# On the Structure of the Minimum Critical Independent Set of a Graph 

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#### Abstract

Let $G=(V, E)$. A set $S \subseteq V$ is independent if no two vertices from $S$ are adjacent, and by $\operatorname{Ind}(G)$ we mean the set of all independent sets of $G$. The number $d(X)=|X|-|N(X)|$ is the difference of $X \subseteq V$, and $A \in \operatorname{Ind}(G)$ is critical if $$
d(A)=\max \{d(I): I \in \operatorname{Ind}(G)\}
$$


Let us recall the following definitions:

$$
\begin{aligned}
\operatorname{ker}(G) & =\cap\{S: S \text { is a critical independent set }\} \\
\text { core }(G) & =\cap\{S: S \text { is a maximum independent set }\}
\end{aligned}
$$

Recently, it was established that $\operatorname{ker}(G) \subseteq \operatorname{core}(G)$ is true for every graph [5], while the corresponding equality holds for bipartite graphs [6].

In this paper we present various structural properties of $\operatorname{ker}(G)$. The main finding claims that
$\operatorname{ker}(G)=\cup\left\{S_{0}: S_{0}\right.$ is an inclusion minimal independent set with $\left.d\left(S_{0}\right)>0\right\}$.

Keywords: independent set, critical set, ker, core, matching

## 1 Introduction

Throughout this paper $G=(V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V=V(G)$ and edge set $E=E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G-W$ we mean either the subgraph $G[V-W]$, if $W \subseteq V(G)$, or the partial subgraph $H=(V, E-W)$ of $G$, for $W \subseteq E(G)$. In either case, we use $G-w$, whenever $W=\{w\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v)=\{w: w \in V$ and $v w \in E\}$, while the closed neighborhood of $v \in V$ is $N[v]=N(v) \cup\{v\}$; in order to avoid ambiguity,
we use also $N_{G}(v)$ instead of $N(v)$. The neighborhood of $A \subseteq V$ is denoted by $N(A)=$ $N_{G}(A)=\{v \in V: N(v) \cap A \neq \emptyset\}$, and $N[A]=N(A) \cup A$.

A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent, and by $\operatorname{Ind}(G)$ we mean the set of all the independent sets of $G$.

An independent set of maximum size will be referred to as a maximum independent set of $G$, and the independence number of $G$ is $\alpha(G)=\max \{|S|: S \in \operatorname{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets, and core $(G)=\cap\{S: S \in \Omega(G)\}$ 4].

A matching is a set of non-incident edges of $G$; a matching of maximum cardinality is a maximum matching, and its size is denoted by $\mu(G)$.

The number $d(X)=|X|-|N(X)|, X \subseteq V(G)$, is called the difference of the set $X$. The number $d_{c}(G)=\max \{d(X): X \subseteq V\}$ is called the critical difference of $G$, and a set $U \subseteq V(G)$ is critical if $d(U)=d_{c}(G)$ [7]. The number $i d_{c}(G)=\max \{d(I): I \in \operatorname{Ind}(G)\}$ is called the critical independence difference of $G$. If $A \subseteq V(G)$ is independent and $d(A)=i d_{c}(G)$, then $A$ is called critical independent [7]. Clearly, $d_{c}(G) \geq i d_{c}(G)$ is true for every graph $G$.

Theorem 1.1 77 The equality $d_{c}(G)=i d_{c}(G)$ holds for every graph $G$.
For a graph $G$, let denote $\operatorname{ker}(G)=\cap\{S: S$ is a critical independent set $\}$. It is known that $\operatorname{ker}(G) \subseteq \operatorname{core}(G)$ is true for every graph [5], while the equality holds for bipartite graphs [6].

For instance, the graph $G$ from Figure 1 has $X=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ as a critical set, since $N(X)=\left\{v_{3}, v_{4}, v_{5}\right\}$ and $d(X)=1=d_{c}(G)$, while $I=\left\{v_{1}, v_{2}, v_{3}, v_{6}, v_{7}\right\}$ is a critical independent set, because $d(I)=1=i d_{c}(G)$; other critical sets are $\left\{v_{1}, v_{2}\right\},\left\{v_{1}, v_{2}, v_{3}\right\}$, $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{6}, v_{7}\right\}$. In addition, $\operatorname{ker}(G)=\left\{v_{1}, v_{2}\right\}$, and core $(G)$ is a critical set.


Figure 1: $\operatorname{core}(G)=\left\{v_{1}, v_{2}, v_{6}, v_{10}\right\}$.
It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable [1, 7,

Theorem 1.2 For a graph $G=(V, E)$, the following assertions are true:
(i) [5] the function $d$ is supermodular, i.e.,

$$
d(A \cup B)+d(A \cap B) \geq d(A)+d(B) \text { for every } A, B \subseteq V ;
$$

(ii) [5] $G$ has a unique minimal critical independent set, namely, $\operatorname{ker}(G)$.
(iii) [3] there is a matching from $N(S)$ into $S$, for every critical independent set $S$.

In this paper we characterize $\operatorname{ker}(G)$. In addition, a number of properties of $\operatorname{ker}(G)$ are presented as well.

## 2 Results

Deleting a vertex from a graph may decrease, leave unchanged or increase its critical difference. For instance, $d_{c}\left(G-v_{1}\right)=d_{c}(G)-1, d_{c}\left(G-v_{13}\right)=d_{c}(G)$, while $d_{c}\left(G-v_{3}\right)=d_{c}(G)+1$, where $G$ is depicted in Figure

Proposition 2.1 Let $G=(V, E)$ and $v \in V$. Then the following assertions hold:
(i) $d_{c}(G-v)=d_{c}(G)-1$ if and only if $v \in \operatorname{ker}(G)$;
(ii) if $v \in \operatorname{ker}(G)$, then $\operatorname{ker}(G-v) \subseteq \operatorname{ker}(G)-\{v\}$.

Proof. (i) Let $v \in V$ and $H=G-v$.
If $v \notin \operatorname{ker}(G)$, then $\operatorname{ker}(G) \subseteq V(G)-\{v\}$. Hence

$$
d_{c}(G-v) \geq|\operatorname{ker}(G)|-\left|N_{H}(\operatorname{ker}(G))\right| \geq|\operatorname{ker}(G)|-\left|N_{G}(\operatorname{ker}(G))\right|=d_{c}(G) .
$$

Consequently, we infer that $d_{c}(G-v)<d_{c}(G)$ implies $v \in \operatorname{ker}(G)$.
Conversely, assume that $v \in \operatorname{ker}(G)$. Each $u \in N(v)$ satisfies $|N(u) \cap \operatorname{ker}(G)| \geq 2$, because otherwise, $d(\operatorname{ker}(G)-\{v\})=d(\operatorname{ker}(G))$ and this contradicts the minimality of $\operatorname{ker}(G)$. Therefore, $N(\operatorname{ker}(G)-\{v\})=N(\operatorname{ker}(G))$ and hence

$$
\begin{aligned}
& d(\operatorname{ker}(G)-\{v\})=|\operatorname{ker}(G)-\{v\}|-|N(\operatorname{ker}(G)-\{v\})|= \\
& \quad=|\operatorname{ker}(G)|-1-|N(\operatorname{ker}(G))|=d_{c}(G)-1
\end{aligned}
$$

If there is some independent set $A$ in $G-v$, such that $d(A)=d_{c}(G)$, then $A$ is critical in $G$ and, hence we get the following contradiction: $v \in \operatorname{ker}(G) \subseteq A \subseteq V-\{v\}$. Therefore, $\operatorname{ker}(G)-\{v\}$ is a critical independent set of $G-v$ and

$$
d_{c}(G-v)=d(\operatorname{ker}(G)-\{v\})=d_{c}(G)-1
$$

(ii) Assume that $\operatorname{ker}(G-v) \neq \emptyset$. In part (i), we saw that $\operatorname{ker}(G)-\{v\}$ is a critical independent set of $G-v$. Hence, we get that $\operatorname{ker}(G-v) \subseteq \operatorname{ker}(G)-\{v\}$.

Remark 2.2 Actually, $\operatorname{ker}(G-v)$ may be different from $\operatorname{ker}(G)-\{v\}$; for instance, if $K_{3,2}=(A, B, E),|A|=3$, then $\operatorname{ker}\left(K_{3,2}\right)=A$ and $\operatorname{ker}\left(K_{3,2}-v\right)=\emptyset \neq \operatorname{ker}\left(K_{3,2}\right)-\{v\}$, for every $v \in A$. It is also possible $\operatorname{ker}(G)-\{v\}=\emptyset$, while $\operatorname{ker}(G-v) \neq \emptyset$; e.g., $G=C_{4}$.

By Theorem $1.2\left(\right.$ iiii), there is a matching from $N(S)$ into $S=\left\{v_{1}, v_{2}, v_{3}\right\}$, for instance, $M=\left\{v_{2} v_{5}, v_{3} v_{4}\right\}$, since $S$ is critical independent for the graph $G$ from Figure 1 . On the other hand, there is no matching from $N(S)$ into $S-v_{3}$. The case of the critical independence set $\operatorname{ker}(G)$ is more specific.

Theorem 2.3 Let $A$ be a critical independent set in a graph $G$. Then the following statements are equivalent:
(i) $A=\operatorname{ker}(G)$;
(ii) there is no set $B \subseteq N(A), B \neq \emptyset$ such that $|N(B) \cap A|=|B|$;
(iii) for each $v \in A$ there exists a matching from $N(A)$ into $A-v$.

Proof. (i) $\Longrightarrow$ (ii) By Theorem $1.2($ iii), there is a matching, say $M$, from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$. Suppose, to the contrary, that there is some non-empty set $B \subseteq N(\operatorname{ker}(G))$ such that

$$
|M(B)|=|N(B) \cap \operatorname{ker}(G)|=|B|
$$

It contradicts the fact that, by Theorem $1.2(i i), \operatorname{ker}(G)$ is a minimal critical independent set, because

$$
d(\operatorname{ker}(G)-N(B))=d(\operatorname{ker}(G)), \text { while } \operatorname{ker}(G)-N(B) \varsubsetneqq \operatorname{ker}(G)
$$

(ii) $\Longrightarrow$ (i) Suppose $A-\operatorname{ker}(G) \neq \emptyset$. By Theorem 1.2 (iii), there is a matching, say $M$, from $N(A)$ into $A$. Since there are no edges connecting vertices belonging to $\operatorname{ker}(G)$ with vertices from $N(A)-N(\operatorname{ker}(G))$, we obtain that $M(N(A)-N(\operatorname{ker}(G))) \subseteq A-\operatorname{ker}(G)$. Moreover, we have that $|N(A)-N(\operatorname{ker}(G))|=|A-\operatorname{ker}(G)|$, otherwise

$$
\begin{aligned}
|A|-|N(A)| & =(|\operatorname{ker}(G)|-|N(\operatorname{ker}(G))|)+(|A-\operatorname{ker}(G)|-|N(A)-N(\operatorname{ker}(G))|)> \\
& >(|\operatorname{ker}(G)|-|N(\operatorname{ker}(G))|)=d_{c}(G)
\end{aligned}
$$

It means that the set $N(A)-N(\operatorname{ker}(G))$ contradicts the hypothesis of (ii), because

$$
|N(A)-N(\operatorname{ker}(G))|=|A-\operatorname{ker}(G)|=|N(N(A)-N(\operatorname{ker}(G))) \cap A|
$$

Consequently, the assertion is true.
(ii) $\Longrightarrow$ (iii) By Theorem $1.2($ iii), there is a matching, say $M$, from $N(A)$ into $A$. Suppose, to the contrary, that there is no matching from $N(A)$ into $A-v$. Hence, by Hall's Theorem, it implies the existence of a set $B \subseteq N(A)$ such that $|N(B) \cap A|=|B|$, which contradicts the hypothesis of (ii).
(iii) $\Longrightarrow$ (ii) Assume, to the contrary, that there is a non-empty subset $B$ of $N(A)$ such that $|N(B) \cap A|=|B|$. Let $v \in N(B) \cap A$. Hence, we obtain that

$$
|N(B) \cap A-v|<|B| .
$$

Then, by Hall's Theorem, it is impossible to find a matching from $N(A)$ into $A-v$, in contradiction with the hypothesis of (iii).

Since $\operatorname{ker}(G)$ is a critical set, Theorem $1.2($ iii) assures that there is a matching from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$. The following result shows that there are at least two such matchings.

Corollary 2.4 For a graph $G$ the following are true:
(i) every edge $e \in(\operatorname{ker}(G), N(\operatorname{ker}(G)))$ belongs to a matching from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$;
(ii) every edge $e \in(\operatorname{ker}(G), N(\operatorname{ker}(G)))$ is not included in one matching from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$ at least.

Proof. Let $e=x y \in(\operatorname{ker}(G), N(\operatorname{ker}(G)))$, such that $x \in \operatorname{ker}(G)$. By Theorem 2.3(iii) there is a matching $M$ from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)-x$, that matches $y$ with some $z \in \operatorname{ker}(G)-x$. Clearly, $M$ is a matching from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$ that does not contain the edge $e=x y$, while $(M-\{y z\}) \cup\{x y\}$ is a matching from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$, which includes the edge $e=x y$.


Figure 2: $\operatorname{core}\left(G_{1}\right)=\{a, b\}, \operatorname{core}\left(G_{2}\right)=\{q, x, y, z\}, \operatorname{core}\left(G_{3}\right)=\{t, u, v, w\}$.

Let us notice that the graphs $G_{1}, G_{2}$ from Figure 2 have: $\operatorname{ker}\left(G_{1}\right)=\operatorname{core}\left(G_{1}\right)$, $\operatorname{ker}\left(G_{2}\right)=\{x, y, z\} \subset \operatorname{core}\left(G_{2}\right)$, and both core $\left(G_{1}\right)$ and core $\left(G_{2}\right)$ are critical sets of maximum size. The graph $G_{3}$ from Figure 2 has $\operatorname{ker}\left(G_{3}\right)=\{u, v\}$, the set $\{t, u, v\}$ as a critical independent set of maximum size, while core $\left(G_{3}\right)=\{t, u, v, w\}$ is not a critical set. If $S_{\min }$ denotes an inclusion minimal independent set with $d\left(S_{\min }\right)>0$, one can see that: $S_{\min }=\operatorname{ker}\left(G_{1}\right)$ for $G_{1}$, while the graph $G_{2}$ in the same figure has $S_{\text {min }} \in\{\{x, y\},\{x, z\},\{y, z\}\}$ and $\operatorname{ker}\left(G_{2}\right)=\{x, y\} \cup\{x, z\} \cup\{y, z\}$.

In [5] we have shown that $\operatorname{ker}(G)$ is equal to the intersection of all critical, independent or not, sets of $G$.

Theorem 2.5 For every graph $G$

$$
\operatorname{ker}(G)=\cup\left\{S_{0}: S_{0} \text { is an inclusion minimal independent set with } d\left(S_{0}\right)>0\right\}
$$

Proof. Let $A$ be a critical set and $S_{0}$ be an inclusion minimal independent set such that $d\left(S_{0}\right)>0$. Then, Theorem $1.2(i)$ implies

$$
d\left(A \cup S_{0}\right)+d\left(A \cap S_{0}\right) \geq d(A)+d\left(S_{0}\right)>d(A)=d_{c}(G)
$$

Since $S_{0}$ is an inclusion minimal independent set such that $d\left(S_{0}\right)>0$, we obtain that if $A \cap S_{0} \neq S_{0}$, then $d\left(A \cap S_{0}\right) \leq 0$. Hence

$$
d(A)=d_{c}(G) \geq d\left(A \cup S_{0}\right) \geq d(A)+d\left(S_{0}\right)>d(A)
$$

which is impossible. Therefore, $S_{0} \subseteq A$ for every critical set $A$. Consequently,

$$
S_{0} \subseteq \cap\{B: B \text { is a critical set of } G\}=\operatorname{ker}(G)
$$

Thus we obtain
$\cup\left\{S_{0}: S_{0}\right.$ is an inclusion minimal independent set such that d $\left.\left(S_{0}\right)>0\right\} \subseteq \operatorname{ker}(G)$.
Conversely, it is enough to show that every vertex from $\operatorname{ker}(G)$ belongs to some inclusion minimal independent set with positive difference. Let $v \in \operatorname{ker}(G)$. According to Theorem 2.3 (iii) there exists a matching, say $M$, from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)-v$.

Let us build the following sequence of sets

$$
\{v\} \subseteq M(N(v)) \subseteq \ldots \subseteq[M N]^{k}(v) \subseteq \ldots
$$

where $M N$ is a superposition of two mappings $N: 2^{V} \longrightarrow 2^{V}(N(A)$ is the neighborhood of $A$ ) and $M: 2^{N(\operatorname{ker}(G))} \longrightarrow 2^{\operatorname{ker}(G)}(M(A)$ is set of the vertices matched by $M$ with vertices belonging to $A$ ).

Since the set $\operatorname{ker}(G)$ is finite, there is an index $j$ such that $[M N]^{j}(v)=[M N]^{j+1}(v)$. Hence $\left|N\left([M N]^{j}(v)\right)\right|=\left|[M N]^{j}(v)\right|-1$. In other words, we found an independent set, namely, $[M N]^{j}(v)$ such that $v \in[M N]^{j}(v)$ and $d\left([M N]^{j}(v)\right)=1$. Therefore, there must exist an inclusion minimal independent set $X$ such that $v \in X$ and $d(X)=1$.

Remark 2.6 In a graph $G$, the union of all minimum cardinality independent sets $S$ with $d(S)>0$ may be a proper subset of $\operatorname{ker}(G)$; e.g., the graph $G$ in Figure 3, that has $\{x, y\} \subset \operatorname{ker}(G)=\{x, y, u, v, w\}$.


Figure 3: Both $S_{1}=\{x, y\}$ and $S_{2}=\{u, v, w\}$ are inclusion minimal independent sets satisfying $d(S)>0$.

Proposition $2.7 \min \left\{\left|S_{0}\right|: d\left(S_{0}\right)>0, S_{0} \in \operatorname{Ind}(G)\right\} \leq|\operatorname{ker}(G)|-d_{c}(G)+1$.
Proof. Since $\operatorname{ker}(G)$ is a critical independent set, Theorem 1.2 (iii) implies that there is a matching, say $M$, from $N(\operatorname{ker}(G))$ into $\operatorname{ker}(G)$. Let $X=M(N(\operatorname{ker}(G)))$. Then $d(X)=0$. For every $v \in \operatorname{ker}(G)-X$ we have

$$
N(\operatorname{ker}(G)) \subseteq N(X) \subseteq N(X \cup\{v\}) \subseteq N(\operatorname{ker}(G))
$$

Hence we get $|X \cup\{v\}|-|N(X \cup\{v\})|=1$, while $|X \cup\{v\}|=|\operatorname{ker}(G)|-d_{c}(G)+1$.
Remark 2.8 All the inclusion minimal independent sets $S$, with $d(S)>0$, of the graph $H$ from Figure 3 are of the same size. However, there are inclusion minimal independent sets $S$ with $d(S)>0$, of different cardinalities; e.g., the graph $G$ from Figure 3 ,

Proposition 2.9 If $S_{0}$ is an inclusion minimal independent set with $d\left(S_{0}\right)>0$, then $d\left(S_{0}\right)=1$.

Proof. For each $v \in S_{0}$, it follows that $N\left(S_{0}-v\right)=N\left(S_{0}\right)$, otherwise,

$$
\begin{gathered}
d\left(S_{0}-v\right)=\left|S_{0}-v\right|-\left|N\left(S_{0}-v\right)\right|= \\
=\left|S_{0}\right|-1-\left|N\left(S_{0}-v\right)\right| \geq\left|S_{0}\right|-\left|N\left(S_{0}\right)\right|>0
\end{gathered}
$$

i.e., $S_{0}$ is not an inclusion minimal independent set with positive difference.

Since $S_{0}$ is an inclusion minimal independent set with positive difference, we know that $d\left(S_{0}-v\right) \leq 0$. On the other hand, it follows from the equality $N\left(S_{0}-v\right)=N\left(S_{0}\right)$ that

$$
d\left(S_{0}-v\right)=\left|S_{0}-v\right|-\left|N\left(S_{0}-v\right)\right|=\left|S_{0}\right|-1-\left|N\left(S_{0}\right)\right|=d\left(S_{0}\right)-1 \leq 0
$$

Consequently, $0<\left|S_{0}\right|-\left|N\left(S_{0}\right)\right| \leq 1$, which means that $\left|S_{0}\right|-\left|N\left(S_{0}\right)\right|=1$.

Remark 2.10 The converse of Proposition 2.9 is not true. For instance, $S=\{x, y, u\}$ is independent in the graph $G$ from Figure 3 and $d(S)=1$, but $S$ is not minimal with this property.

Proposition 2.11 If $S_{i}, i=1,2, \ldots, k, k \geq 1$, are inclusion minimal independent sets, such that $d\left(S_{i}\right)>0, S_{i} \nsubseteq \bigcup_{j=1, j \neq i}^{k} S_{j}, 1 \leq i \leq k$, then $d\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right) \geq k$.

Proof. For $k=1$ the claim has been treated in Proposition [2.9] where we have achieved a stronger result.

We continue by induction on $k$.
Let $k=2$. Since $S_{1} \neq S_{1} \cap S_{2} \subset S_{1}$, it follows that $d\left(S_{1} \cap S_{2}\right) \leq 0$. Hence, Theorem 1.2 (i) and Proposition 2.9 imply

$$
d\left(S_{1} \cup S_{2}\right) \geq d\left(S_{1} \cup S_{2}\right)+d\left(S_{1} \cap S_{2}\right) \geq d\left(S_{1}\right)+d\left(S_{2}\right)=2
$$

Assume that the assertion is true for each $k \geq 2$, and let $\left\{S_{i}, 1 \leq i \leq k+1\right\}$ be a family of inclusion minimal independent sets with

$$
d\left(S_{i}\right)>0 \text { and } S_{i} \nsubseteq \bigcup_{j=1, j \neq i}^{k+1} S_{j}, 1 \leq i \leq k+1
$$

Since $S_{k+1} \neq\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right) \cap S_{k+1} \subset S_{k+1}$, we obtain that

$$
d\left(\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right) \cap S_{k+1}\right) \leq 0
$$

Further, using the supermodularity of the function $d$ and Proposition 2.9, we get

$$
\begin{gathered}
d\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k} \cup S_{k+1}\right) \geq \\
\geq d\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k} \cup S_{k+1}\right)+d\left(\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right) \cap S_{k+1}\right) \geq \\
\geq d\left(S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right)+d\left(S_{k+1}\right) \geq k+1
\end{gathered}
$$

as required.
Remark 2.12 The sets $S_{1}=\left\{v_{1}, v_{2}\right\}, S_{2}=\left\{v_{2}, v_{3}\right\}, S_{3}=\left\{v_{3}, v_{4}\right\}$ are inclusion minimal independent sets of the graph $H$ from Figure 3, such that

$$
d\left(S_{i}\right)>0, S_{i} \nsubseteq \bigcup_{j=1, j \neq i}^{3} S_{j}, i=1,2,3
$$

Notice that both families $\left\{S_{1}, S_{2}\right\},\left\{S_{1}, S_{3}\right\}$ have two elements, and d $\left(S_{1} \cup S_{2}\right)=2$, while $d\left(S_{1} \cup S_{3}\right)>2$.

## 3 Conclusions

In this paper we investigate structural properties of $\operatorname{ker}(G)$.
Having in view Theorem 2.5, notice that the graph:

- $G_{1}$ from Figure 2 has only one inclusion minimal independent set $S$ such that $d(S)>0$, and $d_{c}\left(G_{1}\right)=1$;
- $G$ from Figure 3 has only two inclusion minimal independent sets $S$ such that $d(S)>0$, and $d_{c}(G)=2$;
- $H$ from Figure 3 has 6 inclusion minimal independent sets $S$ such that $d(S)>0$, and $d_{c}(H)=3$.

These remarks motivate the following.
Conjecture 3.1 The number of inclusion minimal independent set $S$ such that $d(S)>0$ is greater or equal to $d_{c}(G)$.

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