SIMPLY SEQUENTIALLY ADDITIVE LABELINGS OF 2-REGULAR GRAPHS

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Abstract. We conjecture that any 2-regular simple graph has an SSA labeling. We provide several special cases to support our conjecture. Most of our constructions are based on Skolem sequences and on an extension of it. We establish a connection between simply sequentially additive labelings of 2-regular graphs and ordered graceful labelings of spiders.

1. Introduction

In this paper C_n denotes the cycle of length n (C_1 is a loop on one vertex, C_2 has two vertices and two edges joining them) and P_n denotes the path with n edges. For any connected graphs $G_1, \dots, G_k$, we denote by $G_1 \cup \dots \cup G_k$ the graph that has k components: $G_1, \dots, G_k$ (up to isomorphism), and kG stands for $G \cup \dots \cup G$ (k times). We write $S_1 \cup^* \dots \cup^* S_n$ for the union of the sets $S_1, \dots, S_n$, if the sets are pairwise disjoint and we want to emphasize this fact.

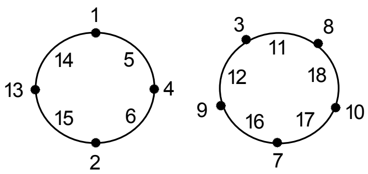

Figure 1: An SSA labeling of $C_4 \cup C_5$

Bange, Barkauskas and Slater [2] defined a *k-sequentially additive labeling* f of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, \dots, k + |V| + |E| - 1\}$ such that for each edge uv, $f(uv) = f(u) + f(v)$ (the required edge label for a loop on vertex u is $2f(u)$); if such a labeling exists, then G is said to be *k-sequentially additive*. We only deal with 1-sequentially additive labelings, see [6] for further results on k-sequentially additive labelings. 1-sequentially additive labelings (and graphs) are called *simply sequentially additive* (or *SSA*). Since the edge labels are uniquely determined by the vertex labels in an SSA labeling, we usually omit the enumeration of edge labels.

f_1 and f_2 (vertex-)labelings of G are considered the same in this paper, if the sets of (injectively) assigned labels are equal and $f_1 \circ f_2^{-1}$ is an automorphism of G.

Bange et al. [2] proved that C_n is simply sequentially additive if and only if $n \equiv 0$ or 1 mod 3. It is easy to see that the divisibility condition is necessary: The sum of the labels in an SSA labeled cycle of length n is $3\sum_{v\in V} f(v) = n(2n+1)$. The same reasoning shows that this condition is necessary for any 2-regular SSA graph on n vertices.

Observation 1. *The number of vertices in a simply sequentially additive 2-regular graph has the form $3k$ or $3k+1$ for some $k \in \mathbb{N}$.*

We conjecture that this condition is sufficient for *simple* 2-regular graphs:

Conjecture 2. *Every 2-regular simple graph on n vertices is simply sequentially additive, if $n \equiv 0$ or $1 \mod 3$.*

Clearly, an SSA graph cannot have multiple edges and it is easy to check that nC_1 $(n > 1)$ is not SSA (we cannot assign label $2n-1$). That is why we forbid C_1 and C_2 components. However, there are 2-regular graphs with loops that are simply sequentially additive (for example, $kC_3 \cup C_1$ is SSA for all $k \in \mathbb{N}$, see [8]), so our conjecture is not sharp.

In this paper we collected previous results supporting our conjecture and we prove it in some other special cases. We develop a method for constructing SSA labeled 2-regular graphs from certain Skolem sequence pairs. In the last section we formulate an interesting conjecture about Skolem sequences, motivated by this work.

As an application, we get ordered graceful labelings of spiders from SSA labelings of 2-regular graphs. (This application was the main motivation of this work.) In addition, we show that the ordered graceful tree conjecture [3] also holds for an other class of symmetrical trees using V-Skolem sequences (that are introduced in this paper to show that kC_4 is SSA, if $k \equiv 0$ or $1 \mod 3$).

2. Small cycles

In this section we verify Conjecture 2 for graphs that have the form kC_3 or kC_4.

Our main tool was introduced by Skolem in [10], we say that the partition $(a_i, b_i)_{i=1,\ldots,k}$ of $\{1,\ldots,2k\}$ is a *Skolem sequence* of order k, if $b_i - a_i = i$ for each i. We say that a_i is the ith *left endpoint* and b_i is the ith *right endpoint* and we denote the set of left endpoints (of a Skolem sequence S) by L_S and the right endpoints by R_S. These names are motivated by the following visualization:

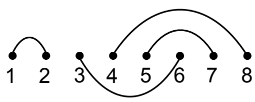

Figure 2: A Skolem sequence of order 4

A Skolem sequence of order k defines a perfect matching graph on $\{1,\ldots,2k\}$ (the edges are $a_1b_1,\ldots,a_kb_k$). If we embed this graph into the plane so that vertex v maps to the point v of real line, then we obtain a drawing where the lengths of edges are $\{1,\ldots,k\}$, see Fig. 2. (The *length* of an edge uv is $|u-v|$.)

It is known [10] that a Skolem sequence of order k exists if and only if $k \equiv 0$ or $1 \mod 4$.

The following result was proved in [8] by Nowakowski and Whitehead. However, they use an other terminology (they showed that spiders are ordered graceful trees, see Section 4), and we improve their construction, so we outline the proof here.

Theorem 3. *(Nowakowski and Whitehead, [8]) kC_3 is simply sequentially additive for all $k \in \mathbb{N}$.*

Sketch proof. The set of labels is $\{1, \dots, 6k\}$. Our SSA labeling is based on Skolem sequences.

If $k \equiv 0$ or $1 \mod 4$, a Skolem sequence $(p_i, q_i)_{i=1,\dots,k}$ of order k exists and we can assume that it partitions the set $\{k+1, \dots, 3k\}$ (we can translate the original sequence by k). Using this sequence we can partition $\{1, \dots, 3k\}$ into triples $T_i = \{a_i, b_i, c_i\}$ $(i = 1, \dots, k)$ so that $a_i + b_i = c_i$ for each i by setting $a_i = i$, $b_i = p_i$, $c_i = q_i$. With the notations $s' = 6k+1-s$ and $S' = \{s' : s \in S\}$ $(s \in \mathbb{N}, S \subset \mathbb{N})$, we assign the labels a_i, b_i and c_i' to the vertices of the ith cycle $(a_i + b_i + c_i' = 6k+1)$. Then the induced edge labels on this cycle are a_i', b_i' and c_i, so the set of assigned labels on this component is $T_i \cup^* T_i'$. Since $T_1 \cup^* \cdots \cup^* T_k = \{1, \dots, 3k\}$ and $\{1, \dots, 3k\}' = \{3k+1, \dots, 6k\}$, this labeling is bijective and it is simply sequentially additive by construction. We note that we have got exponentially many distinct SSA labelings, because the number of distinct Skolem sequences of order k is at least $2^{\lfloor k/3 \rfloor}$ (see [1]) and distinct Skolem sequences generate distinct labelings in this construction.

In the cases when $k \equiv 2$ or $3 \mod 4$ we can partition $\{1, \dots, 3k\}$ into k triples again, but now $k-1$ triples have the form $\{a_i, b_i, a_i + b_i\}$ like above and we have one special triple T whose elements sum to $6k+1$. If we assign the elements of T to the vertices of a 3-cycle, then the set of induced edge labels is T' on this cycle. The remaining components can be labeled in the same way as above. □

In the remaining part of this section, we investigate kC_4 graphs. We first define an analogoue of Skolem sequences that will be also used in Section 4. We say that the partition $(a_i, b_i; c_i)_{i=1,\dots,k}$ of $\{1, \dots, 3k\}$ is a *V-Skolem sequence* of order k $(a_i < b_i < c_i)$, if $\cup_{i=1}^k \{c_i - a_i, c_i - b_i\} = \{1, \dots, 2k\}$. In the terminology of drawings, a V-Skolem sequence is a drawing of kP_2 to $\{1, \dots, 3k\}$ such that for each edge the left endpoint is a leaf, the right endpoint is the center of a P_2 component and the set of edge-lengths is $\{1, \dots, 2k\}$, see Fig. 3.

Theorem 4. *A V-Skolem sequence of order k exists if and only if $k \equiv 0$ or 1* mod 3.

Proof. To see that the divisibility condition is necessary, let $(a_i, b_i; c_i)_{i=1,\dots,k}$ be a V-Skolem sequence of order k. We use the notations $S_1 = \sum_{i=1}^k (a_i + b_i)$ and $S_2 = \sum_{i=1}^k c_i$. Then $S_1 + S_2 = \frac{3k(3k+1)}{2}$ and $2S_2 - S_1 = k(2k+1)$ hold, so $k(2k+1)$ is divisible by 3.

In order to complete the proof, we give V-Skolem sequences of order $3l$ and $3l+1$ for all l.

Case 1: $k \equiv 0 \mod 3$ $(k = 3l)$.

The triples of a proper V-Skolem sequence are (see Fig. 3/1 for the case $l = 3$):

- $(2l+3+4i, 8l-2i; 8l+1+i) : i = 0, \dots, l-1$
- $(2l+1-2i, 2l+2-2i; 2l+4+4i) : i = 0, \dots, l-1$
- $(6l-3-4i, 6l-2-4i; 6l+3+2i) : i = 0, \dots, l-2$
- $(1, 2; 6l+1)$.

Case 2: $k \equiv 1 \mod 3$ $(k = 3l+1)$.

The construction is similar (see Fig. 3/2):

- $(2l+5+4i, 8l+2-2i; 8l+4+i) : i = 0, \dots, l-1$

- $(2l-2i, 2l+1-2i; 2l+4+4i) : i = 0, \ldots, l-1$
- $(6l-2-4i, 6l-1-4i; 6l+5+2i) : i = 0, \ldots, l-1$
- $(1, 6l+2; 6l+3)$. $\square$

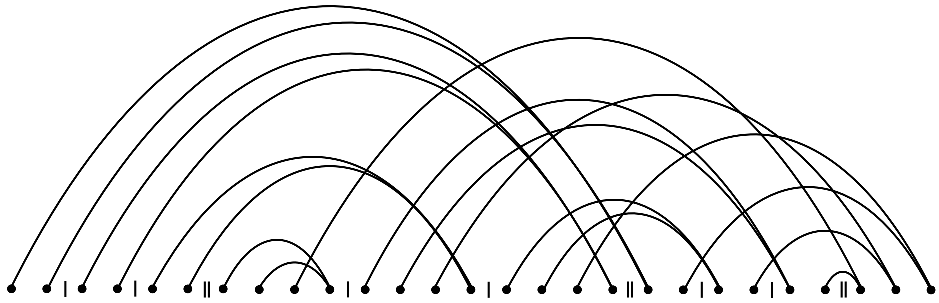

Figure 3/1: V-Skolem sequence of order 9 ($l = 3$)

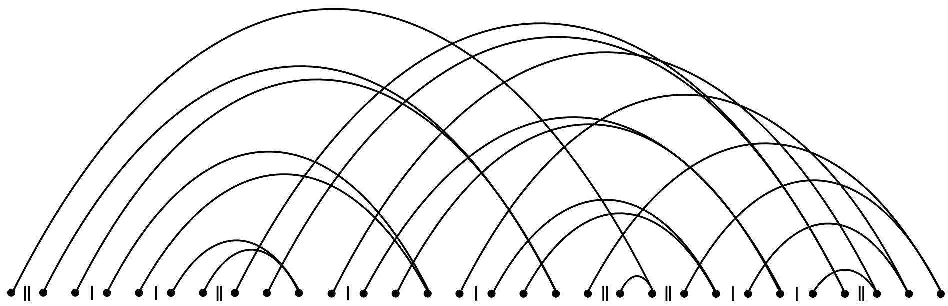

Figure 3/2: V-Skolem sequence of order 10 ($l = 3$)

Now we are ready to prove the main theorem of this section:

Theorem 5. *kC_4 is simply sequentially additive if and only if $k \equiv 0$ or $1 \mod 3$.*

Proof. The only-if part follows directly from Observation 1.

Let $k \equiv 0$ or $1 \mod 3$. By Theorem 4, there exists an $(a_i, b_i, c_i)_{i=1,\ldots,k}$ V-Skolem sequence of order k. Label the vertices of the ith 4-cycle of kC_4 by $5k+1-c_i$, $c_i - a_i$, $8k+1-c_i$ and $c_i - b_i$. We leave the reader to check that we indeed defined an SSA labeling. $\square$

3. Large cycles

In this section we do not distinguish vertices from their labels.

We saw in the proof of Theorem 3 that using Skolem sequences kC_3 can be SSA labeled when $k \equiv 0$ or $1 \mod 4$: the labeling of the ith 3-cycle corresponds to the ith pair of a Skolem sequence S on $\{k+1, \ldots, 3k\}$. Now we consider the k largest labels $(1', \ldots, k')$: i' is assigned as an edge label of the ith 3-cycle. We want to 'glue' some 3-cycles together without breaking the SSA labeling by moving these edges. First we delete these edges and we get an SSA labeled graph (kP_2), then we want to add back k edges so that we get a new 2-regular graph and the labels of the new edges (determined by vertex labels) are $\{1', \ldots, k'\}$. In other words, we search for a matching on vertex set $L_S \cup R'_S$ such that its edge labels are $\{1', \ldots, k'\}$. Let b' be an arbitrary vertex from R'_S. The pair of this vertex, a, must be contained in L_S and it must be lower than b (otherwise the edge label would exceed $6k$). Since

$a + b' = (b - a)'$ we in fact search for a Skolem sequence T on $\{k+1, \dots, 3k\}$ such that $L_T = L_S$ (and $R_T = R_S$). By choosing $T = S$ we get back Theorem 3.

If S, T are Skolem sequences of order k such that $L_S = L_T$ and $R_S = R_T$, we say that (S, T) is a *double Skolem sequence* of order k. The *graph of* (S, T) is an undirected graph on vertex set $\{1, \dots, 2k\}$ whose edge set is the union of perfect matchings defined by S and T. The components of this graph are cycles of even length and L, R define a 2-coloring ($L := L_S = L_T$, $R := R_S = R_T$).

We summarize the above discussion in the following theorem:

Theorem 6. *If $C_{2k_1} \cup \dots \cup C_{2k_m}$ is isomorphic to the graph of a double Skolem sequence, then $C_{3k_1} \cup \dots \cup C_{3k_m}$ is simply sequentially additive.*

Proof. Let (S, T) denote the double Skolem sequence (of order $k = k_1 + \dots + k_m$) in question. We can assume that S and T partition the set $\{k+1, \dots, 3k\}$. Let G denote the graph of (S, T), it is given that G is isomorphic to $C_{2k_1} \cup \dots \cup C_{2k_m}$.

We insert one new vertex in each edge of S (we subdivide the edges of S), then we get a graph G' isomorphic to $C_{3k_1} \cup \dots \cup C_{3k_m}$ (because the new vertices are placed on a perfect matching of each component of G). We label the vertices of G' as follows (see Fig. 4, the vertices of L and R are denoted by $\bullet$ and $\circ$, respectively):

$$l(v) = \begin{cases} v, & \text{if } v \in L \subset V(G), \\ v', & \text{if } v \in R \subset V(G), \\ i, & \text{if } v \text{ is the subdividing vertex that corresponds to the } i\text{th pair of } S \end{cases}$$

So the set of vertex labels is $\{1, \dots, k\} \cup L \cup R'$. The set of (induced) edge labels is $\{1', \dots, k'\}$ on T, and the appearing labels on the subdivided edges are $L' \cup R$, so l is an SSA labeling.

If we swap the roles of S and T, we get a second SSA labeling of G' (if $S \neq T$). $\square$

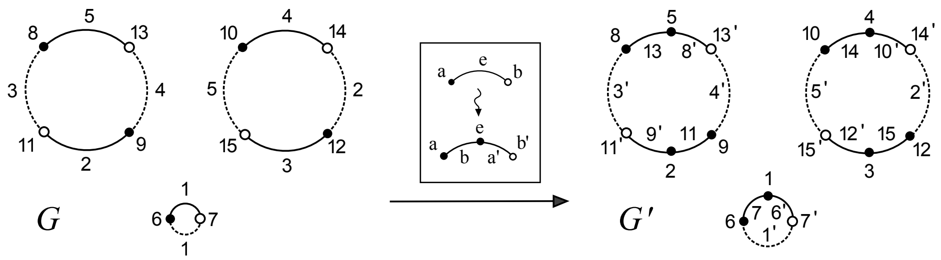

Figure 4

The main benefit of Theorem 6 is that double Skolem sequences can be visualized. We draw the Skolem sequences in the way defined in Section 2, but one of them is represented by 'upper' edges and the other one is represented by 'lower' edges (see Fig. 5). So the drawing of a double Skolem sequence has the following properties:

(1) Both the set of upper edge-lengths and the set of lower edge-lengths are $\{1, \dots, k\}$.
(2) Each vertex v has 2 neighbours, one of them is joined by an upper edge, the other one is joined by a lower edge and
(3) either both are greater than v (when $v \in L$) or both are less than v (when $v \in R$).

Conversely, it is easy to check that each drawing on $\{1, \ldots, 2k\}$ that has these properties defines a double Skolem sequence of order k.

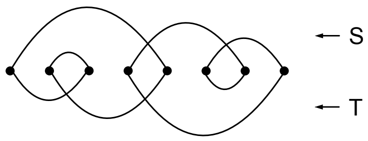

Figure 5: A double Skolem sequence of order 4

In the remaining part of this section we show some applications of Theorem 6.

Theorem 7. *$C_{6k} \cup C_{6k}$ is simply sequentially additive for all $k \in \mathbb{N}$.*

Proof. In view of Theorem 6 all we have to do is to find a double Skolem sequence whose graph is $C_{4k} \cup C_{4k}$.

Fig. 5 shows a proper sequence for the case $k = 1$. It has a symmetry: one cycle component has edge-lengths $\{1, \ldots, 4k\}$ and the other component is obtained from it by a reflection in point $4k + \frac{1}{2}$. We will generalize this by giving a drawing of C_{4k} on the vertex set $V = \{1, 2, \ldots, 2k\} \cup \{2k+1, 2k+3, \ldots, 6k-1\}$ such that the edge-lengths are $\{1, \ldots, 4k\}$ and conditions (2)-(3) are satisfied on this component. Clearly, such a drawing and its mirror image define a suitable double Skolem sequence. (It is important to note that $V \cup^* V' = \{1, \ldots, 8k\}$, where $s' = 8k + 1 - s$.)

Case 1: k is even ($k = 2l$).

A suitable drawing of C_{4k} is shown in Fig. 6/1, it is built up from 3 paths:

- $\mathcal{P}_1 : a_0 \to b_0 \to a_1 \to b_1 \to \cdots \to a_{3l-1} \to b_{3l-1} \to a_{3l}$, where

$$a_i = 4q_i + 2^{r_i}, \text{ where } q_i \in \mathbb{Z},\ r_i \in \{0, 1, 2\} : 3q_i + r_i = i \quad (i = 0, \ldots, 3l),$$

$$b_i = 4i + 3 \quad (i = 0, \ldots, 3l - 1).$$

(Vertex set: $\{1, 2, \ldots, 4l+1\} \cup \{4l+3, 4l+7, \ldots, 12l-1\}$, startpoint: 1, endpoint: $4l + 1$, edge-lengths: $\{1, 2, \ldots, 8l\} \setminus \{4, 8, \ldots, 8l\}$)

- $\mathcal{P}_2$ is a 'spiral':
$(4l + 1) \to (12 - 3) \to (4l + 5) \to (12l - 7) \to \cdots \to (8l - 3) \to (8l + 1)$
(Edge-lengths: $\{4, 8, \ldots, 8l - 4\}$)
- $\mathcal{P}_3$ is an edge of length $8l$: $(8l + 1) \to 1$.

Case 2: k is odd ($k = 2l + 1$).

The construction is similar (Fig. 6/2), we use the same notations as above:

- $\mathcal{P}_1 : a_0 \to b_0 \to a_1 \to b_1 \to \cdots \to a_{3l} \to b_{3l} \to a_{3l+1}$,
- $\mathcal{P}_2 : (4l + 2) \to (12l + 5)$,
- $\mathcal{P}_3 : (12l + 5) \to (4l + 5) \to (12l + 1) \to (4l + 9) \to \cdots \to (8l + 1) \to (8l + 5)$ and
- $\mathcal{P}_4 : (8l + 5) \to 1$.

(In the case when $l = 0$, $\mathcal{P}_3$ has no edges.) $\square$

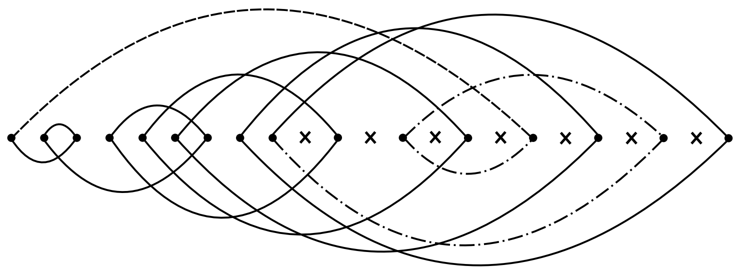

Figure 6/1: A suitable drawing of C_{16} $(l = 2)$

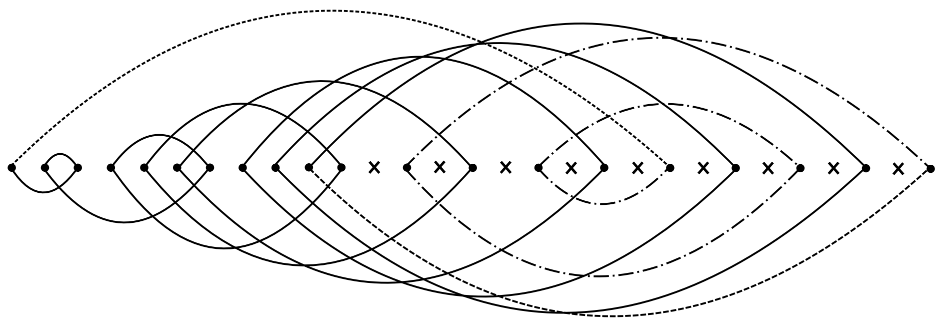

Figure 6/2: A suitable drawing of C_{20} $(l = 2)$

A much simpler drawing gives the following theorem:

Theorem 8. *$C_{6k} \cup C_{6k} \cup C_3$ is simply sequentially additive for all $k \in \mathbb{N}$.*

Proof. A suitable drawing of $C_{4k} \cup C_{4k} \cup C_2$ is shown in Fig. 7. It has a symmetry: one C_{4k} component has edge-lengths $\{1, \dots, 4k\}$ and vertex set $\{1, \dots, 4k+1\} \setminus \{3k+1\}$, the other C_{4k} component is obtained from it by a translation by $4k+1$ (and a reflection in real line). Finally we join the two missed vertices by two edges of length $4k+1$ to get the C_2 component.

The said drawing of C_{4k} is a spiral starting from $2k+1$ that jumps over the point $3k+1$ (and edge-length $2k$), and the endpoints are joined by an edge of length $2k$: $(2k+1) \to 2k \to (2k+2) \to (2k-1) \to \dots \to 3k \to (k+1) \to (3k+2) \to k \to (3k+3) \to (k-1) \to \dots \to (4k+1) \to 1$ and the last edge is $1 \to (2k+1)$. $\square$

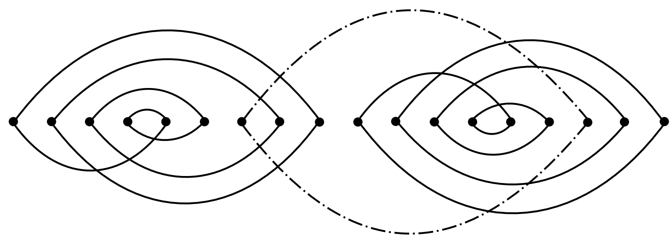

Figure 7: A suitable drawing of $C_8 \cup C_8 \cup C_2$ $(k = 2)$

Both double Skolem sequences we saw in the proofs of Theorems 7-8 had some symmetries. Due to the following lemma in these cases we can decide whether we use the labeling algorithm of Theorem 6 on a 'symmetric' pair of C_{2m} components to get two partially SSA labeled cycles of length $3m$ or we 'glue' them together to get one partially SSA labeled cycle of length $6m$ *with the same set of assigned labels.*

Lemma 9. *C and $\tilde{C}$ are components of a double Skolem sequence on vertex set $\{r+1, \dots, 3r\}$ such that*

(i) *Either $\tilde{C}$ can be obtained from C by reflection*
(ii) *or $\tilde{C}$ can be obtained from C by translation.*

(A reflection is defined on the vertex set by $x \mapsto s - x$ for some $s \in \mathbb{R}$ constant, a translation is defined by $x \mapsto t + x$ for some $t \in \mathbb{R}$ constant; these operations are extended to graphs (to edges) in a natural way.)

Then we can assign the labels $S = V \cup^* V' \cup^* D \cup^* D'$ to the vertices and edges of C_{6m} bijectively so that each edge gets the sum of the labels of its endpoints, where $2m$ is the size of C, $V = V(C) \cup^* V(\tilde{C})$ and D denotes the set of edge-lengths in C (or in $\tilde{C}$). (Based on $C \cup \tilde{C}$, the algorithm of Theorem 6 also assigns the label set S to $C_{3m} \cup C_{3m}$.)

Proof. We use the notations of Fig. 8, where k_i and l_i denote edge-lengths, and the vertices of L and R are denoted by $\bullet$ and $\circ$, respectively. We label the vertices of C_{6m} as follows (in order of the cyclic sequence of vertices on the cycle):

(i) $a_1, k_1, \tilde{a}'_1; \tilde{b}_1, l_1, b'_1; a_2, k_2, \tilde{a}'_2; \tilde{b}_2, l_2, b'_2; \ldots; a_m, k_m, \tilde{a}'_m; \tilde{b}_m, l_m, b'_m$
(ii) $a_1, k_1, \tilde{a}_1; \tilde{b}'_1, l_1, b'_1; a_2, k_2, \tilde{a}_2; \tilde{b}'_2, l_2, b'_2; \ldots; a_m, k_m, \tilde{a}_m; \tilde{b}'_m, l_m, b'_m$

We leave the reader to check that the set of assigned labels is S. (The edge labels are determined by vertex labels.) $\square$

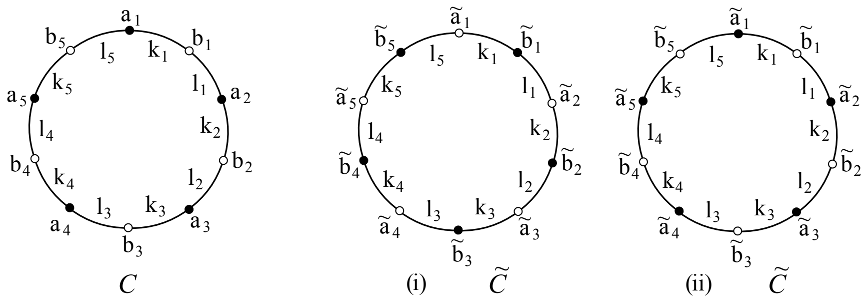

Figure 8: Notations $(m = 5)$

The following theorem is an immediate corollary of Lemma 9 and proofs of Theorems 7-8.

Theorem 10.

(a) C_{12k} *is simply sequentially additive for all* $k \in \mathbb{N}$.
(b) $C_{12k} \cup C_3$ *is simply sequentially additive for all* $k \in \mathbb{N}$.

Theorem 10/(a) is a special case of the result of Bange et al. [2]. We note that our C_{12k}-labeling differs from their labeling.

4. An application: Ordered graceful labelings

A *graceful labeling (or drawing)* of a tree $T(V, E)$ is a bijection (or drawing) from V to $\{0, \ldots, |E|\}$ such that the set of induced edge-lengths is $\{1, \ldots, |E|\}$ (that is equivalent to the fact that there are no edges of the same length). A tree that admits a graceful labelling is called *graceful*. A long-standing conjecture of Rosa [9] states that every tree is graceful.

Cahit [3] defined a stronger labeling, a graceful labeling of a tree is called *ordered graceful* (or *solid graceful*, *gracious*) if, when the edges of the tree are oriented from the endvertex with larger label to the endvertex with smaller label, then every vertex has either indegree 0 or outdegree 0. In the terminology of drawings (see Fig. 9), the extra condition is an analogue of property (3) of double Skolem sequences. Cahit conjectures that every tree is ordered graceful, i.e. every tree has an ordered graceful labeling.

It is known [5] that symmetrical trees (ie., rooted trees in which every level contains vertices of the same degree) are graceful, but the ordered graceful tree conjecture has not been verified for this class of trees. In 1994 Cahit asked [4] whether spiders are ordered graceful. (The *spider* with n legs is the subdivision of S_n, the star with n edges, where each edge of the star is replaced by a path of length 2.) In fact this question had been already answered by Bange, Barkauskas and Slater [2] in 1983 in the language of SSA labelings (C_{3k} and C_{3k+1} are SSA for all k). In 2001 Nowakowski and Whitehead [8] proved that there exists exponentially many graceful labelings of spiders that we could rephrase in Theorem 3. In this section we show the natural correspondence between SSA labelings of 2-regular graphs and certain ordered graceful labelings of spiders.

Now consider an arbitrary ordered graceful labeling g of a spider with n legs such that the center gets the label 0. The labels on the ith leg are a_i (assigned to the leaf) and b_i (assigned to the 'internal' vertex with degree 2). We know that $a_i < b_i$ $(i = 1, \dots, n)$ so the edge-lengths on this leg are b_i and $d_i := b_i - a_i$. Since g is a graceful labeling, $|A| = |B| = n$, $A \cup^* B = \{1, \dots, 2n\}$ and $D = A$, using the notations $A = \{a_i : i = 1, \dots, n\}$, $B = \{b_i : i = 1, \dots, n\}$ and $D = \{d_i : i = 1, \dots, n\}$. Hence $\phi_g = \phi : a_i \mapsto d_i$ $(i = 1, \dots, n)$ is a permutation of A such that $\{a + \phi(a) : a \in A\} = B = A^c := \{1, \dots, 2n\}\setminus A$.

We can identify ϕ with its graph, that is a directed graph on vertex set A such that uv is an edge if and only if $\phi(u) = v$. It is well known, that this is a one–one and onto correspondence between permutations of A and graphs on A that consist of directed cycle components. The condition on ϕ means that the graph of ϕ can be interpreted as an SSA labeling. Conversely, it is straightforward to check that the following theorem holds:

Theorem 11. *Let be given a 2-regular graph G on n vertices with an SSA labeling. This labeling determines 2^K ordered graceful labeling of the n-leg spider S such that the center gets label 0, where K is the number of non-loop cycle components in G. Every such ordered graceful labeling of S can be obtained in this way, and distinct graphs or distinct SSA labelings determine distinct labelings.*

(There are 2^K ways to orient G to get the graph of a permutation.)

Corollary 12. *[8] Spiders are ordered graceful trees.*

If the number of legs has the form $3k$ or $3k+1$, the statement follows from the fact that kC_3 and $kC_3 \cup C_1$ (or C_{3k} and C_{3k+1}) are simply sequentially additive for all k and from Theorem 11. Our results in Sections 2-3 also give new ordered graceful labelings of spiders. All these labelings assign 0 to the root, so the remaining case ($3k + 2$ legs) comes from the following well known observation (see Fig. 9):

Observation 13. *Let be given an ordered graceful drawing g of a tree T. Let T' be the tree obtained by glueing a star S_k to T so that a leaf of S_k is identified with vertex $g^{-1}(0)$ of T. Then T' is ordered graceful.*

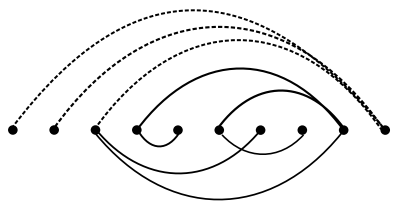

Figure 9

Let $T_d(n)$ denote the symmetric tree that has 2 levels (root, sons and grandsons), the root has n sons and all the sons have d further sons. In this notation $T_1(n)$ is the n-leg spider.

Theorem 14. *[7] $T_2(k)$ is ordered graceful for all k.*

Sketch proof. If $k \equiv 0$ or $1 \mod 4$, we label the root by 0. Then we can partition $\{1, \dots, 3k\}$ into triplets of the form $\{a_i, b_i, a_i + b_i\}_{i=1,\dots,k}$ using Skolem sequences (see Theorem 3). So if we label the ith son of the root by $a_i + b_i$ and its sons by a_i and b_i, we get an ordered graceful labeling.

The case $k = 4l + 2$ follows from Observation 13 and the remaining case, $k = 4l + 3$, can be proved in a similar way. □

Finally we investigate $T_4(k)$. The idea is based on the proof of Theorem 14. If we can partition $\{1, \dots, 5k\}$ into $\{a_i, b_i, c_i, d_i, e_i\}_{i=1,\dots,k}$, where $a_i + b_i = c_i + d_i = e_i$ ($i = 1, \dots, k$), then we can get an ordered graceful labeling of $T_4(k)$: Label the root by 0, its ith son by e_i and its sons by a_i, b_i, c_i and d_i. (Then the lengths of edges incident to the ith son are a_i, b_i, c_i, d_i and e_i.) Such a partitioning can be constructed using V-Skolem sequences:

Corollary 15. *$\{1, \dots, 5k\}$ can be partitioned into $\{a_i, b_i, c_i, d_i, e_i\}_{i=1,\dots,k}$ so that $a_i + b_i = c_i + d_i = e_i$ ($i = 1, \dots, k$), if and only if $k \equiv 0$ or $1 \mod 3$.*

Proof. If such a partitioning exists, then $1 + \cdots + 5k$ is divisible by 3, so the divisibility condition follows.

By Theorem 4, a V-Skolem sequence of order k exists, if $k \equiv 0$ or $1 \mod 3$: $(p_i, q_i, r_i)_{i=1,\dots,k}$. Then $\{r_i - p_i, p_i + 2k, r_i - q_i, q_i + 2k, r_i + 2k\}_{i=1,\dots,k}$ partitions $\{1, \dots, 5k\}$ as required.

The following theorem follows by combining the preceding remark and Observation 13:

Theorem 16. *$T_4(k)$ is ordered graceful for all k.*

5. Further problems

Motivated by Theorem 6, it is interesting to compute Skolem sequences whose set of left endpoints is L and set of right endpoint is R (where L and R are fixed sets, $|L| = |R| = k$, $L \cup^* R = \{1, \dots, 2k\}$ for some order k, $k \equiv 0$ or $1 \mod 4$). Every pair of such Skolem sequences determines a double Skolem sequence, which leads to a simply sequentially additive 2-regular graph due to Theorem 6. We denote the number of Skolem sequences in question by $\#S(L, R)$.

The following conjecture has been verified by computer up to order 13:

Conjecture 17. *$\#S(L, R)$ is even for all L–R partitions, if the order is greater than 1.*

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