ for $1 \leq \ell \leq s-2$.

Observe that at least one of the edges of $c$ not containing $v_{0}$ is not a bridge of $H$. To show this pick $x \notin c$ such that $\left\{x, v_{k}\right\}$ is an edge of $H$. We may assume that $v_{k+1} \neq v_{0}\left(\right.$ or $\left.v_{k-1} \neq v_{0}\right)$. Since $G^{\prime}=G \backslash\left\{v_{k}\right\}$ is connected, there is a path $\mathcal{P}$ of $G^{\prime}$ joining $x$ and $v_{k+1}$ (or $v_{k-1}$ ). This readily yields a cycle of $H$ containing an edge of $c$ which is not a bridge of $H$. Hence at least one of the blocks $K_{1}, \ldots, K_{s-2}$, say $K_{i}$, contains vertices outside $c$.

Next we show that two distinct blocks $B_{1}, B_{2}$ of $H$ cannot intersect outside c. We proceed by contradiction assuming that $V\left(B_{1}\right) \cap V\left(B_{2}\right)=\{z\}$ for some $z$ not in $c$. Let $H_{1}, \ldots, H_{t}$ be the connected components of $H \backslash\{z\}$. Notice that $t \geq 2$ because $\{z\}$ is the intersection of two different blocks of $H$. We may assume that $\left\{v_{1}, \ldots, v_{s-1}\right\}$ are contained in $H_{1}$. Consider the subgraph $H_{1}^{\prime}$ of $G \backslash\{z\}$ obtained from $H_{1}$ by adding the vertex $v_{0}$ and the edges $\left\{v_{0}, v_{1}\right\},\left\{v_{0}, v_{s-1}\right\}$. It follows that the connected components of $G \backslash\{z\}$ are $H_{1}^{\prime}, H_{2}, \ldots, H_{t}$, which is impossible because $G$ is 2-connected.

Let $K_{i}$ be a block of $H$ that contains vertices outside $c$ for some $1 \leq i \leq s-2$. By induction hypothesis $K_{i}$ is a ring graph. Thus by Remark [2.7we can construct $K_{i}$ starting with a primitive cycle $c_{1}$ that contains the edge $\left\{v_{i}, v_{i+1}\right\}$, and then adding appropriate paths. Suppose that $\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}$ is the sequence of paths added to $c_{1}$ to obtain $K_{i}$. If we remove the path $\mathcal{P}_{m}$ from $G$ and use the fact that distinct blocks of $H$ cannot intersect outside $c$, then again by induction hypothesis we obtain a ring graph. It follows that $G$ is a ring graph as well.

An immediate consequence of Theorem 2.13 is:
Corollary 2.14 Let $G$ be a graph. If $\operatorname{rank}(G)=\operatorname{frank}(G)$, then $G$ is planar.
Corollary 2.15 If $G$ is a ring graph and $H$ is an induced subgraph of $G$, then $H$ is a ring graph.

Proof. It follows from part (c) of Theorem 2.13.
Two graphs $H_{1}$ and $H_{2}$ are called homeomorphic if there exists a graph $G$ such that both $H_{1}$ and $H_{2}$ are subdivisions of $G$. A graph is outerplanar if it can be embedded in the plane so that all its vertices lie on a common face; it is usual to choose this face to be the exterior face. The complete bipartite graph with bipartition $\left(V_{1}, V_{2}\right)$ is denoted by $\mathcal{K}_{t, s}$, where $\left|V_{1}\right|=t$ and $\left|V_{2}\right|=s$.

Theorem 2.16 [10, Theorem 11.10] A graph is outerplanar if and only if it has no subgraph homeomorphic to $\mathcal{K}_{4}$ or $\mathcal{K}_{2,3}$ except $\mathcal{K}_{4} \backslash\{e\}$, where $e$ is an edge.

Proposition 2.17 If $G$ is an outerplanar graph, then $\operatorname{rank}(G)=\operatorname{frank}(G)$.

Proof. By Theorem[2.13(c) it suffices to prove that $G$ satisfies PCP and $G$ does not contain a subdivision of $\mathcal{K}_{4}$ as a subgraph. If $G$ contains a subdivision $H$ of $\mathcal{K}_{4}$ as a subgraph, then $G$ contains a subgraph, namely $H$, homeomorphic to $\mathcal{K}_{4}$, but this is impossible by Theorem 2.16. To finish the proof we now show that $G$ has the PCP property. Let $c_{1}=\left\{x_{1}, x_{2}, \ldots, x_{m}=x_{1}\right\}$ and $c_{2}=\left\{y_{1}, y_{2}, \ldots, y_{n}=y_{1}\right\}$ be two distinct primitive cycles having at least one common edge. We may assume that $x_{i}=y_{i}$ for $i=1,2$ and $x_{3} \neq y_{3}$. Notice that $y_{3} \notin c_{1}$ because otherwise $\left\{y_{2}, y_{3}\right\}=\left\{x_{2}, y_{3}\right\}$ is a chord of $c_{1}$. We need only show that $\left\{x_{1}, x_{2}\right\}=c_{1} \cap c_{2}$, because this implies that $c_{1}$ and $c_{2}$ cannot have more than one edge in common. Assume that $\left\{x_{1}, x_{2}\right\} \subsetneq c_{1} \cap c_{2}$. Let $r$ be the minimum integer such that $y_{r}$ belong to $\left(c_{1} \cap c_{2}\right) \backslash\left\{x_{1}, x_{2}\right\}$. Notice that $y_{r} \neq x_{3}$ because otherwise $\left\{x_{2}, x_{3}\right\}$ is a chord of $c_{2}$. Hence $c_{1}$ together with the path $\left\{x_{2}=y_{2}, y_{3}, \ldots, y_{r}\right\}$ give a subgraph $H$ of $G$ which is a subdivision of $\mathcal{K}_{2,3}$, a contradiction to Theorem 2.16.

## 3 Toric ideals of graphs

Let $R=k\left[x_{1}, \ldots, x_{n}\right]$ be a polynomial ring over a field $k$ and let $G$ be a graph on the vertex set $V(G)=\left\{x_{1}, \ldots, x_{n}\right\}$. The edge subring of the graph $G$, denoted by $k[G]$, is the $k$-subalgebra of $R$ generated by the monomials corresponding to the edges of $G$ :

$$
k[G]=k\left[\left\{x_{i} x_{j} \mid x_{i} \text { is adjacent to } x_{j}\right\}\right] \subset R .
$$

There is a graded epimorphism of $k$-algebras

$$
\varphi: B=k\left[t_{1}, \ldots, t_{q}\right] \longrightarrow k[G], \quad\{x, y\} \longmapsto x y,
$$

where $B$ is a polynomial ring graded by $\operatorname{deg}\left(t_{i}\right)=1$ for all $i$ and $k[G]$ has the normalized grading $\operatorname{deg}\left(f_{i}\right)=1$ for all $i$. The kernel of $\varphi$, denoted by $P(G)$, is a graded prime ideal of $B$ called the toric ideal of $G$. The graded structure of $P(G)$ will not play a role in what follows. Later we will emphasize the fact that toric ideals of oriented graphs may not have a graded structure. Having a grading is useful if one studies the projective toric variety defined by $P(G)$ or systems of generators of $P(G)$.

The Krull dimension of $k[G]$ equals the rank of the incidence matrix of $G$ [11]. If $G$ is a connected graph, then by [23, Corollary 6.3] one has:

$$
\operatorname{dim}(k[G])= \begin{cases}n & \text { if } G \text { is not bipartite, and } \\ n-1 & \text { otherwise } .\end{cases}
$$

Since $B / P(G) \simeq k[G]$, we obtain that height of $P(G)$ is $q-n+1$ if $G$ is a connected bipartite graph and that height of $P(G)$ is $q-n$ if $G$ is a connected non-bipartite graph.

Definition 3.1 The toric ideal $P(G)$ is called a complete intersection if it can be generated by $g$ polynomials, where $g$ is the height of $P(G)$. The graph $G$ is called a complete intersection if $P(G)$ is a complete intersection.

The complete intersection property is independent of $k$ [14, Theorem 3.9]. In the area of complete intersection toric ideals there are some recent papers, see [3, 4] and the introduction of [14], where one can find additional properties and references on this active area.

Next we describe a generating set for $P(G)$ that shows how the cycle structure of $G$ determine $P(G)$. Let

$$
c=\left\{x_{0}, x_{1}, \ldots, x_{r}=x_{0}\right\}
$$

be an even cycle of $G$ such that $f_{i}=x_{i-1} x_{i}$. Notice that the binomial

$$
t_{c}=t_{1} t_{3} \cdots t_{r-1}-t_{2} t_{4} \cdots t_{r}
$$

is in $P(G)$. If $G$ is bipartite, then $P(G)$ is minimally generated by the set of all $t_{c}$ such that $c$ is a primitive cycle of $G$, see [21].

The next result can be extended to non connected bipartite graphs.
Theorem 3.2 [19, Theorem 2.5] If $G$ is a bipartite connected graph, then $G$ is a complete intersection if and only if $\operatorname{rank}(G)=\operatorname{frank}(G)$.

This was the first characterization of complete intersection bipartite graphs. For these graphs the equality $\operatorname{rank}(G)=\operatorname{frank}(G)$ can also be interpreted in homological terms [19]. Another characterization is the following:

Theorem 3.3 ([13]) If $G$ is a bipartite graph, then $G$ is a complete intersection if and only if $G$ is planar and satisfies PCP.

The next result is interesting because it shows how to construct all the complete intersection bipartite graphs.

Corollary 3.4 If $G$ is a bipartite graph, then $G$ is a complete intersection if and only if $G$ is a ring graph.

Proof. By Theorem 3.2 $G$ is a complete intersection if and only if $\operatorname{rank}(G)=$ $\operatorname{frank}(G)$ and the result follows from Theorem 2.13,

Notation For $a=\left(a_{1}, \ldots, a_{q}\right) \in \mathbb{N}^{q}$ and $f_{1}, \ldots, f_{q}$ in a commutative ring we set $f^{a}=f_{1}^{a_{1}} \cdots f_{q}^{a_{q}}$. The support of $f^{a}$ is the set $\operatorname{supp}\left(f^{a}\right)=\left\{f_{i} \mid a_{i} \neq 0\right\}$.

Definition 3.5 Let $g_{1}=t^{\alpha_{1}}-t^{\beta_{1}}, \ldots, g_{r}=t^{\alpha_{r}}-t^{\beta_{r}}$ be a sequence of homogeneous binomials of degree at least 2 in the polynomial ring $B=k\left[t_{1}, \ldots, t_{q}\right]$. We say that $\mathcal{B}=\left\{g_{1}, \ldots, g_{r}\right\}$ is a foliation if the following conditions are satisfied:
(a) $t^{\alpha_{i}}$ and $t^{\beta_{i}}$ are square-free monomials for all $i$,
(b) $\operatorname{supp}\left(t^{\alpha_{i}}\right) \cap \operatorname{supp}\left(t^{\beta_{i}}\right)=\emptyset$ for all $i$, and
(c) $\left|\left(\cup_{i=1}^{j} C_{i}\right) \cap C_{j+1}\right|=1$ for $1 \leq j<r$, where $C_{i}=\operatorname{supp}\left(t^{\alpha_{i}}\right) \cup \operatorname{supp}\left(t^{\beta_{i}}\right)$.

Proposition 3.6 If $\mathcal{B}=\left\{g_{1}, \ldots, g_{r}\right\}$ is a foliation, then the ideal $I=(\mathcal{B})$ generated by $\mathcal{B}$ is a complete intersection and $\mathcal{B}$ is a Gröbner basis of I.

Proof. By the constructive nature of $\mathcal{B}$ we can order the variables $t_{1}, \ldots, t_{q}$ such that the leading terms of $g_{1}, \ldots, g_{r}$, with respect to the lexicographical order, are relatively prime. Let $\operatorname{in}\left(g_{i}\right)$ be the leading term of $g_{i}$. Then $\mathcal{B}$ is a Gröbner basis by a result of Buchberger [22, Theorem 2.4.15]. Since $B / I$ and $\left.B / \operatorname{in}\left(g_{1}\right), \ldots, \operatorname{in}\left(g_{r}\right)\right)$ have the same Krull dimension by a result of Macaulay [22, Corollary 2.4.13], we obtain that the height of $I$ is equal to $r$, as required.

Corollary 3.7 If $G$ is a 2-connected bipartite graph with at least four vertices, then the toric ideal $P(G)$ is a complete intersection if and only if it is generated by a foliation.

Proof. It follows from Corollary 3.4 and the definition of a ring graph.

## 4 Toric ideals of oriented graphs

Let $G$ be a connected graph with $n$ vertices and $q$ edges and let $\mathcal{O}$ be an orientation of the edges of $G$, i.e., an assignment of a direction to each edge of $G$. Thus $\mathcal{D}=(G, \mathcal{O})$ is an oriented graph. To each oriented edge $e=\left(x_{i}, x_{j}\right)$ of $\mathcal{D}$, we associate the vector $v_{e}$ defined as follows: the $i$ th entry is -1 , the $j$ th entry is 1 , and the remaining entries are zero. The incidence matrix $A_{\mathcal{D}}$ of $\mathcal{D}$ is the $n \times q$ matrix with entries in $\{0, \pm 1\}$ whose columns are the vectors of the form $v_{e}$, with $e$ an edge of $\mathcal{D}$. For simplicity of notation we set $A=A_{\mathcal{D}}$. The set of column vectors of $A$ will be denoted by $\mathcal{A}=\left\{v_{1}, \ldots, v_{q}\right\}$. It is well known [15] that $A$ defines a matroid $M[A]$ on $\mathcal{A}=\left\{v_{1}, \ldots, v_{q}\right\}$ over the field $\mathbb{Q}$ of rational numbers, which is called the vector matroid of $A$, whose independent sets are the independent subsets of $\mathcal{A}$. A minimal dependent set or circuit of $M[A]$ is a dependent set all of whose proper subsets are independent. A subset $B$ of $\mathcal{A}$ is called a basis of $M[A]$ if $B$ is a maximal independent set. Recall that an integer matrix is called totally unimodular if each $i \times i$ minor (subdeterminant) of the matrix is 0 or $\pm 1$ for all $i \geq 1$.

Lemma 4.1 The circuits of $M[A]$ are precisely the cycles of $G, A$ is totally unimodular, and $\operatorname{rank}(A)=n-1$

Proof. It follows from [9, pp. 343-344] and [18, p. 274].
Let $\alpha \in \mathbb{R}^{q}$. The support of $\alpha$ is defined as $\operatorname{supp}(\alpha)=\left\{i \mid \alpha_{i} \neq 0\right\}$. An elementary vector of $\operatorname{ker}(A)$ is a vector $0 \neq \alpha$ in $\operatorname{ker}(A)$ whose support is minimal with respect to inclusion, i.e., $\operatorname{supp}(\alpha)$ does not properly contain the support of any other nonzero vector in $\operatorname{ker}(A)$. A circuit of $\operatorname{ker}(A)$ is an elementary vector of $\operatorname{ker}(A)$ with relatively prime integral entries (see [24, Section 2]). There is a one to one correspondence

$$
\text { Circuits of } \operatorname{ker}(A) \quad \longrightarrow \quad \text { Circuits of } M[A]=\text { cycles of } G
$$

given by $\alpha=\left(\alpha_{1}, \ldots, \alpha_{q}\right) \rightarrow C(\alpha)=\left\{v_{i} \mid i \in \operatorname{supp}(\alpha)\right\}$. Thus the set of circuits of the kernel of $A$ is the algebraic realization of the set of circuits of the vector matroid $M[A]$.

Consider the edge subring $k[\mathcal{D}]:=k\left[x^{v_{1}}, \ldots, x^{v_{q}}\right] \subset k\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$ of the oriented graph $\mathcal{D}$. There is an epimorphism of $k$-algebras

$$
\varphi: B=k\left[t_{1}, \ldots, t_{q}\right] \longrightarrow k[\mathcal{D}], \quad t_{i} \longmapsto x^{v_{i}}
$$

where $B$ is a polynomial ring. The kernel of $\varphi$, denoted by $P_{\mathcal{D}}$, is called the toric ideal of $\mathcal{D}$. Notice that $P_{\mathcal{D}}$ is no longer a graded ideal, see Proposition 4.7. The toric ideal $P_{\mathcal{D}}$ is a prime ideal of height $q-n+1$ generated by binomials and $k[\mathcal{D}]$ is a normal domain. Thus any minimal generating set of $P_{\mathcal{D}}$ must have at least $q-n+1$ elements, by the principal ideal theorem.

Let $\alpha \in \mathbb{R}^{q}$. Note that $\alpha=\alpha_{+}-\alpha_{-}$, where $\alpha_{+}$and $\alpha_{-}$are two non negative vectors with disjoint support. If $0 \neq \alpha \in \operatorname{ker}(A) \cap \mathbb{Z}^{n}$ we associate the binomial $t_{\alpha}=t^{\alpha_{+}}-t^{\alpha_{-}}$. Notice that $t_{\alpha} \in P_{\mathcal{D}}$. Given a cycle $c$ of $\mathcal{D}$, we split $c$ in two disjoint sets of edges $c_{+}$and $c_{-}$, where $c_{+}$is oriented clockwise and $c_{-}=c \backslash c_{+}$. The binomial

$$
t_{c}=\prod_{v_{i} \in c_{+}} t_{i}-\prod_{v_{i} \in c_{-}} t_{i}
$$

belongs to $P_{\mathcal{D}}$. If $c_{+}=\emptyset$ or $c_{-}=\emptyset$ we set $\prod_{v_{i} \in c_{+}} t_{i}=1$ or $\prod_{v_{i} \in c_{-}} t_{i}=1$.
Definition 4.2 The toric ideal $P_{\mathcal{D}}$ is called a binomial complete intersection if $P_{\mathcal{D}}$ can be generated by $q-n+1$ binomials.

If $P_{\mathcal{D}}$ is homogeneous and is generated by $q-n+1$ polynomials, then $P_{\mathcal{D}}$ is a binomial complete intersection.

Proposition $4.3 P_{\mathcal{D}}$ is generated by the set of all binomials $t_{c}$ such that $c$ is a cycle of $\mathcal{D}$ and this set is a universal Gröbner basis.

Proof. Let $\mathcal{U}_{\mathcal{D}}$ be the set of all binomials of the form $t_{\alpha}$ such that $\alpha$ is a circuit of $\operatorname{ker}(A)$. Since $A$ is totally unimodular, by [20, Proposition 8.11 ], the set $\mathcal{U}_{\mathcal{D}}$ form
a universal Gröbner basis of $P_{\mathcal{D}}$. Notice that the circuits of $\operatorname{ker}(A)$ are in one to one correspondence with the circuits of the vector matroid $M[A]$. To complete the proof it suffices to observe that the circuits of $M[A]$ are precisely the cycles of $G$, see Lemma 4.1.

Proposition 4.4 Let $c=\left\{x_{1}, x_{2}, \ldots, x_{r}, x_{1}\right\}$ be a circuit of $\mathcal{D}$. Suppose that $\left(x_{i}, x_{j}\right)$ or $\left(x_{j}, x_{i}\right)$ is an edge of $\mathcal{D}$, with $i+1<j$. Then $t_{c}$ is a linear combination of $t_{c_{1}}$ and $t_{c_{2}}$, where $c_{1}=\left\{x_{1}, x_{2}, \ldots, x_{i}, x_{j}, x_{j+1}, \ldots, x_{r}, x_{1}\right\}$ and $c_{2}=$ $\left\{x_{i}, x_{i+1}, \ldots, x_{j}, x_{i}\right\}$.

Proof. Suppose without loss of generality that $v_{k}=\left(x_{i}, x_{j}\right)$ is the edge of $\mathcal{D}$ with $i+1<j$. Then we can write $t_{c_{1}}=t^{\alpha_{+}}-t^{\alpha_{-}}$and $t_{c_{2}}=t^{\beta_{+}-} t^{\beta_{-}}$for some $\alpha, \beta$. We may assume that $v_{k} \in c_{1_{+}} \cap c_{2_{+}}$, because otherwise we may multiply $t_{c_{1}}$ or $t_{c_{2}}$ by -1 . As $t_{k}$ divides $t^{\alpha_{+}}$and $t_{k}$ divides $t^{\beta_{+}}$, we get

$$
\begin{aligned}
\left(\frac{t^{\beta_{+}}}{t_{k}}\right) t_{c_{1}}-\left(\frac{t^{\alpha_{+}}}{t_{k}}\right) t_{c_{2}} & =\left(\frac{t^{\beta_{+}}}{t_{k}}\right)\left(t^{\alpha_{+}}-t^{\alpha_{-}}\right)-\left(\frac{t^{\alpha_{+}}}{t_{k}}\right)\left(t^{\beta_{+}-} t^{\beta_{-}}\right) \\
& =\left(\frac{t^{\alpha_{+}}}{t_{k}}\right) t^{\beta_{-}}-\left(\frac{t^{\beta_{+}}}{t_{k}}\right) t^{\alpha_{-}}=t^{\gamma_{1}}-t^{\gamma_{2}}
\end{aligned}
$$

Hence $t^{\gamma_{1}}-t^{\gamma_{2}}$ is in $P_{\mathcal{D}}$, where $\gamma_{1}=\left(\alpha_{+}-e_{k}\right)+\beta_{-}$and $\gamma_{2}=\left(\beta_{+}-e_{k}\right)+\alpha_{-}$. Then $t^{\gamma_{1}}$ is the product of the edges of $\left(c_{1_{+}} \backslash\left\{t_{k}\right\}\right) \cup c_{2_{-}}$, but these are the edges of $c_{+}$. By the same reason $t_{\gamma_{2}}$ is the product of the edges of $c_{-}$. Thus $t_{c}=t^{\gamma_{1}}-t^{\gamma_{2}}$. From the equality above we get that $t_{c}$ is a linear combination of $t_{c_{1}}$ and $t_{c_{2}}$.

As an immediate consequence of Propositions 4.3 and 4.4 we get:
Corollary 4.5 $P_{\mathcal{D}}$ is generated by the set of binomials corresponding to primitive cycles.

We say that a cycle $c$ of $\mathcal{D}$ is oriented if all the arrows of $c$ are oriented in the same direction. If $\mathcal{D}$ does not have oriented cycles, we say that $\mathcal{D}$ is acyclic.

Proposition 4.6 (10]) $\mathcal{D}$ is acyclic if and only if there is a linear ordering of the vertices such that every edge of $\mathcal{D}$ has the form $\left(x_{i}, x_{j}\right)$ with $i<j$.

The ordering of the last proposition is called a topological ordering. The next result is not hard to prove.

Proposition 4.7 If $\mathcal{D}$ has a topological ordering, then $P_{\mathcal{D}}$ is generated by homogeneous binomials with respect to the grading induced by degree $\left(t_{k}\right)=j-i$, where $t_{k}$ maps to $x_{i}^{-1} x_{j}$ and $\left(x_{i}, x_{j}\right)$ is an edge.

Corollary 4.8 If $\mathcal{D}$ is acyclic, then $P_{\mathcal{D}}$ is a complete intersection if and only if $P_{\mathcal{D}}$ is generated by $q-n+1$ binomials corresponding to primitive cycles.

Proof. Since $P_{\mathcal{D}}$ is a graded ideal, it suffices to recall that all the homogeneous minimal sets of generators of $P_{\mathcal{D}}$ have the same number of elements.

In general the binomial complete intersection property of $P_{\mathcal{D}}$ depends on the orientation of $G$. However we have:

Corollary 4.9 If $G$ is a ring graph, then $P_{\mathcal{D}}$ is a complete intersection for any orientation of $G$. The converse holds if $G$ is bipartite.

Proof. By Corollary 4.5, $P_{\mathcal{D}}$ is generated by $q-n+1$ binomials. To show the converse assume that $G$ is bipartite. Let $\left(V_{1}, V_{2}\right)$ be a bipartition of $G$. Consider the oriented graph $\mathcal{D}$ obtained from $G$ by orienting all the edges of $G$ from $V_{1}$ to $V_{2}$, i.e., all the arrows of $G$ have tail at $V_{1}$ and head at $V_{2}$. Since every vertex of $\mathcal{D}$ is either a source or a sink it follows that $P(G)=P_{\mathcal{D}}$. Hence $P(G)$ is a complete intersection and $G$ is a ring graph by Corollary 3.4.

An interesting problem that remains unsolved is to characterize the graphs with the property that $P_{\mathcal{D}}$ is a binomial complete intersection for all orientations of $G$. Apart from ring graphs it has been shown that complete graphs have this property [17, 16].

## A special orientation

Let $G$ be a connected graph. Here we show that there is always an orientation of $G$ such that $P_{\mathcal{D}}$ is a complete intersection generated by the binomials that correspond to a cycle basis of a certain spanning tree of $G$.

Definition 4.10 Let $S$ be a set of vertices of a graph $G$. The neighbor set of $S$, denoted by $N_{G}(S)$ or simply by $N(S)$ if $G$ is understood, is the set of vertices of $G$ that are adjacent with at least one vertex of $S$.

Lemma 4.11 If $H$ is a subgraph of a connected graph $G$ and $N_{G}(V(H)) \subset V(H)$, then $V(G)=V(H)$.

Proof. Fix a vertex $x \in V(H)$. Let $y \in V(G)$. Since $G$ is connected, there is a path $\mathcal{P}=\left\{b_{1}=x, b_{2}, \ldots, b_{\ell}=y\right\}$ from $x$ to $y$. Using that $\left\{b_{j}, b_{j+1}\right\} \in E(G)$ for $1 \leq j<\ell-1$ and that $b_{1} \in V(H)$, by induction we get that $b_{j} \in V(H)$ for all $j$. Thus $y \in V(H)$.

We begin by constructing a proper nested sequence $A_{1}, \ldots, A_{m}$ of subtrees of $G$ labeled by $V\left(A_{j}\right)=\left\{y_{1}^{j}, \ldots, y_{r_{j}}^{j}\right\}$ such that $A_{m}$ is a spanning tree of $G$ and $V\left(A_{i}\right) \subsetneq V\left(A_{i+1}\right)$ for $i<m$. First we construct the sequence $A_{1}, \ldots, A_{m}$ and then we show that it has the required properties. Let $A_{1}$ be a path of $G$ maximal with respect to inclusion. Set $V\left(A_{1}\right)=\left\{y_{1}^{1}, y_{2}^{1}, \ldots, y_{r_{1}}^{1}\right\}$. We define

$$
i_{1}=\max \left\{u \in \mathbb{N} \mid N_{G}\left(y_{1}^{1}, \ldots, y_{u}^{1}\right) \subset V\left(A_{1}\right)\right\},
$$

where $N_{G}(B)$ is the neighbor set of $B$. If $i_{1}=r_{1}$, then $N_{G}\left(V\left(A_{1}\right)\right) \subset V\left(A_{1}\right)$ and by Lemma 4.11 we get $V\left(A_{1}\right)=V(G)$, in this case $A_{1}$ is the required spanning tree and we set $m=1$. If $i_{1}<r_{1}$, we define $a_{1}=y_{i_{1}+1}^{1}$. By induction we define the sequence of subgraphs $A_{1}, \ldots, A_{m}$. Suppose that $A_{j}$ has been defined, where $V\left(A_{j}\right)=\left\{y_{1}^{j}, \ldots, y_{r_{j}}^{j}\right\}$. We define

$$
i_{j}=\max \left\{u \in \mathbb{N} \mid N_{G}\left(y_{1}^{j}, \ldots, y_{u}^{j}\right) \subset V\left(A_{j}\right)\right\} .
$$

If $i_{j}=r_{j}$, then by Lemma 4.11 we get $V\left(A_{j}\right)=V(G)$, in this case we set $m=j$ and $A_{1}, \ldots, A_{j}$ is the desired sequence. If $i_{j}<r_{j}=\left|V\left(A_{j}\right)\right|$, we define $a_{j}=y_{i_{j}+1}^{j}$. Let $\mathcal{L}_{j}$ be a maximal path with respect to inclusion such that $V\left(\mathcal{L}_{j}\right) \cap V\left(A_{j}\right)=$ $\left\{a_{j}\right\}$ and $V\left(\mathcal{L}_{j}\right)=\left\{z_{1}^{j}, z_{2}^{j}, \ldots, z_{s_{j}}^{j}=a_{j}\right\}$, the final vertex of $\mathcal{L}_{j}$ is $a_{j}$. We define $A_{j+1}$ as follows: $V\left(A_{j+1}\right)=V\left(A_{j}\right) \cup V\left(\mathcal{L}_{j}\right)=\left\{y_{1}^{j+1}, \ldots, y_{r_{j}+s_{j}-1}^{j+1}\right\}$, where

$$
y_{i}^{j+1}= \begin{cases}y_{i}^{j} & \text { if } i \leq i_{j}  \tag{1}\\ z_{i-i_{j}}^{j} & \text { if } i_{j}+1 \leq i \leq i_{j}+s_{j} \\ y_{i-s_{j}+1}^{j} & \text { if } i_{j}+s_{j}+1 \leq i \leq r_{j}+s_{j}-1\end{cases}
$$

$E\left(A_{j+1}\right)=E\left(A_{j}\right) \cup E\left(\mathcal{L}_{j}\right)$, and $r_{j+1}=r_{j}+s_{j}-1$.
Lemma $4.12 i_{k+1}>i_{k}$ for $1 \leq k \leq m-1$.
Proof. By construction $y_{i}^{k+1}=y_{i}^{k}$ for $1 \leq i \leq i_{k}$ and $y_{i_{k}+1}^{k+1}=z_{1}^{k}$ (see Eq.(1)). By the maximality of $\mathcal{L}_{j}$ we have

$$
N_{G}\left(y_{1}^{k+1}, y_{2}^{k+1}, \ldots, y_{i_{k}}^{k+1}, y_{i_{k}+1}^{k+1}\right) \subset V\left(A_{k+1}\right),
$$

thus $i_{k+1}>i_{k}$ by definition of $i_{k+1}$.
Suppose that the process finish at step $m$, i.e., $i_{m}=r_{m}$. We now prove that $A_{1}, \ldots, A_{m}$ has the required properties:

Lemma 4.13 $A_{i}$ is a tree for $1 \leq i \leq m$ and $A_{m}$ is a spanning tree of $G$.
Proof. By induction on $i$. For $i=1$ the assertion is clear. Suppose that $A_{i}$ is a tree. Recall that $\mathcal{L}_{i}$ is a tree and $V\left(\mathcal{L}_{i}\right) \cap V\left(A_{i}\right)=\left\{a_{i}\right\}$. On the other hand $V\left(A_{i+1}\right)=V\left(A_{i}\right) \cup V\left(\mathcal{L}_{i}\right)$ and $E\left(A_{i+1}\right)=E\left(A_{i}\right) \cup E\left(\mathcal{L}_{i}\right)$, then $A_{i+1}$ is connected and does not has cycles. By Lemma 4.11 we get that $V\left(A_{m}\right)=V(G)$ and $A_{m}$ is a spanning tree.

Orientation of the tree $A_{m}$ and the graph $G$. Let $\tau=\left(A_{m}, \mathcal{O}\right)$ be the oriented tree obtained from $A_{m}$ using the following orientation:

$$
\left(y_{i}^{m}, y_{j}^{m}\right) \in E(\tau) \text { if and only if }\left\{y_{i}^{m}, y_{j}^{m}\right\} \in E\left(A_{m}\right) \text { and } j>i .
$$

By Lemma 4.13 we have $V(G)=V\left(A_{m}\right)=\left\{y_{1}^{m}, y_{2}^{m}, \ldots, y_{r_{m}}^{m}\right\}$ and we orient $G$ to obtain the oriented graph $\mathcal{D}=(G, \mathcal{O})$ in the following way:

$$
\left(y_{i}^{m}, y_{j}^{m}\right) \in \mathcal{D} \text { if and only if }\left\{y_{i}^{m}, y_{j}^{m}\right\} \in E(G) \text { and } j>i
$$

Example 4.14 The construction of the spanning tree $A_{m}$ and the orientation $\mathcal{O}$ of $G$ is illustrated below.


Notation For each $f_{i} \in E(\mathcal{D}) \backslash E(\tau)$ the unique cycle of the subgraph $\tau \cup\left\{f_{i}\right\}$ is denoted by $c\left(\tau, f_{i}\right)$.

Proposition 4.15 For each $f_{i} \in E(\mathcal{D}) \backslash E(\tau)$ all the edges of $c\left(\tau, f_{i}\right) \backslash\left\{f_{i}\right\}$ are oriented in the same direction and $f_{i}$ is oriented in the opposite direction.

Proof. By induction on $m$, the number of subtrees $A_{1}, \ldots, A_{m}$. If $m=1$ the result is easy to verify because $A_{1}$ is a spanning path of $G$. Assume $m>1$. Consider the subgraphs

$$
\bar{G}=G \backslash\left\{y_{1}^{1}, \ldots, y_{i_{1}}^{1}\right\}, \quad \bar{A}_{i}=A_{i} \backslash\left\{y_{1}^{1}, \ldots, y_{i_{1}}^{1}\right\}, i \geq 2
$$

We set $\overline{\mathcal{D}}=(\bar{G}, \overline{\mathcal{O}})$ and $\bar{\tau}=\left(\bar{A}_{m}, \overline{\mathcal{O}}\right)$, where $\overline{\mathcal{O}}$ is the orientation induced from $\mathcal{O}$. Notice that $\bar{G}$ is connected because $\bar{A}_{m}$ is a spanning tree of $\bar{G}$. Using the equality

$$
V\left(\bar{A}_{2}\right)=\left\{y_{i_{1}+1}^{2}, \ldots, y_{r_{1}+s_{1}-1}^{2}\right\}=\left\{z_{1}^{1}, \ldots, z_{s_{1}}^{1}, y_{i_{1}+2}^{1}, \ldots, y_{r_{1}}^{1}\right\}
$$

and $z_{s_{1}}=y_{i_{1}+1}^{1}$ it is not hard to see that $\bar{A}_{2}$ is a maximal path of $\bar{G}$ and the result follows by induction. Indeed a fundamental cycle of $\tau$ is equal to $c\left(\tau, f_{i}\right)=c\left(\bar{\tau}, f_{i}\right)$
with $f_{i} \in E(\overline{\mathcal{D}}) \backslash E(\bar{\tau})$ or $c\left(\tau, f_{i}\right)=c\left(\tau^{\prime}, f_{i}\right)$ with $f_{i} \in E(H) \backslash E\left(\tau^{\prime}\right)$ where $H$ is the induced subgraph on $\left\{y_{1}^{1}, \ldots, y_{i_{1}}^{1}\right\}$ and $\tau^{\prime}$ is the spanning path of $H$ given by $y_{1}^{1}, \ldots, y_{i_{1}}^{1}$. In the first case we apply induction to obtain that the edges of $c\left(\bar{\tau}, f_{i}\right)$ are properly oriented, in the second case it is easy to verify that $c\left(\tau^{\prime}, f_{i}\right)$ has the required orientation.

Theorem 4.16 $P_{\mathcal{D}}=\left(\left\{t_{c\left(\tau, f_{i}\right)} \mid f_{i} \in E(\mathcal{D}) \backslash E(\tau)\right\}\right)$.
Proof. Set $E(\mathcal{D}) \backslash E(\tau)=\left\{f_{1}, \ldots, f_{q-n+1}\right\}$. Suppose without loss of generality that $t_{1}, \ldots, t_{q-n+1}$ are the variables associated to $f_{1}, \ldots, f_{q-n+1}$ respectively. By Proposition $4.15 t_{c\left(\tau, f_{i}\right)}=t_{i}-t^{\beta_{i}}$, where $t^{\beta_{i}}$ is a product of variables associated to edges in $\tau$. Let $I$ be the ideal generated by the set $\left\{t_{c\left(\tau, f_{i}\right)} \mid f_{i} \in E(\mathcal{D}) \backslash E(\tau)\right\}$ in $B=k\left[t_{1}, \ldots, t_{q}\right]$. Let $h=t^{\alpha}-t^{\beta}$ be a binomial in $P_{\mathcal{D}}$. Thus $\overline{t_{i}}=\overline{t^{\beta_{i}}}$ in $B / I$ for $i=1, \ldots, q-n+1$. Then $\bar{h}=\overline{t^{\gamma}}-\overline{t^{\omega}}$, where $t^{\gamma}$ and $t^{\omega}$ are products of variables associated to edges of $\tau$. As $I \subset P_{\mathcal{D}}$, then $t^{\gamma}-t^{\omega} \in P_{\mathcal{D}}=\operatorname{ker}(\varphi)$. But $\tau$ is a tree, thus $t^{\gamma}=t^{\omega}$, and $\bar{h}=\overline{0}$ in $B / I$. Since $P_{\mathcal{D}}$ is generated by binomials, $P_{\mathcal{D}}=I$.

Corollary 4.17 Assume that $\mathcal{D}$ is the oriented graph constructed above. Then $P_{\mathcal{D}}$ is a homogeneous ideal generated by $q-n+1$ binomials corresponding to primitive cycles.

Proof. By Theorem 4.16 it follows that $P_{\mathcal{D}}$ does not contains binomials of the form $1-t^{a}$, i.e., $\mathcal{D}$ is acyclic. Thus we may apply Corollary 4.8,

A tournament $\mathcal{D}$ is a complete graph $\mathcal{K}_{n}$ with a given orientation.
Proposition 4.18 ([10]) If $\mathcal{D}$ is a tournament, then $\mathcal{D}$ has a spanning oriented path.

Proposition 4.19 ([12]) If $\mathcal{D}$ is an acyclic tournament, then $P_{\mathcal{D}}$ is a complete intersection minimally generated by a Gröbner basis.

Proof. Let $\tau$ be a spanning oriented path of $\mathcal{D}$, i.e., $\tau=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ and $\left(x_{i}, x_{i+1}\right)$ is an edge of $\mathcal{D}$ for all $i<n$. Since $\mathcal{D}$ is acyclic, using the proof of Theorem 4.16, it follows that

$$
P_{\mathcal{D}}=\left(\left\{t_{c\left(\tau, f_{i}\right)} \mid f_{i} \in E(\mathcal{D}) \backslash E(\tau)\right\}\right)
$$

where for each $f_{i} \in E(\mathcal{D}) \backslash E(\tau)$, the unique cycle of the subgraph $\tau \cup\left\{f_{i}\right\}$ is denoted by $c\left(\tau, f_{i}\right)$.

Similarly we can prove the following generalization:
Proposition 4.20 If $\mathcal{D}$ is an acyclic oriented graph with a spanning oriented path, then $P_{\mathcal{D}}$ is a complete intersection minimally generated by a Gröbner basis.

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