# Star subdivisions and connected even factors in the square of a graph 

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#### Abstract

. For any positive integer $s$, a $[2,2 s]$-factor in a graph $G$ is a connected even factor with maximum degree at most $2 s$. We prove that if every induced $S\left(K_{1,2 s+1}\right)$ in a graph $G$ has at least 3 edges in a block of degree at most two, then $G^{2}$ has a $[2,2 s]$-factor. This extends the results of Hendry and Vogler and of Abderrezzak et al.


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## 1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and we consider only finite undirected simple graphs, unless otherwise stated.

Let $G=(V, E)$ be a graph with vertex set $V$ and edge set $E$. Let $\alpha(G)$ denote the independence number of $G$, i.e., the cardinality of a largest independence set in $G$. For any vertex $x$ of $G$, let $d_{G}(x)$ denote the degree of $x$ in $G, N_{G}(x)$ the set of all neighbors of $x$ in $G, N_{G}[x]=N_{G}(x) \cup\{x\}$. The square of a graph $G$, denoted by $G^{2}$, is the graph with $V\left(G^{2}\right)=V(G)$ in which two vertices are adjacent if their distance in $G$ is at most two. Thus $G \subseteq G^{2}$.

[^0]For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. For a positive integer $s$, the graph $S\left(K_{1,2 s+1}\right)$ is obtained from the complete bipartite graph $K_{1,2 s+1}$ by subdividing each edge once. The graph $G$ is said to be $S\left(K_{1,2 s+1}\right)$-free if it does not contain any induced copy of $S\left(K_{1,2 s+1}\right)$.

A connected graph that has no cut vertices is called a block. A block of a graph $G$ is a subgraph of $G$ that is a block and is maximal with respect to this property. The degree of a block $B$ in a graph $G$, denoted by $d(B)$, is the number of cut vertices of $G$ belonging to $V(B)$.

A factor in a graph $G$ is a spanning subgraph of $G$. A connected even factor in $G$ is a connected factor in $G$ with all vertices of even degree. A $[2,2 s]$-factor in $G$ is a connected even factor in $G$ in which degree of every vertex is at most $2 s$. A graph is hamiltonian if it has a spanning cycle. In other word, a graph is hamiltonian if and only if it has a [2, 2]-factor.

The following result concerns the existence of a $[2,2]$-factor in the square of a 2 connected graph.

Theorem A [3]. Let $G$ be a 2-connected graph. Then $G^{2}$ is hamiltonian.
Gould and Jacobson in [4] conjectured that for the hamiltonicity of $G^{2}$, the connectivity condition can be relaxed for $S\left(K_{1,3}\right)$-free graphs. Their conjecture was proved by Hendry and Vogler in [5].

Theorem B [5]. Let $G$ be a connected $S\left(K_{1,3}\right)$-free graph. Then $G^{2}$ is hamiltonian, i.e., has a $[2,2]$-factor.

Moreover, Abderrezzak, Flandrin and Ryjáček in [1] proved the following result in which graphs may contain an induced $S\left(K_{1,3}\right)$ of a special type.

Theorem C [1]. Let $G$ be a connected graph such that every induced $S\left(K_{1,3}\right)$ in $G$ has at least three edges in a block of degree at most two. Then $G^{2}$ is hamiltonian, i.e., has a [2, 2]-factor.

It is a natural question if there exists a $[2,2 s]$-factor in the square of a graph if one replaces $S\left(K_{1,3}\right)$ by $S\left(K_{1,2 s+1}\right)$ in Theorems B and C. In this paper, we will give a positive answer to this question; we will extend Theorems $B$ and $C$ as follows.

Theorem 1. Let $G$ be a connected $S\left(K_{1,2 s+1}\right)$-free graph of order at least three and $s$ a positive integer. Then $G^{2}$ has a $[2,2 s]$-factor.

Since the square of an $S\left(K_{1,2 s+1}\right)$ itself has no [2,2s]-factor, Theorem 1 is the best possible in a sense.

Theorem 2. Let $s$ be a positive integer and $G$ be a connected graph such that every induced $S\left(K_{1,2 s+1}\right)$ has at least three edges in a block of degree at most two. Then $G^{2}$ has a $[2,2 s]$-factor.

Note that Theorem 2 is a strengthening of Theorem [1, but we state Theorem 1 separately because it will be used in the proof of Theorem 2.

## 2 Preliminaries and auxiliary results

As noted in Section 1, for graph-theoretic notation not explained in this paper, we refer the reader to [2].

A graph $G$ is even if every vertex of $G$ has even degree. In the subsequent sections, we frequently take the symmetric difference of two subgraphs of a graph. Let $H, H^{\prime}$ be subgraphs of a graph $G$. The graph $H \Delta H^{\prime}$ has vertex set $V(H) \cup V\left(H^{\prime}\right)$ and its edge set is the symmetric difference of $E(H)$ and $E\left(H^{\prime}\right)$. Note that if $H$ and $H^{\prime}$ are both even graphs, then $H \Delta H^{\prime}$ is also an even graph.

A trail between vertices $u_{0}$ and $u_{r}$ is a finite sequence $T=u_{0} e_{1} u_{1} e_{2} u_{2} \cdots e_{r} u_{r}$, whose terms are alternately vertices and edges, with $e_{i}=u_{i-1} u_{i}, 1 \leq i \leq r$, where the edges are distinct. A trail $T$ is closed if $u_{0}=u_{r}$, and it is spanning if $V(T)=V(G)$. An $s$-trail between $u_{0}$ and $u_{r}$ is a trail starting at $u_{0}$, ending at $u_{r}$ and in which every vertex is visited at most $s$ times. In other words, a [2,2s]-factor in a graph $G$ can be viewed as a spanning closed $s$-trail in $G$ and vice versa. We define the degree of a vertex $x$ in an $s$-trail as the number of edges incident with $x$ in the corresponding $[2,2 s]$-factor.

We use the following fact (see [6], Corollary 2.3.1 for a proof).
Theorem D [6]. Let $k \geq 2$ be an integer and $G$ a $k$-connected graph. If $\alpha(G)>k$ then $V(G)$ can be covered with $\alpha(G)-k$ disjoint paths.

From the proof of this Theorem it follows that the statement is true without the restrictions on $k$, in particular for $k=0$.

Corollary 3. Let $G$ be a graph. Then there are at most $\alpha(G)$ disjoint paths covering $V(G)$.

Let $G_{1}, G_{2}$ be graphs such that $V\left(G_{1}\right) \cap V\left(G_{2}\right)=\{x\}$. The symbol $G=G_{1} x G_{2}$ denotes a graph $G$ with $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$.

Given a subgraph $K$ of a graph $H$, we define $\partial_{H}(K)$ as the set of all edges of $H$ with exactly one endvertex in $V(K)$. Thus $\partial_{H}(K)$ is a (not necessarily minimal) edge-cut.

Lemma 4. Let $H$ be a connected graph and $P=x y z$ a path of length two such that $V(H) \cap V(P)=\{x\}$. If $(H x P)^{2}$ has a $[2,2 s]$-factor, then one of the following holds:
(a) $H^{2}$ contains a spanning closed $s$-trail $T$ such that the degree of $x$ in $T$ is at most $2 s-2$, or
(b) $H^{2}$ contains a spanning $s$-trail $T$ between $x$ and some $x^{\prime} \in N_{H}(x)$.

Proof. Let $F$ be a $[2,2 s]$-factor of $(H x P)^{2}$ and let $K_{0}, \ldots, K_{\ell}$ be all the components of $F \backslash\{y, z\}$, where $x \in V\left(K_{0}\right)$. Furthermore, define $W=N_{F}(y) \backslash\{z\}$ and $W_{i}=$ $W \cap V\left(K_{i}\right)(i=0, \ldots, \ell)$. Observe that each $W_{i}$ is nonempty. Clearly, the induced subgraph $Q$ of $H^{2}$ on $W \cup\{x\}$ is complete.

Since $F$ covers $z$, it includes the edges $y z$ and $x z$. For $0 \leq i \leq \ell$, every edge in $\partial_{F}\left(K_{i}\right)$ is incident with $y$, except for the edge $x z \in \partial_{F}\left(K_{0}\right)$. Since

$$
\partial_{F}\left(K_{i}\right)=\partial_{H}\left(K_{i}\right) \cap E(F)
$$

and the intersection of any edge-cut with an eulerian subgraph has even cardinality, we conclude that for $0 \leq i \leq \ell$,

$$
\left|W_{i}\right| \text { is odd if and only if } i=0 \text {. }
$$

If $w \in W_{i}$ and $w \neq x$, then the degree of $w$ in $K_{i}$ is odd and does not exceed $2 s-1$. The same is true for $w=x$ provided that $x \notin W$, since then $x z$ is the only edge of $\partial_{F}\left(K_{0}\right)$ incident with $x$. On the other hand, if $x \in W$, then both $x z$ and $x y$ have this property, so the degree of $x$ in $K_{0}$ is even and does not exceed $2 s-2$.

For each $i, 0 \leq i \leq \ell$, choose a matching $M_{i}$ that covers all except one or two vertices of $W_{i}$ (one if $i=0$, two otherwise) and uses as few edges as possible from $F$. We argue that the symmetric difference $K_{i} \Delta M_{i}$ is connected. We may assume that $M_{i}$ uses at least one edge of $F$, otherwise there is nothing to prove. For a fixed $i$, let $X \subseteq W_{i}$ be the set consisting of vertices incident with edges in $E\left(M_{i}\right) \cap E(F)$, together with the vertices of $W_{i}$ left uncovered by $M_{i}$. By the choice of $M_{i}, K_{i}[X]$ must be complete and $|X| \geq 3$. All the edges of $K_{i}$ that are removed as a result of taking the symmetric difference are edges of $K_{i}[X]$. Since any graph obtained by removing a matching from a complete graph on at least 3 vertices is connected, the claim follows.

Observe that for $i \geq 1$, each $K_{i} \Delta M_{i}$ contains exactly two vertices of odd degree (and the degree does not exceed $2 s-1$ ). The same is true for $i=0$ unless $x \in W$ and $x$
is not incident with $M_{0}$, in which case $K_{0} \triangle M_{0}$ is eulerian and the degree of $x$ in this graph is at most $2 s-2$. It follows that if $\ell=0$, then we can set $T:=K_{0} \triangle M_{0}$ and we are done ( $T$ satisfies condition (a) if $x \in W \backslash V\left(M_{0}\right)$ and condition (b) otherwise).

If $\ell \geq 1$, then let $u_{0}$ be the vertex of $W_{0} \backslash V\left(M_{0}\right)$, and for $i \geq 1$, let $W_{i} \backslash$ $V\left(M_{i}\right)=\left\{u_{i}, v_{i}\right\}$. Taking the union of all the graphs $K_{i} \triangle M_{i}$ and adding the edges $u_{0} v_{1}, u_{1} v_{2}, \ldots, u_{\ell-1} v_{\ell}$, we obtain a connected graph $T$ in which the only vertices of odd degree are $x$ and $u_{\ell}$, and which satisfies condition (b) in the lemma.

Using a similar argument as in the proof of Lemma 4, one can prove the following.
Lemma 5. Let $H$ be a connected graph and $P=x y$ an edge such that $V(H) \cap V(P)=$ $\{x\}$. If $(H x P)^{2}$ has a $[2,2 s]$-factor, then $H^{2}$ has a spanning $s$-trail $T$ between $x^{\prime} \in N_{H}[x]$ and some vertex $x^{\prime \prime} \in N_{H}(x)$.

The following theorem will be used in the proof of Theorem 2
Theorem E [3]. Let $y$ and $z$ be arbitrarily chosen vertices of a 2-connected graph $G$. Then $G^{2}$ has a hamiltonian cycle $C$ such that the edges of $C$ incident with $y$ are in $G$ and at least one of the edges of $C$ incident with $z$ is in $G$. If $y$ and $z$ are adjacent in $G$, then these are three different edges.

## 3 Proofs

The purpose of this section is to prove Theorem 2. As mentioned in Section 1, the proof makes use of Theorem 1 which we derive next.

Proof of Theorem 1, This proof is inspired by the proof in [5]. We prove our result by induction on $|V(G)|$. Clearly $G^{2}$ is hamiltonian (hence has a [2, 2]-factor) for graphs with $|V(G)| \leq 6$, since $G$ is $S\left(K_{1,3}\right)$-free. By Theorem A. we may assume that $G$ has cut vertices. If all cut vertices have degree two, then $G$ is a path and hence $G^{2}$ is hamiltonian. So we may assume that there is a cut vertex $u$ such that $d_{G}(u)=d \geq 3$. Since $G$ is connected, we may take a spanning tree $S$ of $G$ such that $S$ contains all edges of $G$ incident with $u$. We label the neighbors of $u$ by $u_{1}, u_{2}, \cdots, u_{d}$ in such a way that $d_{G}\left(u_{i}\right) \geq 2$ for $1 \leq i \leq m$ and $d_{G}\left(u_{i}\right)=1$ for $m+1 \leq i \leq d$. For $i \leq m$, let $G_{i}$ be the subgraph of $G$ induced by the vertices in the component of the forest $S-u$ containing $u_{i}$; we fix a neighbour $u_{i}^{\prime}$ of $u$ that is not contained in the same component of $G-u$ as $u_{i}$ (note that there must be such a vertex since $u$ is a cut vertex of $G$ ), and let $H_{i}=G\left[V\left(G_{i}\right) \cup\left\{u, u_{i}^{\prime}\right\}\right]$. Then $H_{i}$ is a proper $S\left(K_{1,2 s+1}\right)$-free subgraph of $G$ since $H_{i}$ is
an induced subgraph of $G$ and $d_{G}(u) \geq 3$. Note that $H_{i}$ is connected. By the inductive hypothesis, $H_{i}^{2}$ has a $[2,2 s]$-factor. Note that $d_{H_{i}^{2}}\left(u_{i}^{\prime}\right)=2$.

By Lemma 4 it follows that at least one of the following facts holds.
(a) there exists a spanning closed $s$-trail $T_{i}$ in $G_{i}^{2}$ such that $d_{T_{i}}\left(u_{i}\right) \leq 2 s-2$;
(b) there exists a spanning $s$-trail $T_{i}$ in $G_{i}^{2}$ between $u_{i}$ and some $z_{i} \in N_{G_{i}}\left(u_{i}\right)$.

Without loss of generality we may assume that $\left\{u_{1}, u_{2}, \ldots, u_{m^{\prime}}\right\} \subseteq\left\{u_{1}, u_{2}, \ldots, u_{m}\right\}$ is the set of all vertices $u_{i}$ such that $G_{i}$ has an $s$-trail of type (b), for a suitable $m^{\prime} \leq m$. Construct the graph $H$ from $G\left[\left\{u_{1}, u_{2}, \ldots, u_{m^{\prime}}, z_{1}, z_{2}, \ldots, z_{m^{\prime}}\right\}\right]$ by contracting edges $u_{i} z_{i}$ to a vertex $w_{i}$ for $i=1, \ldots, m^{\prime}$. Since $G$ is $S\left(K_{1,2 s+1}\right)$-free, $\alpha(H) \leq 2 s$. By Corollary 3, there are $\ell \leq \alpha(H)$ vertex-disjoint paths $P_{1}, P_{2}, \ldots, P_{\ell}$ covering $V(H)$. Without loss of generality, we may assume that $P_{i}=w_{s_{i-1}+1} w_{s_{i-1}+2} \ldots w_{s_{i}}$, for $i=1, \ldots, \ell$ (where $s_{0}=0$ and $s_{\ell}=m^{\prime}$ ). Since we contracted edges $u_{j} z_{j}$ to vertices $w_{j}$, both $u_{j}$ and $z_{j}$ have a neighbor in $\left\{u_{j+1}, z_{j+1}\right\}$ in $G^{2}$ for $i=1, \ldots, \ell$, and $j=s_{i-1}+1, \ldots, s_{i}-1$. Hence from the paths $P_{i}(i=1, \ldots, \ell)$ and $s$-trails $T_{j}\left(i=1, \ldots, m^{\prime}\right)$ we can obtain the following $s$-trails $F_{i}$ in $G^{2}$ :

- for a trivial (one-vertex) path $P_{i}, F_{i}=T_{i}$,
- for a nontrivial path $P_{i}$, we construct $F_{i}$ by joining the trails $T_{s_{i-1}+1}, \ldots, T_{s_{i}}$ with the edges $x_{j} x_{j+1}$, where $x_{j} \in\left\{u_{j}, z_{j}\right\}$ and $x_{j+1} \in\left\{u_{j+1}, z_{j+1}\right\}$ with respect to $P_{i}$. Clearly $d_{F_{i}}\left(u_{s_{i-1}+1}\right)<2 s, d_{F_{i}}\left(x_{s_{i}}\right)<2 s$ and $F_{i}$ spans all the vertices of $G_{s_{i-1}+1} \cup \cdots \cup G_{s_{i}}$.

Note that the number of $s$-trails $F_{i}$ is $\ell \leq 2 s$.
Let $T=u_{m^{\prime}+1} T_{m^{\prime}+1} u_{m^{\prime}+1} u_{m^{\prime}+2} T_{m^{\prime}+2} u_{m^{\prime}+2} \ldots u_{m} T_{m} u_{m} u_{m+1} \ldots u_{d}$ be an $s$-trail containing all vertices of $G_{m^{\prime}+1} \cup \cdots \cup G_{m}$ and all neighbours of $u$ of degree one in $G$. We set $F^{\prime}=u_{1} F_{1} x_{s_{1}} u x_{s_{2}} F_{2} u_{s_{1}+1} u_{s_{2}+1} F_{3} \ldots x_{s_{\ell}} F_{\ell} u_{s_{\ell-1}+1} u_{m^{\prime}+1}$ for even $\ell$ and $F^{\prime}=$ $u_{1} F_{1} x_{s_{1}} u x_{s_{2}} F_{2} u_{s_{1}+1} u_{s_{2}+1} F_{3} \ldots u_{s_{\ell-1}+1} F_{\ell} x_{s_{\ell}} u u_{m^{\prime}+1}$ for odd $\ell$. In both cases, $F^{\prime}$ is an $s$-trail containing all vertices of $G_{1} \cup \cdots \cup G_{m^{\prime}}$. Finally, we construct a trail $F=$ $u_{1} F^{\prime} u_{m^{\prime}+1} T u_{d} u_{1}$. Clearly, $d_{F}(u)=\ell \leq 2 s$ and $F$ corresponds to a [2,2s]-factor in $G^{2}$.

Corollary 6. Let $G$ be a simple connected graph with $\Delta(G) \leq 2 s$. Then $G^{2}$ has a [ $2,2 s]$-factor.

Before we present the proof of Theorem 2, we give some additional definitions. Let $x$ be a cut vertex of $G$, and $H^{\prime}$ be a component of $G-x$. Then the subgraph
$H=G\left[V\left(H^{\prime}\right) \cup\{x\}\right]$ is called a branch of $G$ at $x$. Let $F$ be a connected subgraph of $G$ and $x$ some vertex of $F$. Let $P_{i}(x)$ denote a path on $i$ vertices with end vertex $x$. The subgraph $F$ is called to be nontrivial at $x$ if it contains a $P_{3}(x)$ as a proper induced subgraph (i.e., $F$ is trivial at $x$ if $F=P_{3}(x)$ or $\left.V(F) \subseteq N[x]\right)$.

Now we present the proof of Theorem 2.
Proof of Theorem 2. We prove this theorem by contradiction. Suppose that Theorem 2 is not true and choose a graph $G$ in such a way that
(1) $G$ is connected and every induced $S\left(K_{1,2 s+1}\right)$ in $G$ has at least three edges in a block of degree at most two;
(2) $G^{2}$ has no $[2,2 s]$-factor;
(3) $|V(G)|$ is minimized with respect to (1) and (2).

The following fact is necessary for our proof.
Claim 1. Let $x$ be a cut vertex of $G$ and $F_{1}, F_{2}$ two connected subgraphs of $G$ such that $F_{1}, F_{2}$ belong to different branches of $G$ at $x$. Assume that $F_{2}$ is nontrivial at $x$, i.e., $F_{2}$ contains an induced $P_{3}(x)=x y z$ as a proper induced subgraph. Then the graph $G^{\prime}=F_{1} x P_{3}(x)$ also satisfies (1).

Proof of Claim 1. If not, there exists in $G^{\prime}$ some $S\left(K_{1,2 s+1}\right)$ that has no connected part of order at least 4 in a block of degree at most two. But if so, it is the same in $G$, since any $S\left(K_{1,2 s+1}\right)$ in $G^{\prime}$ is also an induced $S\left(K_{1,2 s+1}\right)$ of $G$.

Since in our proof we have assumed that $G^{2}$ has no [2,2s]-factor, we know from Theorem 1 that $G$ contains some $S\left(K_{1,2 s+1}\right)$ as an induced subgraph. By (1), the $S\left(K_{1,2 s+1}\right)$ has at least 3 edges in some block $H$ of $G$ of degree at most 2 . Notice that $|V(H)| \geq 5$.

Case 1: $d(H)=1$. Let $c$ be the cut vertex of $G$ belonging to $H$ and let $R$ be the union of all branches of $G$ at $c$ which intersect $H$ only at $c$.

If $H$ is trivial at $c$, then $V(H)-\{c\}=\left\{b_{1}, b_{2}, \ldots, b_{h}\right\} \subseteq N(c)$. The graph $G^{\prime}=$ $R c\left(c b_{1}\right)$ satisfies condition (1). So by minimality of $G$, the graph $G^{2}$ has a $[2,2 s]$-factor and, by Lemma $5, R^{2}$ has a spanning $s$-trail $T$ between some $c^{\prime} \in N_{R}[c]$ and some $c^{\prime \prime} \in N_{R}(c)$. Let $F=c^{\prime} T c^{\prime \prime} b_{1} \ldots b_{h} c^{\prime}$. It is easy to see that $F$ is a $[2,2 s]$-factor in $G^{2}$, a contradiction.

Hence $H$ is nontrivial at $c$, i.e., it contains a proper induced path $P_{3}(c)=c b_{1} b_{2}$. By Theorem E, $H^{2}$ contains a hamiltonian path $b_{1} P_{H^{2}} c$ connecting $b_{1}$ and $c$. On the other hand the graph $G^{\prime \prime}=R c P_{3}(c)$ is connected and, by Claim 1, $G^{\prime \prime}$ satisfies condition (1).

Since $\left|V\left(G^{\prime \prime}\right)\right|<|V(G)|,\left(G^{\prime \prime}\right)^{2}$ has a $[2,2 s]$-factor and by Lemma 4 , one of the following subcases occur.

If the graph $R^{2}$ has a spanning closed $s$-trail $T^{\prime}$ in which $d_{T^{\prime}}(c) \leq 2 s-2$, then $F=c T^{\prime} c b_{1} P_{H^{2}} c$ is a $[2,2 s]$-factor in $G^{2}$, a contradiction.

If the graph $R^{2}$ has a spanning $s$-trail $T^{\prime \prime}$ between $c$ and some neighbor $c^{\prime \prime \prime} \in N_{R}(c)$, then $F=c T^{\prime \prime} c^{\prime \prime \prime} b_{1} P_{H^{2}} c$ is a $[2,2 s]$-factor in $G^{2}$, contradicting condition (2).

Case 2: $d(H)=2$. Let $c_{1}$ and $c_{2}$ be two cut vertices of $G$ belonging to $H$ and let $B_{i}, i=1,2$, be the union of all branches of $G$ at $c_{i}$ not containing $H$. This means that $G=\left(B_{1} c_{1} H\right) c_{2} B_{2}$. The subgraph $H$ is a block and thus, by Theorem E, $V(H)$ can be covered by two vertex-disjoint paths $a_{1} P_{H}^{1} a_{2}$ and $c_{2} P_{H}^{2} c_{1}$ in $H^{2}$, where $a_{1} \in N\left(c_{1}\right)$ and $a_{2} \in N\left(c_{2}\right)$. We distinguish, up to symmetry, the following three subcases.

Subcase 2.1: $B_{1}$ is trivial at $c_{1}$ and $B_{2}$ is trivial at $c_{2}$.
If $V\left(B_{1}\right)=\left\{b_{1}, b_{2}, \ldots, b_{k}, c_{1}\right\} \subseteq N\left[c_{1}\right], k \geq 1$, and $B_{2}=P_{3}\left(c_{2}\right)=c_{2} d_{1} d_{2}$, then $F=c_{1} b_{1} b_{2} \ldots b_{k} a_{1} P_{H}^{1} a_{2} d_{1} d_{2} c_{2} P_{H}^{2} c_{1}$ is even a hamiltonian cycle in $G^{2}$, which contradicts the fact that $G^{2}$ has no [2,2s]-factor.

The proof is similar if $B_{1}=P_{3}\left(c_{1}\right)$ and $V\left(B_{2}\right) \subseteq N\left[c_{2}\right]$.
If $V\left(B_{1}\right)=\left\{b_{1}, b_{2}, \ldots, b_{k}, c_{1}\right\} \subseteq N\left[c_{1}\right]$ and $V\left(B_{2}\right)=\left\{d_{1}, d_{2}, \ldots, d_{l}, c_{2}\right\} \subseteq N\left[c_{2}\right]$, then $F=c_{1} b_{1} b_{2} \ldots b_{k} a_{1} P_{H}^{1} a_{2} d_{1} d_{2} \ldots d_{l} c_{2} P_{H}^{2} c_{1}$ is also a hamiltonian cycle in $G^{2}$, contradicting (2).

Finally, if $B_{1}=P_{3}\left(c_{1}\right)=c_{1} b_{1} b_{2}$ and $B_{2}=P_{3}\left(c_{2}\right)=c_{2} d_{1} d_{2}$, then again the cycle $F=c_{1} b_{2} b_{1} a_{1} P_{H}^{1} a_{2} d_{1} d_{2} c_{2} P_{H}^{2} c_{1}$ gives a similar contradiction.

Subcase 2.2: $B_{1}$ is nontrivial at $c_{1}$ and $B_{2}$ is trivial at $c_{2}$.
Since $\left|V(H) \cup V\left(B_{2}\right)\right|>3$, there exists some vertex in $V(H) \cup V\left(B_{2}\right)$ (for example each vertex in $\left.V\left(B_{2}\right) \backslash\left\{c_{2}\right\}\right)$ nonadjacent to $c_{1}$, the subgraph $G^{\prime}=H c_{2} B_{2}$ is nontrivial. Then $G^{\prime}$ contains a path $P_{3}\left(c_{1}\right)=c_{1} n_{1} n_{2}$ as a proper induced subgraph. Now let $G_{1}=B_{1} c_{1} n_{1} n_{2}$. By Claim 11, $G_{1}$ satisfies condition (1). By minimality of $G$, the graph $G_{1}^{2}$ has a $[2,2 s]$-factor and thus, by Lemma 4 , we have the following two possibilities.
a) The graph $B_{1}^{2}$ has a spanning closed $s$-trail $T$ in which $d_{T}\left(c_{1}\right) \leq 2 s-2$.

If $V\left(B_{2}\right)=\left\{b_{1}, b_{2}, \ldots, b_{k}, c_{2}\right\} \subseteq N\left[c_{2}\right], k \geq 1$, then $F=c_{1} T c_{1} a_{1} P_{H}^{1} a_{2} b_{1} b_{2} \ldots b_{k} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$, a contradiction with (2).

If $B_{2}=P_{3}\left(c_{2}\right)=c_{2} d_{1} d_{2}$, then $F=c_{1} T c_{1} a_{1} P_{H}^{1} a_{2} d_{1} d_{2} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$, which contradicts condition (2).
b) The graph $B_{1}^{2}$ has a spanning $s$-trail $T^{\prime}$ between $c_{1}$ and some neighbor $c_{1}^{\prime} \in$ $N_{B_{1}}\left(c_{1}\right)$.

If $V\left(B_{2}\right)=\left\{b_{1}, b_{2}, \ldots, b_{k}, c_{2}\right\} \subseteq N\left[c_{2}\right], k \geq 1$, then $F=c_{1} T^{\prime} c_{1}^{\prime} a_{1} P_{H}^{1} a_{2} b_{1} b_{2} \ldots b_{k} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$ and contradicts (2).


Figure 1: An example showing that a condition in Theorem 2 cannot be relaxed.

If $B_{2}=P_{3}\left(c_{2}\right)=c_{2} d_{1} d_{2}$, then $F=c_{1} T^{\prime} c_{1}^{\prime} a_{1} P_{H}^{1} a_{2} d_{1} d_{2} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$, a contradiction with (2).

Subcase 2.3: $B_{1}$ is nontrivial at $c_{1}$ and $B_{2}$ is nontrivial at $c_{2}$.
Let $G_{1}$ be the same graph as in Subcase 2.2 and in a similar way as in Subcase 2.2 let $G_{2}=B_{2} c_{2} m_{1} m_{2}$, where a path $c_{2} m_{1} m_{2}$ is a proper induced subgraph of $H c_{1} B_{1}$. Then, by Claim 1, both $G_{1}$ and $G_{2}$ satisfy condition (1). By minimality of $G$, the graphs $G_{1}^{2}$ and $G_{2}^{2}$ have a $[2,2 s]$-factor and thus, by Lemma 4 , we have the following two possibilities.
a) The graph $B_{1}^{2}$ has a spanning closed $s$-trail $T$ in which $d_{T}\left(c_{1}\right) \leq 2 s-2$.

If the graph $B_{2}^{2}$ has a spanning closed $s$-trail $T^{\prime}$ in which $d_{T^{\prime}}\left(c_{2}\right) \leq 2 s-2$, then $F=c_{1} T c_{1} a_{1} P_{H}^{1} a_{2} c_{2} T^{\prime} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$ and contradicts (2).

If the graph $B_{2}^{2}$ has a spanning $s$-trail $T^{\prime \prime}$ between $c_{2}$ and some neighbor $c_{2}^{\prime} \in$ $N_{B_{2}}\left(c_{2}\right)$, then $F=c_{1} T c_{1} a_{1} P_{H}^{1} a_{2} c_{2}^{\prime} T^{\prime \prime} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$, contradicting condition (2).
b) The graph $B_{1}^{2}$ has a spanning $s$-trail $T^{*}$ between $c_{1}$ and some neighbor $c_{1}^{\prime} \in$ $N_{B_{1}}\left(c_{1}\right)$.

If the graph $B_{2}^{2}$ has a spanning closed $s$-trail $T^{* *}$ in which $d_{T^{* *}}\left(c_{2}\right) \leq 2 s-2$, then $F=c_{1} T^{*} c_{1}^{\prime} a_{1} P_{H}^{1} a_{2} c_{2} T^{* *} c_{2} P_{H}^{2} c_{1}$ is a [2, 2s]-factor in $G^{2}$, a contradiction.

If the graph $B_{2}^{2}$ has a spanning $s$-trail $T^{\bullet}$ between $c_{2}$ and some neighbor $c_{2}^{\prime} \in$ $N_{B_{2}}\left(c_{2}\right)$, then $F=c_{1} T^{*} c_{1}^{\prime} a_{1} P_{H}^{1} a_{2} c_{2}^{\prime} T^{\bullet} c_{2} P_{H}^{2} c_{1}$ is a $[2,2 s]$-factor in $G^{2}$ and contradicts (2).

The graph $G$ in Figure 1 shows that (for $s=1$ ) the constant 3 in Theorem 2 cannot be decreased. Although every induced $S\left(K_{1,2 s+1}\right)$ in $G$ has at least two edges in a block of degree at most two, $G^{2}$ has no $[2,2 s]$-factor.

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