On the Structure of the Minimum Critical Independent Set of a Graph

Vadim E. Levit Ariel University Center of Samaria, Israel levitv@ariel.ac.il

Eugen Mandrescu Holon Institute of Technology, Israel eugen_m@hit.ac.il

Abstract

Let G = (V, E). A set $S \subseteq V$ is *independent* if no two vertices from S are adjacent, and by Ind(G) we mean the set of all independent sets of G. The number d(X) = |X| - |N(X)| is the *difference* of $X \subseteq V$, and $A \in Ind(G)$ is *critical* if

 $d(A) = \max\{d(I) : I \in \operatorname{Ind}(G)\} [7].$

Let us recall the following definitions:

 $\ker(G) = \cap \{S : S \text{ is a critical independent set} \} [5],$

 $\operatorname{core}(G) = \cap \{S : S \text{ is a maximum independent set}\}$ [4].

Recently, it was established that $\ker(G) \subseteq \operatorname{core}(G)$ is true for every graph [5], while the corresponding equality holds for bipartite graphs [6].

In this paper we present various structural properties of $\ker(G)$. The main finding claims that

 $\ker(G) = \bigcup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.$

Keywords: independent set, critical set, ker, core, matching

1 Introduction

Throughout this paper G = (V, E) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set V = V(G) and edge set E = E(G). If $X \subseteq V$, then G[X] is the subgraph of G spanned by X. By G - W we mean either the subgraph G[V - W], if $W \subseteq V(G)$, or the partial subgraph H = (V, E - W) of G, for $W \subseteq E(G)$. In either case, we use G - w, whenever $W = \{w\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the closed neighborhood of $v \in V$ is $N[v] = N(v) \cup \{v\}$; in order to avoid ambiguity, we use also $N_G(v)$ instead of N(v). The *neighborhood* of $A \subseteq V$ is denoted by $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by Ind(G) we mean the set of all the independent sets of G.

An independent set of maximum size will be referred to as a maximum independent set of G, and the independence number of G is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets, and $\text{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [4].

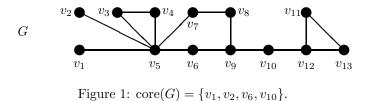
A matching is a set of non-incident edges of G; a matching of maximum cardinality is a maximum matching, and its size is denoted by $\mu(G)$.

The number d(X) = |X| - |N(X)|, $X \subseteq V(G)$, is called the *difference* of the set X. The number $d_c(G) = \max\{d(X) : X \subseteq V\}$ is called the *critical difference* of G, and a set $U \subseteq V(G)$ is *critical* if $d(U) = d_c(G)$ [7]. The number $id_c(G) = \max\{d(I) : I \in \operatorname{Ind}(G)\}$ is called the *critical independence difference* of G. If $A \subseteq V(G)$ is independent and $d(A) = id_c(G)$, then A is called *critical independent* [7]. Clearly, $d_c(G) \ge id_c(G)$ is true for every graph G.

Theorem 1.1 [7] The equality $d_c(G) = id_c(G)$ holds for every graph G.

For a graph G, let denote $\ker(G) = \bigcap \{S : S \text{ is a critical independent set}\}$. It is known that $\ker(G) \subseteq \operatorname{core}(G)$ is true for every graph [5], while the equality holds for bipartite graphs [6].

For instance, the graph G from Figure 1 has $X = \{v_1, v_2, v_3, v_4\}$ as a critical set, since $N(X) = \{v_3, v_4, v_5\}$ and $d(X) = 1 = d_c(G)$, while $I = \{v_1, v_2, v_3, v_6, v_7\}$ is a critical independent set, because $d(I) = 1 = id_c(G)$; other critical sets are $\{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4, v_6, v_7\}$. In addition, ker $(G) = \{v_1, v_2\}$, and core(G) is a critical set.



It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable [1, 7].

Theorem 1.2 For a graph G = (V, E), the following assertions are true:

(i) [5] the function d is supermodular, i.e.,

 $d(A \cup B) + d(A \cap B) \ge d(A) + d(B) \text{ for every } A, B \subseteq V;$

(ii) [5] G has a unique minimal critical independent set, namely, ker(G).

(iii) [3] there is a matching from N(S) into S, for every critical independent set S.

In this paper we characterize $\ker(G)$. In addition, a number of properties of $\ker(G)$ are presented as well.

2 Results

Deleting a vertex from a graph may decrease, leave unchanged or increase its critical difference. For instance, $d_c(G - v_1) = d_c(G) - 1$, $d_c(G - v_{13}) = d_c(G)$, while $d_c(G - v_3) = d_c(G) + 1$, where G is depicted in Figure 1.

Proposition 2.1 Let G = (V, E) and $v \in V$. Then the following assertions hold: (i) $d_c (G - v) = d_c (G) - 1$ if and only if $v \in \ker(G)$; (ii) if $v \in \ker(G)$, then $\ker(G - v) \subseteq \ker(G) - \{v\}$.

Proof. (i) Let $v \in V$ and H = G - v. If $v \notin \ker(G)$, then $\ker(G) \subseteq V(G) - \{v\}$. Hence

$$d_{c}(G - v) \ge |\ker(G)| - |N_{H}(\ker(G))| \ge |\ker(G)| - |N_{G}(\ker(G))| = d_{c}(G)$$

Consequently, we infer that $d_c(G-v) < d_c(G)$ implies $v \in \ker(G)$.

Conversely, assume that $v \in \ker(G)$. Each $u \in N(v)$ satisfies $|N(u) \cap \ker(G)| \ge 2$, because otherwise, $d(\ker(G) - \{v\}) = d(\ker(G))$ and this contradicts the minimality of $\ker(G)$. Therefore, $N(\ker(G) - \{v\}) = N(\ker(G))$ and hence

$$d (\ker(G) - \{v\}) = |\ker(G) - \{v\}| - |N (\ker(G) - \{v\})| =$$

= |\ker(G)| - 1 - |N (\ker(G))| = d_c (G) - 1.

If there is some independent set A in G-v, such that $d(A) = d_c(G)$, then A is critical in G and, hence we get the following contradiction: $v \in \ker(G) \subseteq A \subseteq V - \{v\}$. Therefore, $\ker(G) - \{v\}$ is a critical independent set of G - v and

$$d_c (G - v) = d (\ker(G) - \{v\}) = d_c (G) - 1.$$

(*ii*) Assume that $\ker(G - v) \neq \emptyset$. In part (*i*), we saw that $\ker(G) - \{v\}$ is a critical independent set of G - v. Hence, we get that $\ker(G - v) \subseteq \ker(G) - \{v\}$.

Remark 2.2 Actually, $\ker(G - v)$ may be different from $\ker(G) - \{v\}$; for instance, if $K_{3,2} = (A, B, E), |A| = 3$, then $\ker(K_{3,2}) = A$ and $\ker(K_{3,2} - v) = \emptyset \neq \ker(K_{3,2}) - \{v\}$, for every $v \in A$. It is also possible $\ker(G) - \{v\} = \emptyset$, while $\ker(G - v) \neq \emptyset$; e.g., $G = C_4$.

By Theorem 1.2(*iii*), there is a matching from N(S) into $S = \{v_1, v_2, v_3\}$, for instance, $M = \{v_2v_5, v_3v_4\}$, since S is critical independent for the graph G from Figure 1. On the other hand, there is no matching from N(S) into $S - v_3$. The case of the critical independence set ker(G) is more specific.

Theorem 2.3 Let A be a critical independent set in a graph G. Then the following statements are equivalent:

- (i) $A = \ker(G);$
- (ii) there is no set $B \subseteq N(A)$, $B \neq \emptyset$ such that $|N(B) \cap A| = |B|$;
- (iii) for each $v \in A$ there exists a matching from N(A) into A v.

Proof. (i) \implies (ii) By Theorem 1.2(iii), there is a matching, say M, from $N(\ker(G))$ into $\ker(G)$. Suppose, to the contrary, that there is some non-empty set $B \subseteq N(\ker(G))$ such that

$$|M(B)| = |N(B) \cap \ker(G)| = |B|.$$

It contradicts the fact that, by Theorem 1.2(ii), ker(G) is a minimal critical independent set, because

$$d(\ker(G) - N(B)) = d(\ker(G))$$
, while $\ker(G) - N(B) \subsetneq \ker(G)$.

 $(ii) \Longrightarrow (i)$ Suppose $A - \ker(G) \neq \emptyset$. By Theorem 1.2(*iii*), there is a matching, say M, from N(A) into A. Since there are no edges connecting vertices belonging to $\ker(G)$ with vertices from $N(A) - N(\ker(G))$, we obtain that $M(N(A) - N(\ker(G))) \subseteq A - \ker(G)$. Moreover, we have that $|N(A) - N(\ker(G))| = |A - \ker(G)|$, otherwise

$$|A| - |N(A)| = (|\ker(G)| - |N(\ker(G))|) + (|A - \ker(G)| - |N(A) - N(\ker(G))|) > (|\ker(G)| - |N(\ker(G))|) = d_c(G).$$

It means that the set $N(A) - N(\ker(G))$ contradicts the hypothesis of *(ii)*, because

$$|N(A) - N(\ker(G))| = |A - \ker(G)| = |N(N(A) - N(\ker(G))) \cap A|.$$

Consequently, the assertion is true.

 $(ii) \implies (iii)$ By Theorem 1.2(*iii*), there is a matching, say M, from N(A) into A. Suppose, to the contrary, that there is no matching from N(A) into A - v. Hence, by Hall's Theorem, it implies the existence of a set $B \subseteq N(A)$ such that $|N(B) \cap A| = |B|$, which contradicts the hypothesis of (*ii*).

 $(iii) \implies (ii)$ Assume, to the contrary, that there is a non-empty subset B of N(A) such that $|N(B) \cap A| = |B|$. Let $v \in N(B) \cap A$. Hence, we obtain that

$$|N(B) \cap A - v| < |B|.$$

Then, by Hall's Theorem, it is impossible to find a matching from N(A) into A - v, in contradiction with the hypothesis of *(iii)*.

Since $\ker(G)$ is a critical set, Theorem 1.2*(iii)* assures that there is a matching from $N(\ker(G))$ into $\ker(G)$. The following result shows that there are at least two such matchings.

Corollary 2.4 For a graph G the following are true:

(i) every edge $e \in (\ker(G), N(\ker(G)))$ belongs to a matching from $N(\ker(G))$ into $\ker(G)$;

(ii) every edge $e \in (\ker(G), N(\ker(G)))$ is not included in one matching from $N(\ker(G))$ into $\ker(G)$ at least.

Proof. Let $e = xy \in (\ker(G), N(\ker(G)))$, such that $x \in \ker(G)$. By Theorem 2.3(*iii*) there is a matching M from $N(\ker(G))$ into $\ker(G) - x$, that matches y with some $z \in \ker(G) - x$. Clearly, M is a matching from $N(\ker(G))$ into $\ker(G)$ that does not contain the edge e = xy, while $(M - \{yz\}) \cup \{xy\}$ is a matching from $N(\ker(G))$ into $\ker(G)$ into $\ker(G)$, which includes the edge e = xy.

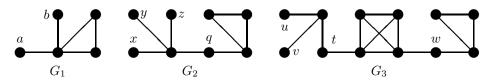


Figure 2: $\operatorname{core}(G_1) = \{a, b\}, \operatorname{core}(G_2) = \{q, x, y, z\}, \operatorname{core}(G_3) = \{t, u, v, w\}.$

Let us notice that the graphs G_1 , G_2 from Figure 2 have: $\ker(G_1) = \operatorname{core}(G_1)$, $\ker(G_2) = \{x, y, z\} \subset \operatorname{core}(G_2)$, and both $\operatorname{core}(G_1)$ and $\operatorname{core}(G_2)$ are critical sets of maximum size. The graph G_3 from Figure 2 has $\ker(G_3) = \{u, v\}$, the set $\{t, u, v\}$ as a critical independent set of maximum size, while $\operatorname{core}(G_3) = \{t, u, v, w\}$ is not a critical set. If S_{\min} denotes an inclusion minimal independent set with $d(S_{\min}) > 0$, one can see that: $S_{\min} = \ker(G_1)$ for G_1 , while the graph G_2 in the same figure has $S_{\min} \in \{\{x, y\}, \{x, z\}, \{y, z\}\}$ and $\ker(G_2) = \{x, y\} \cup \{x, z\} \cup \{y, z\}$.

In [5] we have shown that $\ker(G)$ is equal to the intersection of all critical, independent or not, sets of G.

Theorem 2.5 For every graph G

 $\ker(G) = \bigcup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.$

Proof. Let A be a critical set and S_0 be an inclusion minimal independent set such that $d(S_0) > 0$. Then, Theorem 1.2(i) implies

$$d(A \cup S_0) + d(A \cap S_0) \ge d(A) + d(S_0) > d(A) = d_c(G).$$

Since S_0 is an inclusion minimal independent set such that $d(S_0) > 0$, we obtain that if $A \cap S_0 \neq S_0$, then $d(A \cap S_0) \leq 0$. Hence

$$d(A) = d_c(G) \ge d(A \cup S_0) \ge d(A) + d(S_0) > d(A),$$

which is impossible. Therefore, $S_0 \subseteq A$ for every critical set A. Consequently,

 $S_0 \subseteq \cap \{B : B \text{ is a critical set of } G\} = \ker(G).$

Thus we obtain

 $\cup \{S_0 : S_0 \text{ is an inclusion minimal independent set such that } d(S_0) > 0\} \subseteq \ker(G).$

Conversely, it is enough to show that every vertex from $\ker(G)$ belongs to some inclusion minimal independent set with positive difference. Let $v \in \ker(G)$. According to Theorem 2.3(*iii*) there exists a matching, say M, from $N(\ker(G))$ into $\ker(G) - v$.

Let us build the following sequence of sets

$$\{v\} \subseteq M\left(N\left(v\right)\right) \subseteq \dots \subseteq [MN]^{k}\left(v\right) \subseteq \dots,$$

where MN is a superposition of two mappings $N : 2^V \longrightarrow 2^V (N(A))$ is the neighborhood of A) and $M : 2^{N(\ker(G))} \longrightarrow 2^{\ker(G)} (M(A))$ is set of the vertices matched by M with vertices belonging to A). Since the set ker(G) is finite, there is an index j such that $[MN]^j(v) = [MN]^{j+1}(v)$. Hence $\left|N\left([MN]^j(v)\right)\right| = \left|[MN]^j(v)\right| - 1$. In other words, we found an independent set, namely, $[MN]^j(v)$ such that $v \in [MN]^j(v)$ and $d\left([MN]^j(v)\right) = 1$. Therefore, there must exist an inclusion minimal independent set X such that $v \in X$ and d(X) = 1.

Remark 2.6 In a graph G, the union of all minimum cardinality independent sets S with d(S) > 0 may be a proper subset of ker (G); e.g., the graph G in Figure 3, that has $\{x, y\} \subset \text{ker}(G) = \{x, y, u, v, w\}.$

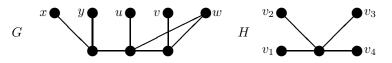


Figure 3: Both $S_1 = \{x, y\}$ and $S_2 = \{u, v, w\}$ are inclusion minimal independent sets satisfying d(S) > 0.

Proposition 2.7 min $\{|S_0| : d(S_0) > 0, S_0 \in \text{Ind}(G)\} \le |\ker(G)| - d_c(G) + 1.$

Proof. Since ker(G) is a critical independent set, Theorem 1.2(*iii*) implies that there is a matching, say M, from N(ker(G)) into ker(G). Let X = M(N(ker(G))). Then d(X) = 0. For every $v \in \text{ker}(G) - X$ we have

$$N(\ker(G)) \subseteq N(X) \subseteq N(X \cup \{v\}) \subseteq N(\ker(G)).$$

Hence we get $|X \cup \{v\}| - |N(X \cup \{v\})| = 1$, while $|X \cup \{v\}| = |\ker(G)| - d_c(G) + 1$. ■

Remark 2.8 All the inclusion minimal independent sets S, with d(S) > 0, of the graph H from Figure 3 are of the same size. However, there are inclusion minimal independent sets S with d(S) > 0, of different cardinalities; e.g., the graph G from Figure 3.

Proposition 2.9 If S_0 is an inclusion minimal independent set with $d(S_0) > 0$, then $d(S_0) = 1$.

Proof. For each $v \in S_0$, it follows that $N(S_0 - v) = N(S_0)$, otherwise,

$$d(S_0 - v) = |S_0 - v| - |N(S_0 - v)| =$$

= |S_0| - 1 - |N(S_0 - v)| \ge |S_0| - |N(S_0)| > 0.

i.e., S_0 is not an inclusion minimal independent set with positive difference.

Since S_0 is an inclusion minimal independent set with positive difference, we know that $d(S_0 - v) \leq 0$. On the other hand, it follows from the equality $N(S_0 - v) = N(S_0)$ that

$$d(S_0 - v) = |S_0 - v| - |N(S_0 - v)| = |S_0| - 1 - |N(S_0)| = d(S_0) - 1 \le 0.$$

Consequently, $0 < |S_0| - |N(S_0)| \le 1$, which means that $|S_0| - |N(S_0)| = 1$.

Remark 2.10 The converse of Proposition 2.9 is not true. For instance, $S = \{x, y, u\}$ is independent in the graph G from Figure 3 and d(S) = 1, but S is not minimal with this property.

Proposition 2.11 If $S_i, i = 1, 2, ..., k, k \ge 1$, are inclusion minimal independent sets, such that $d(S_i) > 0, S_i \nsubseteq \bigcup_{j=1, j \ne i}^k S_j, 1 \le i \le k$, then $d(S_1 \cup S_2 \cup ... \cup S_k) \ge k$.

Proof. For k = 1 the claim has been treated in Proposition 2.9, where we have achieved a stronger result.

We continue by induction on k.

Let k = 2. Since $S_1 \neq S_1 \cap S_2 \subset S_1$, it follows that $d(S_1 \cap S_2) \leq 0$. Hence, Theorem 1.2(*i*) and Proposition 2.9 imply

$$d(S_1 \cup S_2) \ge d(S_1 \cup S_2) + d(S_1 \cap S_2) \ge d(S_1) + d(S_2) = 2$$

Assume that the assertion is true for each $k \ge 2$, and let $\{S_i, 1 \le i \le k+1\}$ be a family of inclusion minimal independent sets with

$$d(S_i) > 0$$
 and $S_i \not\subseteq \bigcup_{j=1, j \neq i}^{k+1} S_j, 1 \le i \le k+1$

Since $S_{k+1} \neq (S_1 \cup S_2 \cup ... \cup S_k) \cap S_{k+1} \subset S_{k+1}$, we obtain that

$$d\left(\left(S_1 \cup S_2 \cup \dots \cup S_k\right) \cap S_{k+1}\right) \le 0.$$

Further, using the supermodularity of the function d and Proposition 2.9, we get

$$d(S_1 \cup S_2 \cup \dots \cup S_k \cup S_{k+1}) \ge$$

$$\ge d(S_1 \cup S_2 \cup \dots \cup S_k \cup S_{k+1}) + d((S_1 \cup S_2 \cup \dots \cup S_k) \cap S_{k+1}) \ge$$

$$\ge d(S_1 \cup S_2 \cup \dots \cup S_k) + d(S_{k+1}) \ge k+1,$$

as required. \blacksquare

Remark 2.12 The sets $S_1 = \{v_1, v_2\}, S_2 = \{v_2, v_3\}, S_3 = \{v_3, v_4\}$ are inclusion minimal independent sets of the graph H from Figure 3, such that

$$d(S_i) > 0, S_i \not\subseteq \bigcup_{j=1, j \neq i}^3 S_j, i = 1, 2, 3.$$

Notice that both families $\{S_1, S_2\}$, $\{S_1, S_3\}$ have two elements, and $d(S_1 \cup S_2) = 2$, while $d(S_1 \cup S_3) > 2$.

3 Conclusions

In this paper we investigate structural properties of $\ker(G)$.

Having in view Theorem 2.5, notice that the graph:

- G_1 from Figure 2 has only one inclusion minimal independent set S such that d(S) > 0, and $d_c(G_1) = 1$;
- G from Figure 3 has only two inclusion minimal independent sets S such that d(S) > 0, and $d_c(G) = 2$;
- *H* from Figure 3 has 6 inclusion minimal independent sets *S* such that d(S) > 0, and $d_c(H) = 3$.

These remarks motivate the following.

Conjecture 3.1 The number of inclusion minimal independent set S such that d(S) > 0 is greater or equal to $d_c(G)$.

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