A short proof of the tree-packing theorem

Tomáš Kaiser*

Abstract

We give a short elementary proof of Tutte and Nash-Williams' characterization of graphs with k edge-disjoint spanning trees.

We deal with graphs that may have parallel edges and loops; the vertex and edge sets of a graph H are denoted by $V(H)$ and $E(H)$, respectively. Let G be a graph. If $\mathcal{P}$ is a partition of $V(G)$, we let $G/\mathcal{P}$ be the graph on the set $\mathcal{P}$ with an edge joining distinct vertices $X, Y \in \mathcal{P}$ for every edge of G with one end in X and another in Y. Tutte [7] and Nash-Williams [4] proved the following classical result:

Theorem 1. *A graph G contains k pairwise edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$, the graph $G/\mathcal{P}$ has at least $k(|\mathcal{P}|-1)$ edges.*

Necessity of the condition in Theorem 1 is immediate. An elegant proof of sufficiency is based on the matroid union theorem (see, e.g., [5, Corollary 51.1a]) which yields the more general matroid base packing theorem of Edmonds [2]. A relatively short elementary proof of sufficiency in Theorem 1, due to W. Mader (personal communication from R. Diestel), is given in [1, Theorem 2.4.1].

In this paper, we give another elementary proof that is also short and perhaps somewhat more straightforward. The argument directly translates to an efficient algorithm to find either k disjoint spanning trees, or a proof that none exist.

To give the reader an idea of the approach, let us briefly sketch the proof of sufficiency, restricting to the case $k = 2$. Let T be a spanning tree of G, and let $\overline{T} = G - E(T)$. We may assume that $\overline{T}$ is disconnected as a spanning subgraph of G (otherwise, we have two disjoint spanning trees). We seek a partition $\mathcal{P}$ of $V(G)$ such that each class of $\mathcal{P}$ induces a connected subgraph in both T and

*Department of Mathematics and Institute for Theoretical Computer Science (ITI), University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: kaisert@kma.zcu.cz. Supported by project 1M0545 and Research Plan MSM 4977751301 of the Czech Ministry of Education, and by project GAČR 201/09/0197 of the Czech Science Foundation.

$\overline{T}$. In order to find it, we start with the trivial partition $\{V(G)\}$ and iteratively refine it (in a suitable way) until we reach the desired partition $\mathcal{P}$.

Let $E_{\mathcal{P}}$ denote the set of edges of G joining different classes of $\mathcal{P}$. The fact that $T[X]$ is connected for each $X \in \mathcal{P}$ enables us to count the edges of T in $E_{\mathcal{P}}$. Meanwhile, the density condition yields a lower bound on $|E_{\mathcal{P}}|$ and implies $\left|E(\overline{T}) \cap E_{\mathcal{P}}\right| \geq |\mathcal{P}| - 1$. Since $\overline{T}$ is disconnected, and since $\overline{T}[X]$ is connected for all $X \in \mathcal{P}$, this forces a cycle in $\overline{T}$ intersecting at least two classes of $\mathcal{P}$. We can replace some edge of T by an edge of this cycle, so as to obtain a new spanning tree T'. When done correctly, the exchange 'improves' the spanning tree T in a well-defined way. Thus, if the initial spanning tree T is chosen as optimal, then the basic assumption that $\overline{T}$ is disconnected must fail, which gives us the desired disjoint spanning trees.

A variant of this approach was used by Kaiser and Vrána [3] in connection with the conjecture of Thomassen [6] that 4-connected line graphs are hamiltonian. In that context, the method is applied to hypergraphs instead of graphs and gives a connectivity condition under which a hypergraph admits a 'spanning hypertree' whose complement is, in a way, close to being connected. A significant difference from the above setup is that the situation in [3] is asymmetric (unlike the packing of two spanning trees in a graph). It would be interesting to identify more general conditions allowing for the application of the method.

As noted by D. Král' (personal communication), a matroid-theoretic reformulation of the argument of the present paper yields a proof of the matroid base packing theorem mentioned above.

Before we start with the detailed proof of Theorem 1, we introduce some terminology. Let $k \geq 1$. A *k-decomposition* $\mathcal{T}$ of a graph G is a k-tuple $(T_1, \ldots, T_k)$ of spanning subgraphs of G such that $\{E(T_i) : 1 \leq i \leq k\}$ is a partition of $E(G)$.

We define the sequence $(\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_\infty)$ of partitions of $V(G)$ *associated with* $\mathcal{T}$ as follows. (See the illustration in Figure 1.) First, $\mathcal{P}_0 = \{V(G)\}$. For $i \geq 0$, if there exists $c \in \{1, \ldots, k\}$ such that the induced subgraph $T_c[X]$ is disconnected for some $X \in \mathcal{P}_i$, then let c_i be the least such c, and let $\mathcal{P}_{i+1}$ consist of the vertex sets of all components of $T_{c_i}[X]$, where X ranges over all the classes of $\mathcal{P}_i$. Otherwise, the process ends by setting $\mathcal{P}_\infty = \mathcal{P}_i$. In this case, we also set $c_j = k+1$ and $\mathcal{P}_j = \mathcal{P}_i$ for all $j \geq i$.

The *level* $\ell(e)$ of an edge $e \in E(G)$ (with respect to $\mathcal{T}$) is defined as the largest i (possibly ∞) such that both ends of e are contained in one class of $\mathcal{P}_i$. To keep the notation simple, the symbols $\mathcal{P}_i$ and $\ell(e)$ (as well as $\mathcal{P}_\infty$ and c_i) will relate to a k-decomposition $\mathcal{T}$, while $\mathcal{P}'_i$ and $\ell'(e)$ will relate to a k-decomposition $\mathcal{T}'$. Thus, for instance, the level $\ell'(e)$ of an edge e with respect to $\mathcal{T}'$ is defined using the partitions $\mathcal{P}'_i$ associated with $\mathcal{T}'$.

When $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $V(G)$, we say that $\mathcal{P}$ *refines* $\mathcal{Q}$ (and write $\mathcal{P} \leq \mathcal{Q}$) if every class of $\mathcal{P}$ is a subset of a class of $\mathcal{Q}$. When $\mathcal{P} \leq \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$, we write $\mathcal{P} < \mathcal{Q}$.

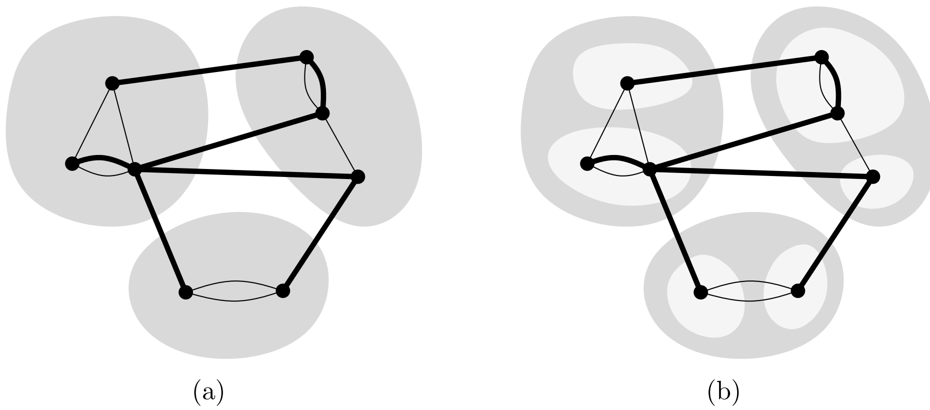

Figure 1: The sequence of partitions associated with a 2-decomposition $\mathcal{T} = (T_1, T_2)$ of G. The edges of T_1 are shown bold. (a) The partition $\mathcal{P}_1$ (dark grey regions). (b) The partition $\mathcal{P}_2$ (light grey regions). Note that $\mathcal{P}_2 = \mathcal{P}_\infty$.

We define a strict partial order $\prec$ on k-decompositions of G. Given two k-decompositions $\mathcal{T}$ and $\mathcal{T}'$, we set $\mathcal{T} \prec \mathcal{T}'$ if there is some (finite) $j \geq 0$ such that both of the following conditions hold:

(i) for $0 \leq i < j$, $\mathcal{P}_i = \mathcal{P}'_i$ and $c_i = c'_i$,

(ii) either $\mathcal{P}_j < \mathcal{P}'_j$, or $\mathcal{P}_j = \mathcal{P}'_j$ and $c_j < c'_j$.

Proof of Theorem 1. The necessity of the condition is clear. To prove the sufficiency, we proceed by induction on k. The claim is trivially true for $k = 0$, so assume $k \geq 1$ and choose a k-decomposition $\mathcal{T} = (T_1, \dots, T_k)$ of G such that $T_1, \dots, T_{k-1}$ are trees and, subject to this condition, $\mathcal{T}$ is maximal with respect to $\prec$.

If T_k is connected, then we are done. Otherwise, suppose that T_k has at least two components (i.e., $|\mathcal{P}_1| \geq 2$). We prove that there exists an edge of finite level (with respect to $\mathcal{T}$) contained in a cycle of T_k. Let $\mathcal{P} = \mathcal{P}_\infty$. Recall that for $1 \leq i < k$ and $X \in \mathcal{P}$, the graph $T_i[X]$ is connected. Hence $T_i/\mathcal{P}$ is a tree and has exactly $|\mathcal{P}| - 1$ edges. By hypothesis, $G/\mathcal{P}$ has at least $k(|\mathcal{P}| - 1)$ edges, so $T_k/\mathcal{P}$ has at least $|\mathcal{P}| - 1$ edges. Since $T_k/\mathcal{P}$ has $|\mathcal{P}|$ vertices and is disconnected, it must contain a cycle. Thus T_k contains a cycle, since $T_k[X]$ is connected for each $X \in \mathcal{P}$. At least two edges of the cycle join different classes of $\mathcal{P}$, and therefore their level is finite, as required.

Let $e \in E(T_k)$ be an edge of minimum level that is contained in a cycle of T_k, and set $m = \ell(e)$. (See Figure 2 for an illustration with $m = 1$.) Let P be the class of $\mathcal{P}_m$ containing both ends of e. Since e joins different components of $T_{c_m}[P]$, we have $c_m \neq k$, and the unique cycle C in $T_{c_m} + e$ contains an edge with only one end in P. Thus, for an edge e' of C of lowest possible level we have $\ell(e') < m$. Let Q be the class of $\mathcal{P}_{\ell(e')}$ containing both ends of e'. Observe that

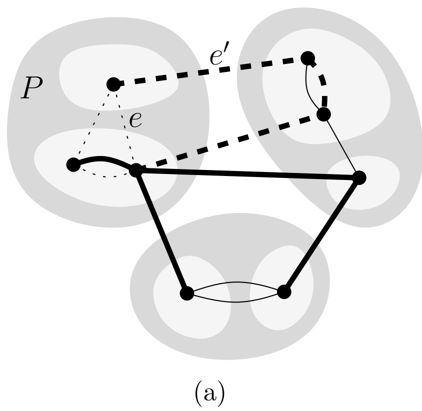

Figure 2: The exchange step for the 2-decomposition $\mathcal{T}$ of Figure 1. (a) A cycle in T_2 containing e (dotted) and the cycle C in T_1+e (dashed). (b) The spanning tree T_1' (bold) obtained from T_1 by exchanging e for the edge e' of C. The partitions $\mathcal{P}_1'$ and $\mathcal{P}_2'$ associated with the resulting 2-decomposition $\mathcal{T}'$ are shown in dark grey and light grey, respectively. Note that $\mathcal{P}_2'$ is equal to $\mathcal{P}_\infty'$ and that $\mathcal{T} \prec \mathcal{T}'$.

$V(C) \subseteq Q$. We will exchange e for e' in the members of the k-decomposition to eventually obtain the desired contradiction.

Let $\mathcal{T}'$ be the k-decomposition obtained from $\mathcal{T}$ by replacing T_{c_m} with $T_{c_m} + e - e'$ and T_k with $T_k - e + e'$. The i-th element of $\mathcal{T}'$, where $1 \leq i \leq k$, is denoted by T_i'. To relate the sequences of partitions associated with $\mathcal{T}$ and $\mathcal{T}'$, we prove the following two claims.

Claim 1. *If $T_c[X]$ is connected, for some $X \subseteq V(G)$ and $1 \leq c \leq k$, then $T_c'[X]$ is connected unless one of the following holds:*

(a) $c = c_m$, and X contains both ends of e', and $Q \not\subseteq X$, or

(b) $c = k$, and X contains both ends of e, and $P \not\subseteq X$.

To prove the claim, suppose that $T_c'[X]$ is disconnected. We have $c \in \{c_m, k\}$, since otherwise $T_c = T_c'$. Consider $c = c_m$. Since $E(T_{c_m}) - E(T_{c_m}') = \{e'\}$, both ends of e' lie in X. Furthermore, $Q \not\subseteq X$, since otherwise $T_{c_m}'[X]$ would contain the path $C - e'$ joining the ends of e', which would make $T_{c_m}'[X]$ connected. A similar argument for the case $c = k$ completes the proof of Claim 1.

Claim 2. *For all $i \leq m$, it holds that $c_i' = c_i$ and $\mathcal{P}_i' = \mathcal{P}_i$.*

We proceed by induction on i. The case $i = 0$ follows from $\mathcal{P}_0 = \mathcal{P}_0' = \{V(G)\}$ and $c_0 = c_0' = k$. Let us thus assume that the assertion holds for some i, $0 \leq i < m$, and prove it for $i + 1$.

We first prove that $\mathcal{P}_{i+1} = \mathcal{P}_{i+1}'$. Let S be an arbitrary class of $\mathcal{P}_{i+1}$; we assert that $T_{c_i'}'[S]$ is connected. Since $T_{c_i}[S]$ is connected and since $c_i' = c_i$ by the inductive hypothesis, we can use Claim 1 (with $X = S$ and $c = c_i$). Condition (a) in the claim cannot hold, because every class of $\mathcal{P}_{i+1}$ containing both ends of e' contains Q as a subset. For a similar reason, condition (b) fails. Consequently, $T'_{c_i}[S]$ is connected, and hence S is a subset of some class of $\mathcal{P}'_{i+1}$. Since S was arbitrary, it follows that $\mathcal{P}_{i+1} \leq \mathcal{P}'_{i+1}$. Now by the choice of $\mathcal{T}$ (and the inductive assumption), we cannot have $\mathcal{P}_{i+1} < \mathcal{P}'_{i+1}$. We conclude that $\mathcal{P}_{i+1} = \mathcal{P}'_{i+1}$.

Next, we prove that $c'_{i+1} = c_{i+1}$. Let $R \in \mathcal{P}'_{i+1}$ and $c < c_{i+1}$. By the above, $R \in \mathcal{P}_{i+1}$. The definition of c_{i+1} implies that $T_c[R]$ is connected. Using Claim 1 as above, we find that $T'_c[R]$ is also connected. Consequently, $c'_{i+1} \geq c_{i+1}$, and by the maximality of $\mathcal{T}$ once again, we must have $c'_{i+1} = c_{i+1}$. The proof of Claim 2 is complete.

It is now easy to finish the proof of Theorem 1. Since $\mathcal{P}'_m = \mathcal{P}_m$ and $c'_m = c_m$, the classes of $\mathcal{P}'_{m+1}$ are the vertex sets of components of $T'_{c_m}[U]$, where $U \in \mathcal{P}_m$. Observe that for $U \in \mathcal{P}_m - \{P\}$, we have $T'_{c_m}[U] = T_{c_m}[U]$, and so the components of $T'_{c_m}[U]$ coincide with those of $T_{c_m}[U]$. The graph $T'_{c_m}[P]$ is obtained from $T_{c_m}[P]$ by adding the edge e that connects two components of $T_{c_m}[P]$. It follows that $\mathcal{P}_{m+1} < \mathcal{P}'_{m+1}$, contradicting the choice of $\mathcal{T}$. $\square$

Acknowledgment

I am indebted to Douglas West and two anonymous referees who suggested a number of improvements to the paper.

References

[1] R. Diestel, *Graph Theory*, 3rd Edition, Springer, 2005.

[2] J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 73–77.

[3] T. Kaiser and P. Vrána, Hamilton cycles in 5-connected line graphs, *European J. Combin.*, doi:10.1016/j.ejc.2011.09.015.

[4] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.

[5] A. Schrijver, *Combinatorial Optimization*, Springer, 2003.

[6] C. Thomassen, Reflections on graph theory, *J. Graph Theory* **10** (1986), 309–324.

[7] W. T. Tutte, On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* **36** (1961), 221–230.