

ON PSEUDO-CONVEX PARTITIONS OF A PLANAR POINT SET

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ABSTRACT. Aichholzer et al. [*Graphs and Combinatorics*, Vol. 23, 481-507, 2007] introduced the notion of pseudo-convex partitioning of planar point sets and proved that the pseudo-convex partition number $\psi(n)$ satisfies, $\frac{3}{4}\lfloor\frac{n}{4}\rfloor \leq \psi(n) \leq \lceil\frac{n}{4}\rceil$. In this paper we prove that $\psi(13) = 3$, which immediately improves the upper bound on $\psi(n)$ to $\lceil\frac{3n}{13}\rceil$, thus answering a question posed by Aichholzer et al. in the same paper.

1. INTRODUCTION

In 1978 Erdős [4] asked whether for every positive integer k , there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k -gon* or a *k -hole*. Esther Klein showed $H(4) = 5$ and Harborth [6] proved that $H(5) = 10$. Horton [7] showed that it is possible to construct arbitrarily large set of points without a 7-hole, proving that $H(k)$ does not exist for $k \geq 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [5] and independently by Nicolás [11]. Later Valtr [14] gave a simpler version of Gerken's proof.

Any two empty convex polygons are said to be *disjoint* if their convex hulls do not intersect. Let $H(k, \ell)$, $k \leq \ell$ denote the smallest integer such that any set of $H(k, \ell)$ points in the plane, no three on a line, contains both a k -hole and a ℓ -hole which are disjoint. Clearly, $H(3, 3) = 6$ and Horton's result [7] implies that $H(k, \ell)$ does not exist for all $\ell \geq 7$. It is known that $H(3, 4) = 7$ [13], $H(3, 5) = 10$ [9], $H(4, 4) = 9$ [10], and $H(4, 5) = 12$ [3]. Recently, Hosono and Urabe [8] showed that $H(5, 5) \geq 17$, and Bhattacharya and Das [2] proved $H(5, 5) \leq 19$.

The problem of partitioning planar point sets with disjoint holes was first addressed by Urabe [13]. For any set S of points in the plane, denote the *convex hull* of S by $CH(S)$, and cardinality of S by $|S|$. Given a set S of n points in the plane, no three on a line, a *disjoint convex partition* of S is a partition of S into subsets S_1, S_2, \dots, S_t , with $\sum_{i=1}^t |S_i| = n$, such that for each $i \in \{1, 2, \dots, t\}$, $CH(S_i)$ forms a $|S_i|$ -gon and $CH(S_i) \cap CH(S_j) = \emptyset$, for any pair of distinct indices i, j . Observe that in any disjoint convex partition of S , the set S_i forms a $|S_i|$ -hole and the holes formed by the sets S_i and S_j are disjoint for any pair of distinct indices i, j . Let $\kappa(S)$ denote the minimum number of disjoint holes in any disjoint convex partition of S . Define $\kappa(n) = \max_S \kappa(S)$, where the maximum is taken over all sets S of n points. $\kappa(n)$ is called the *convex partition number* for all sets S of fixed size n , and it is bounded by $\lceil\frac{n-1}{4}\rceil \leq \kappa(n) \leq \lceil\frac{5n}{18}\rceil$. The lower bound was given by Urabe [13] and the upper bound by Hosono and Urabe [10]. The lower bound was later improved to $\lceil\frac{n+1}{4}\rceil$ by Xu and Ding [15].

A pseudo-triangle is a simple polygon with exactly three vertices having interior angles less than 180° , and is considered to be the natural counterpart of a convex polygon. These have been studied

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recently in the context of pseudo-triangulations, which are tessellation of the plane with pseudo-triangles. They provide sparser tessellations than triangulations but retain many of the desirable properties of triangulations. Pseudo-triangulations have received considerable attention in the last few years for applications in areas like motion planning, collision detection, ray shooting, rigidity, or visibility (refer to Rote et al. [12] for a survey of the different properties of pseudo-triangulations and its various applications).

A pseudo-triangle with ℓ vertices is called a ℓ -pseudo-triangle, and a set is said to contain an empty ℓ -pseudo-triangle if there exists a subset of ℓ points forming a pseudo-triangle which contains no point of the set in its interior. Any two empty pseudo-triangles, or a hole and an empty pseudo-triangle are said to be disjoint if their vertex sets as well as their interiors are disjoint. Recently, Aichholzer et al. [1] introduced the problem of partitioning planar point sets with disjoint holes or empty pseudo-triangles. Given a set S of n points in the plane, no three on a line, a *pseudo-convex partition* of S is a partition of S into subsets S_1, S_2, \dots, S_t , with $\sum_{i=1}^t |S_i| = n$, such that for each $i \in \{1, 2, \dots, t\}$, the set S_i forms a $|S_i|$ -hole or a $|S_i|$ -pseudo-triangle, the holes or pseudo-triangles formed by the sets S_i and S_j are disjoint for any pair of distinct indices i, j . If $\psi(S)$ denotes the minimum number of disjoint holes or empty pseudo-triangles in any pseudo convex partition of S , then the *pseudo-convex partition number* is defined as $\psi(n) = \max_S \psi(S)$, where the maximum is taken over all sets S of n points.

Aichholzer et al. [1] showed that the pseudo-convex partition number $\psi(n)$ satisfies: $\frac{3}{4} \lfloor \frac{n}{4} \rfloor \leq \psi(n) \leq \lceil \frac{n}{4} \rceil$. The upper bound follows from the simple observation that every set of 4 points forms either an 4-hole or an empty 4-pseudo-triangle. In fact, using computer-aided search, Aichholzer et al. [1] obtained bounds on the pseudo-convex partition number $\psi(n)$ for small point sets. However, they were unable to find the exact value of $\psi(13)$, and mentioned the possibility of a non-trivial upper bound on $\psi(n)$ by conjecturing that $\psi(13) = 3$. In this paper, we answer this question in the affirmative, thus proving that $\psi(n) \leq \lceil \frac{3n}{13} \rceil$. Our proof is geometric and does not rely on computer-aided search over the order type database. We identify a number of simple necessary conditions that allows the desired partitioning, and then proceed to show that each set of 13 points must satisfies one of these conditions.

2. NOTATIONS AND DEFINITIONS

We first introduce the definitions and notations required for the remaining part of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the *convex hull* of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\tilde{\mathcal{I}}(CH(S))$, respectively. A region R in the plane is said to be *empty* in S if R contains no elements of S in its interior. Moreover, for any set T , $|T|$ denotes the cardinality of T .

By $\mathcal{P} := p_1 p_2 \dots p_m$ we denote the region bounded by the simple polygon with vertices $\{p_1, p_2, \dots, p_m\}$ ordered anti-clockwise. Let $\mathcal{V}(\mathcal{P})$ denote the set of vertices $\{p_1, p_2, \dots, p_m\}$ and $\mathcal{I}(\mathcal{P})$ the interior of \mathcal{P} . A finite set of points Z is said to *span* a simple polygon \mathcal{P} if $\mathcal{V}(\mathcal{P}) = Z$.

The j -th *convex layer* of S , denoted by $L\{j, S\}$, is the set of points that lie on the boundary of $CH(S \setminus \{\bigcup_{i=1}^{j-1} L\{i, S\}\})$, where $L\{1, S\} = \mathcal{V}(CH(S))$. If $p, q \in S$ are such that pq is an edge of the convex hull of the j -th layer, then the open halfplane bounded by the line pq and not containing any point of $S \setminus \{\bigcup_{i=1}^{j-1} L\{i, S\}\}$ will be referred to as the *outer* halfplane induced by the edge pq (see Figure 1(a)).

For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ (respectively $\mathcal{H}_c(pq, r)$) denotes the open (respectively closed) halfplane bounded by the line pq containing the point r . Similarly, $\overline{\mathcal{H}}(pq, r)$ (respectively

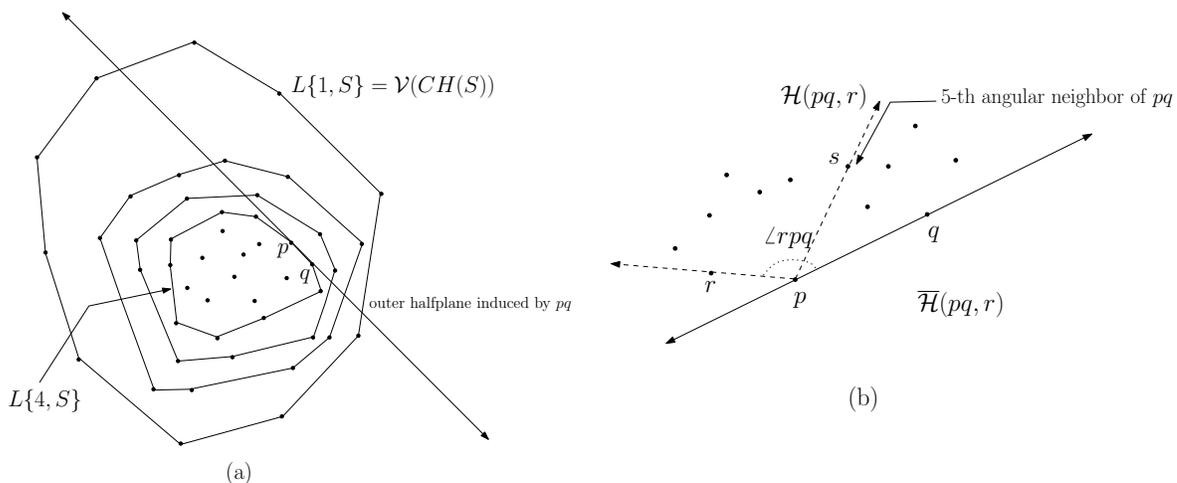


FIGURE 1. (a) Convex layers and outer halfplane, and (b) Cone and nearest angular neighbors.

$\overline{\mathcal{H}}_c(pq, r)$ is the open (respectively closed) halfplane bounded by pq not containing the point r . Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$. A point $s \in Cone(rpq) \cap S$ is called the *nearest angular neighbor* of \overrightarrow{pq} in $Cone(rpq)$ if $Cone(spq)$ is empty in S . Similarly, for any convex region R a point $s \in R \cap S$ is called the *nearest angular neighbor* of \overrightarrow{pq} in R if $Cone(spq) \cap R$ is empty in S . More generally, for any positive integer k , a point $s \in S$ is called the *k -th angular neighbor* of \overrightarrow{pq} whenever $Cone(spq) \cap R$ contains exactly $k - 1$ points of S in its interior (see Figure 1(b)).

3. PSEUDO-CONVEX PARTITIONING

Aichholzer et al. [1] showed that $\psi(n) \leq \lceil \frac{n}{4} \rceil$. They also observed that $3 \leq \psi(13) \leq 4$, and mention the possibility of a better upper bound of $\lceil \frac{3n}{13} \rceil$ on $\psi(n)$ by conjecturing that $\psi(13) = 3$.

In the following theorem we settle this conjecture in the affirmative.

Theorem 3.1. *Every set of 13 points in the plane, in general position, can be partitioned into three sets each of which span either a hole or an empty pseudo-triangle which are mutually disjoint. In other words, $\psi(13) = 3$.*

Theorem 3.1 immediately establishes a non-trivial upper bound on $\psi(n)$, as suggested by Aichholzer et al. [1]:

Theorem 3.2. $\psi(n) \leq \lceil \frac{3n}{13} \rceil$.

Proof. Let S be a set of n points in the plane, no three of which are collinear. By a horizontal sweep we can divide the plane into $\lceil \frac{n}{13} \rceil$ disjoint strips, of which $\lfloor \frac{n}{13} \rfloor$ contain 13 points each and one remaining strip R , with $|R| < 13$. The strips having 13 points, can be partitioned into three disjoint holes or empty pseudo-triangles by Theorem 3.1. Since $|R| < 13$, at most $\lceil \frac{3|R|}{13} \rceil$ disjoint holes or empty pseudo-triangles are needed to partition R , thus proving that $\psi(n) \leq \lceil \frac{3n}{13} \rceil$. \square

4. PROOF OF THEOREM 3.1

Let S be a set of 13 points in the plane in general position. A partition of S into three disjoint subsets S_1, S_2, S_3 is called *admissible* if each S_i , $i \in \{1, 2, 3\}$, is either empty or it forms an $|S_i|$ -hole

or empty $|S_i|$ -pseudo-triangle, such that the holes or pseudo-triangles formed by the sets S_i and S_j are disjoint for any pair of distinct indices $i \neq j$. The set S is said to be *admissible* if there exists an admissible partition of S . To prove Theorem 3.1 we need to exhibit an admissible partition of S , for all sets of 13 points in the plane, in general position.

Observe that any set of 4 points in the plane always spans a convex quadrilateral or a 4-pseudo-triangle. Hence, we have the following observation.

Observation 1. For every integer $k \geq 1$, we have $\psi(4k) \leq k$. □

Observation 2. S is admissible if some outer halfplane induced by an edge of the second convex layer contains more than two points of $\mathcal{V}(CH(S))$.

Proof. Suppose some outer halfplane induced by an edge of the second layer contains more than two points of $\mathcal{V}(CH(S))$. This means that there exists two points $p, q \in S$, such that the line segment pq is an edge of the second convex layer and $|\overline{\mathcal{H}}(pq, r) \cap S| \geq 3$, where $r \in L\{2, S\} \setminus \{p, q\}$. Then $\overline{\mathcal{H}}_c(pq, r) \cap S$ spans a k -hole, with $k \geq 5$. The remaining eight points of S all lie in the halfplane $\mathcal{H}(pq, r)$. As $|\mathcal{H}(pq, r) \cap S| \leq 8$, the points in $\mathcal{H}(pq, r) \cap S$ can be partitioned using at most two disjoint holes or empty pseudo-triangles. □

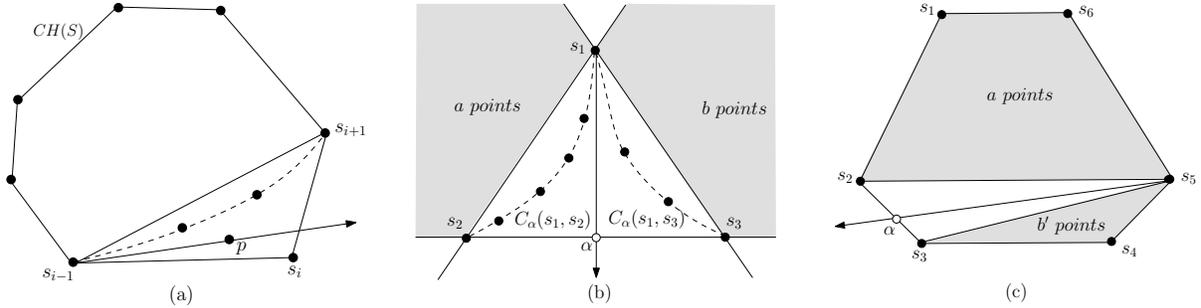


FIGURE 2. Illustration for the proof of (a) Observation 3, (b) Observation 4, and (c) Observation 5.

Observation 3. Let $\mathcal{V}(CH(S)) = \{s_1, s_2, \dots, s_k\}$, with the vertices in counter-clockwise order. If there exists a point $s_i \in \mathcal{V}(CH(S))$, such that $|\mathcal{I}(s_{i-1}s_i s_{i+1}) \cap S| \geq 2$, then S is admissible, where the indices are taken modulo k .

Proof. Suppose $|\mathcal{I}(s_{i-1}s_i s_{i+1}) \cap S| = a \geq 2$, for some $i = 1, 2, \dots, k$. Let $p \in S$ be the first angular neighbor of $\overline{s_{i-1}s_i}$ in $Cone(s_i s_{i-1} s_{i+1})$. Let $Z = (\mathcal{H}_c(s_{i-1}s_{i+1}, s_i) \cap S) \setminus \{p, s_i\}$, and suppose $\mathcal{V}(CH(Z)) = \{s_{i-1}, z_1, \dots, z_b, s_{i+1}\}$. As $a \geq 2$, $b \geq 1$, and $s_{i+1}z_1 \dots z_b s_{i-1} p s_i$ is an empty ℓ -pseudo-triangle with $\ell \geq 5$. Let $S' = S \setminus \{s_{i+1}, z_1, \dots, z_b, s_{i-1}, p, s_i\}$, that is, the points of S without the points of the considered ℓ -pseudo-triangle. As $|S'| \leq 8$ and $CH(S')$ does not intersect the ℓ -pseudo-triangle, S' can be partitioned using at most two holes or empty pseudo-triangles from Observation 1. □

Observation 4. S is admissible if there exists three distinct points $s_1, s_2, s_3 \in S$ satisfying the following two conditions:

- (A) $|\overline{\mathcal{H}}(s_1 s_2, s_3) \cap \overline{\mathcal{H}}(s_1 s_3, s_2) \cap S| = 0$ and $|\overline{\mathcal{H}}(s_2 s_3, s_1) \cap S| = 0$.
- (B) $|\overline{\mathcal{H}}(s_1 s_2, s_3) \cap S| \leq 4$ and $|\overline{\mathcal{H}}(s_1 s_3, s_2) \cap S| \leq 4$.

Proof. Let $|\overline{\mathcal{H}}(s_1s_2, s_3) \cap S| = a \leq 4$ and $|\overline{\mathcal{H}}(s_1s_3, s_2) \cap S| = b \leq 4$. Then, $|\mathcal{I}(s_1s_2s_3) \cap S| = 10 - (a+b)$. Then, there exists a point $\alpha \notin S$ on the line segment s_2s_3 such that $|\mathcal{I}(s_1s_2\alpha) \cap S| = 5 - a$ and $|\mathcal{I}(s_1s_3\alpha) \cap S| = 5 - b$ (see Figure 2(b)). Let $C_\alpha(s_1, s_2) = CH(\{\mathcal{I}(s_1s_2\alpha) \cap S\} \cup \{s_1, s_2\})$ and $C_\alpha(s_1, s_3) = CH(\{\mathcal{I}(s_1s_3\alpha) \cap S\} \cup \{s_1, s_3\})$. Now, since both $a, b \leq 4$, we have $|\mathcal{V}(C_\alpha(s_1, s_2))| \geq 3$ and $|\mathcal{V}(C_\alpha(s_1, s_3))| \geq 3$. Thus, $S_1 = \mathcal{V}(C_\alpha(s_1, s_2)) \cup \mathcal{V}(C_\alpha(s_1, s_3))$ spans an empty ℓ -pseudo-triangle, with $\ell \geq 5$. Moreover, both $S_2 = \mathcal{I}(C_\alpha(s_1, s_2)) \cup \{\overline{\mathcal{H}}(s_1s_2, s_3) \cap S\}$, and $S_3 = \mathcal{I}(C_\alpha(s_1, s_3)) \cup \{\overline{\mathcal{H}}(s_1s_3, s_2) \cap S\}$ lie inside two disjoint convex regions containing at most 4 points each. Therefore, the partition $S = S_1 \cup S_2 \cup S_3$ is admissible. \square

If there exist three distinct points $s_1, s_2, s_3 \in S$ satisfying conditions (A) and (B) of Observation 4, then the three points are said to form a *heart* with the line segment s_2s_3 as *base* and the point s_1 as *pivot*. The set S is admissible if any three of its points form a *heart*. Observe that if $s_1, s_2, s_3 \in \mathcal{V}(CH(S))$ and some edge of the triangle $s_1s_2s_3$ is also an edge of $CH(S)$, then condition (A) is automatically satisfied. In such cases, s_1, s_2, s_3 form a *heart* whenever condition (B) holds.

Equipped with the above three observations, we now proceed to prove the admissibility of S . The proof of the admissibility of S is presented in three separate sections. The first section deals with the cases $|CH(S)| \leq 5$, the second section with the case $|CH(S)| = 6$, and the third considers the cases $|CH(S)| \geq 7$.

Let $\mathcal{V}(CH(S)) = \{s_1, s_2, \dots, s_k\}$, with the vertices taken in the counter-clockwise order. While indexing a set of points from $\mathcal{V}(CH(S))$, we identify indices modulo k .

4.1. $|CH(S)| \leq 5$.

Lemma 4.1. *S is admissible whenever $|CH(S)| \leq 5$.*

Proof. Observe that if $|CH(S)| = 3$, then the admissibility of S is a direct consequence of Observation 3. Now, we consider the following two cases based on the size of $|CH(S)|$:

Case 1: $|CH(S)| = 4$. This implies that $|\tilde{\mathcal{I}}(CH(S))| = 9$. Therefore, $|\mathcal{I}(s_2s_3s_4) \cap S| \geq 2$ or $|\mathcal{I}(s_1s_2s_4) \cap S| \geq 2$. The admissibility of S then follows from Observation 3.

Case 2: $|CH(S)| = 5$. Suppose that $|\mathcal{I}(s_1s_2s_3) \cap S| = a$, and $|\mathcal{I}(s_1s_4s_5) \cap S| = b$. If $a \geq 2$ or $b \geq 2$, the admissibility of S is guaranteed from Observation 3. Therefore, assume that both $a, b \leq 1$. This implies that $|\overline{\mathcal{H}}(s_1s_3, s_4) \cap S| \leq 2$ and $|\overline{\mathcal{H}}(s_1s_4, s_3) \cap S| \leq 2$. Thus, the three points s_1, s_3, s_4 satisfy Conditions (A) and (B) of Observation 4 and form a *heart* with s_3s_4 as *base* and s_1 as *pivot*. Thus, the admissibility of S follows. \square

4.2. $|CH(S)| = 6$. For any $i \in \{1, 2, 3\}$, the diagonal $d := s_i s_{i+3}$ of the hexagon $s_1s_2s_3s_4s_5s_6$ is called an (a, b) -*splitter* of $CH(S)$, where $a \leq b$ are integers, if either $|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap \tilde{\mathcal{I}}(CH(S))| = a$ and $|\overline{\mathcal{H}}(s_i s_{i+3}, s_{i+1}) \cap \tilde{\mathcal{I}}(CH(S))| = b$, or $|\overline{\mathcal{H}}(s_i s_{i+3}, s_{i+1}) \cap \tilde{\mathcal{I}}(CH(S))| = a$ and $|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap \tilde{\mathcal{I}}(CH(S))| = b$.

We now have the following observation:

Observation 5. If any one of the three diagonal s_2s_5 , s_1s_4 , and s_3s_6 is not a $(3, 4)$ -splitter of $CH(S)$, then S is admissible.

Proof. It suffices to prove that S is admissible whenever s_2s_5 is not a $(3, 4)$ -splitter of $CH(S)$. Suppose the diagonal s_2s_5 is a (a, b) -splitter of $CH(S)$, with $a \leq 2$, and $b \geq 5$. Refer to Figure 2(c). W.l.o.g. assume that $|\mathcal{I}(s_1s_2s_5s_6) \cap S| = a$ and $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = b$. Now, since $b \geq 5$, by the pigeon-hole principle, either $\mathcal{I}(s_2s_3s_5)$ or $\mathcal{I}(s_3s_4s_5)$ contains at least 3 points of S . However, if $|\mathcal{I}(s_3s_4s_5) \cap S| \geq 2$, then Observation 3 guarantees the admissibility of S . Therefore, assume

that $|\mathcal{I}(s_3s_4s_5) \cap S| = b' \leq 1$. This implies that $|\overline{\mathcal{H}}(s_5s_3, s_2) \cap S| \leq 2$ and $|\overline{\mathcal{H}}(s_5s_2, s_3) \cap S| \leq 4$, and the three points s_2, s_3, s_5 form a *heart* with s_2s_3 as *base* and s_5 as *pivot* (see Figure 2(c)). The admissibility of S thus follows from Observation 4. \square

In light of Observation 5, it suffices to assume that the three diagonals s_2s_5, s_1s_4 , and s_3s_6 are (3,4)-splitters of $CH(S)$. Consider the partition of the interior of $CH(S)$ by the three diagonals into 7 disjoint regions R_i as shown in Figure 3(a). Let $|R_i|$ denote the number of points of S inside region R_i .

Now, we have the following observation:

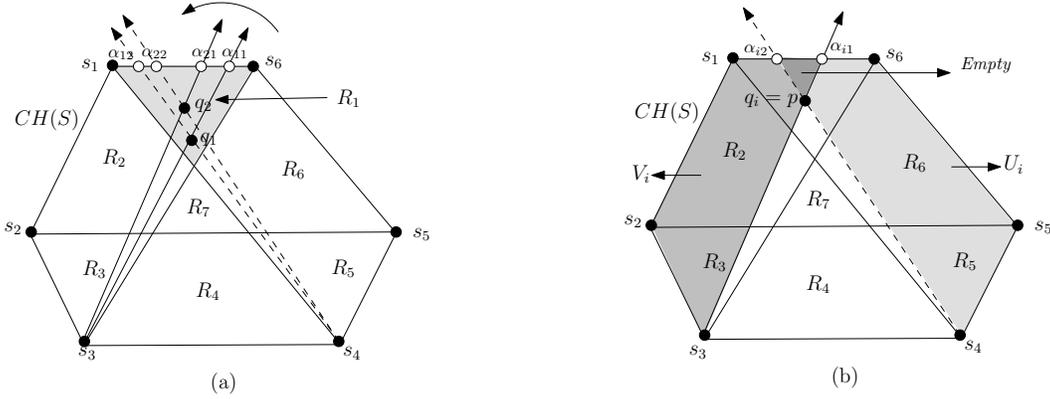


FIGURE 3. Every diagonal of $CH(S)$ forms a (3,4)-splitter of the hexagon $s_1s_2s_3s_4s_5s_6$: (a) Illustration for the proof of Observation 6, (b) The points p, s_3, s_4 form a *heart* with s_3s_4 as *base* and p as *pivot*.

Observation 6. If $|R_4| + |R_7| \geq 2$ and $|R_1| \geq 1$, then there exists a point $p \in R_1 \cap S$ such that the three points p, s_3, s_4 form a heart with s_3s_4 as base with p as pivot.

Proof. Let $|R_4| + |R_7| = a \geq 2$. Now, since both the diagonals s_1s_4 and s_3s_6 are (3,4)-splitters of $CH(S)$, we have $|R_2| + |R_3| = b \leq 2$, $|R_5| + |R_6| = b' \leq 2$, and $|R_1| + |R_5| + |R_6| \geq 3$. Let $q_1 \in R_1 \cap S$ be the $(3-b)$ -th angular neighbor of $\overrightarrow{s_3s_6}$ in R_1 . Let $U_1 = (\text{Cone}(q_1s_3s_4) \setminus \mathcal{I}(q_1s_3s_4)) \cap S$ and $V_1 = \mathcal{H}(s_3q_1, s_2) \cap S$, and α_{11} and α_{12} are the points where the rays $\overrightarrow{s_3q_1}$ and $\overrightarrow{s_4q_1}$ intersect the boundary $CH(S)$, respectively (Figure 3(a)). Therefore, $|U_1| \leq 4$ and $|V_1| \leq 4$. Now, if $\text{Cone}(\alpha_{11}q_1\alpha_{12}) \cap S$ is empty, then $q_1 (= p)$ and the result follows.

Otherwise, suppose that $\text{Cone}(\alpha_{11}q_1\alpha_{12}) \cap S$ is non-empty. Let $q_2 \in R_1 \cap S$ be the nearest angular neighbor of $\overrightarrow{s_3q_1}$ in $\text{Cone}(\alpha_{11}q_1\alpha_{12})$. Let α_{21} and α_{22} be the points where the rays $\overrightarrow{s_3q_2}$ and $\overrightarrow{s_4q_2}$ intersect the boundary $CH(S)$, respectively. Define, $U_2 = (\text{Cone}(q_2s_3s_4) \setminus \mathcal{I}(q_2s_3s_4)) \cap S$ and $V_2 = \mathcal{H}(s_3q_2, s_2) \cap S$. Observe that $U_2 \subseteq U_1$ and $V_2 \subseteq V_1$, and hence, $|U_2| \leq 4$ and $|V_2| \leq 4$. Therefore, if $\text{Cone}(\alpha_{21}q_2\alpha_{22}) \cap S$ is empty, then $q_2 (= p)$ is the required point.

If $\text{Cone}(\alpha_{21}q_2\alpha_{22}) \cap S$ is non-empty, we repeat the same procedure again, until we get a point $p (= q_i) \in R_1 \cap S$ with $|U_i| \leq 4$, $|V_i| \leq 4$, and $|\text{Cone}(\alpha_{i1}q_i\alpha_{i2}) \cap S| = 0$, where q_i is the nearest angular neighbor of $\overrightarrow{s_3q_{i-1}}$ in $R_1 \cap S$, $U_i = (\text{Cone}(q_i s_3 s_4) \setminus \mathcal{I}(q_i s_3 s_4)) \cap S$, $V_i = \mathcal{H}(s_3 q_i, s_2) \cap S$, and α_{i1} and α_{i2} are the points where the rays $\overrightarrow{s_3 q_i}$ and $\overrightarrow{s_4 q_i}$ intersect the boundary $CH(S)$, respectively (see Figure 3(b)). \square

The admissibility of S when $|CH(S)| = 6$ is now proved by considering the following three cases:

Case 1: $|R_4| + |R_7| \geq 2$ and $|R_1| \geq 1$. In this case, Observation 6 guarantees the existence of a point $p \in R_1 \in S$ such that the three points p, s_3, s_4 form a *heart* with s_3s_4 as *base* and p as *pivot* (see Figure 3(b)).

Case 2: $|R_4| + |R_7| \geq 2$ and $|R_1| = 0$. This implies that $|R_2| + |R_3| \leq 2$, since s_1s_4 is a (3, 4)-splitter of $CH(S)$. Thus, $|R_1| + |R_2| + |R_3| \leq 2$, which contradicts the assumption that diagonal s_3s_6 is a (3, 4)-splitter of $CH(S)$.

Case 3: If the previous two cases do not hold, then by symmetry we must have $|R_2| + |R_7| \leq 1$, $|R_4| + |R_7| \leq 1$, and $|R_6| + |R_7| \leq 1$. Therefore, $|R_2| + |R_4| + |R_6| + |R_7| \leq 3$ which implies that $|R_1| + |R_3| + |R_5| \geq 4$. By the pigeon-hole principle, one of the three regions R_1, R_3 , and R_5 contains at least two points S . W.l.o.g., assume $|R_1| \geq 2$. This implies that $|R_2| + |R_3| \leq 2$, and hence $|R_4| + |R_7| \geq 1$. Combining this with the given inequality we get, $|R_4| + |R_7| = 1$, $|R_2| + |R_3| = 2$, and $|R_1| = 2$. For $\{p\} \in (R_4 \cup R_7) \cap S$, the three points s_1, p, s_6 form a *heart* with s_1s_6 as *base* and p as *pivot*.

4.3. $|CH(S)| = 7$. In this section we proof the admissibility of S when $|CH(S)| = 7$. As before assume that $L\{2, S\} = \{p_1, p_2, \dots, p_m\}$, where the vertices are taken in counter-clockwise order, and the indices are to be identified modulo m .

Lemma 4.2. *S is admissible whenever $|CH(S)| = 7$.*

Proof. If $|L\{2, S\}| = 3$, the admissibility of S follows easily from Observation 2. Therefore, $4 \leq |L\{2, S\}| \leq 6$. We consider these three cases separately as follows:

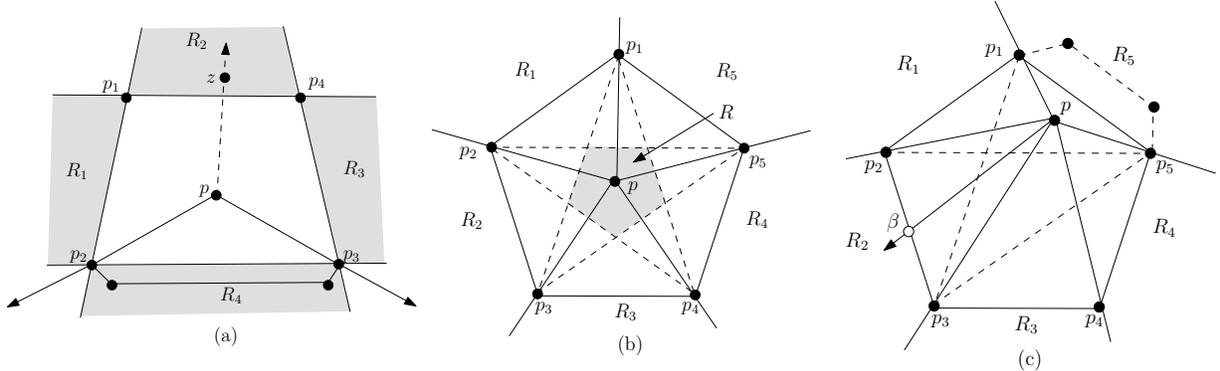


FIGURE 4. Illustration for the proof of Lemma 4.2: (a) $|L\{2, S\}| = 4$, (b) $|L\{2, S\}| = 5$ and $p \in R$, and (c) $|L\{2, S\}| = 5$ and $p \notin R$.

Case 1: $|L\{2, S\}| = 4$. Let R_1, R_2, R_3, R_4 be the 4 shaded regions outside the second convex layer, as shown in Figure 4(a). Note that if $|R_1| + |R_3| \leq 2$, then $S_2 = ((R_1 \cup R_3) \cap S) \cup L\{3, S\}$, and forms a convex quadrilateral or a 4-pseudo-triangle. Hence, $S_1 = \overline{\mathcal{H}}_c(p_1p_4, p_2) \cap S$, S_2 , and $S_3 = \overline{\mathcal{H}}_c(p_2p_3, p_1) \cap S$ is an admissible partition of S . Therefore, we may assume that $|R_1| + |R_3| \geq 3$, and, by symmetry, $|R_2| + |R_4| \geq 3$. As $\sum_{i=1}^4 |R_i| \leq 7$, one of the above inequalities must be an equality. W.l.o.g. assume $|R_2| + |R_4| = 3$. Moreover, by Observation 2 it suffices to assume that $|R_2| = 1$ and $|R_4| = 2$, and let $p \in L\{3, S\}$, be the nearest angular neighbor of $\overrightarrow{p_2p_3}$ inside the second convex layer. Then $S_2 = (R_4 \cap S) \cup \{p_2, p_3, p\}$ spans a 5-hole. If $R_2 \cap S = \{z\}$, then depending on the position of the point $q \in L\{3, S\}$, either $|Cone(zpp_2) \cap S| = 4$ or $|Cone(zpp_3) \cap S| = 4$. Therefore, the partition $S_1 = Cone(zpp_2) \cap S$,

S_2 and $S_3 = (\text{Cone}(zpp_3) \cap S) \cup \{z\}$ or the partition $S_1 = (\text{Cone}(zpp_2) \cap S) \cup \{z\}$, S_2 and $S_3 = (\text{Cone}(zpp_3) \cap S)$ is admissible for S , respectively.

Case 2: $|L\{2, S\}| = 5$. Let $L\{3, S\} = \{p\}$ and consider the partition of the exterior of the second convex layer into disjoint regions $R_i = \text{Cone}(p_i pp_{i+1}) \setminus \mathcal{I}(p_i pp_{i+1})$, for $i \in \{1, 2, 3, 4, 5\}$. Let R be shaded region inside the second convex layer as shown in Figure 4(b).

Case 2.1: $p \in R$. Observe that $\sum_{i=1}^5 |R_i| = 7$ and by Observation 2 for every $i \in \{1, 2, 3, 4, 5\}$, $|R_i| \leq 2$. Therefore, w.l.o.g. assume $|R_1| = 2$ (Figure 4(b)). If $|R_2| + |R_3| = 2$ and $|R_4| + |R_5| = 3$, then $S_1 = (R_1 \cap S) \cup \{p, p_1, p_2\}$, $S_2 = \text{Cone}(p_1 pp_4) \cap S$, and $S_3 = (\text{Cone}(p_2 pp_4) \cap S) \cup \{p_4\}$ is an admissible partition of S . Similarly, for $|R_2| + |R_3| = 3$ and $|R_4| + |R_5| = 2$. Otherwise, either $R_2 \cup R_3$ or $R_4 \cup R_5$ has more than 3 points of S . W.l.o.g., assume that $|R_2| + |R_3| \geq 4$. Observation 2 implies that $|R_2| = |R_3| = 2$, and either the partition $S_1 = (R_2 \cap S) \cup \{p, p_2, p_3\}$, $S_2 = \text{Cone}((p_2 pp_5) \cap S) \cup \{p_5\}$ and $S_3 = \text{Cone}(p_3 pp_5) \cap S$ or the partition $S_1 = (R_2 \cap S) \cup \{p, p_2, p_3\}$, $S_2 = \text{Cone}(p_2 pp_5) \cap S$ and $S_3 = (\text{Cone}(p_3 pp_5) \cap S) \cup \{p_5\}$, is admissible for S .

Case 2.2: $p \notin R$. W.l.o.g. let $p \in \mathcal{I}(p_1 p_2 p_5)$. If $|R_1| + |R_5| \leq 3$, then $S_2 = ((R_1 \cup R_5) \cap S) \cup \{p_1\}$ spans a $|S_2|$ -hole or an empty $|S_2|$ -pseudo-triangle. Moreover, $S_1 = ((R_2 \cup R_4) \setminus \overline{\mathcal{H}}(p_3 p_4, p_1)) \cap S \cup \{p, p_2, p_3, p_4, p_5\}$ spans a $|S_1|$ -hole. Therefore, the partition of S given by S_1 , S_2 , and $S_3 = \overline{\mathcal{H}}(p_3 p_4, p_1) \cap S$ is admissible. Otherwise, $|R_1| + |R_5| \geq 4$, which implies that $|R_1| = |R_5| = 2$, from Observation 2. If $|R_2| = 0$, then $S_2 = (R_5 \cap S) \cup \{p, p_1, p_5\}$ forms a 5-hole and an admissible partition of S is given by $S_1 = (\text{Cone}(p_1 pp_3) \cap S) \cup \{p_3\}$, S_2 , and $S_3 = \text{Cone}(p_5 pp_3) \cap S$. Therefore, assume that $|R_2| \geq 1$. Then there exists a point $\beta \notin S$ on the line segment $p_2 p_3$ such that $|\text{Cone}(p_1 p \beta) \cap S| = |\text{Cone}(p_5 p \beta) \cap S| = 4$ (see Figure 4(c)), and the partition $S_1 = (R_5 \cap S) \cap \{p, p_1, p_5\}$, $S_2 = \text{Cone}(p_1 p \beta) \cap S$, and $S_3 = \text{Cone}(p_5 p \beta) \cap S$ is admissible.

Case 3: $|L\{2, S\}| = 6$. Consider the subdivision of the exterior of the second convex layer into 12 regions R_i as shown in Figure 5(a). Note that the regions R_i and R_{i+2} , for $i \in \{2, 4, 6, 8, 10, 12\}$ might intersect. Observation 2 implies that S is admissible unless $|\bigcup_{b=0}^2 R_{i+b}| \leq 2$, for $i \in \{2, 4, 6, 8, 10, 12\}$. Adding these inequalities and using the fact that $|\bigcup_{b=1}^{12} R_b| = 7$, we get $|R_1| + |R_3| + |R_5| + |R_7| + |R_9| + |R_{11}| \geq 2$. This implies that for some $i \in \{1, 3, 5, 7, 9, 11\}$, $|R_i| \neq 0$. W.l.o.g. assume that $|R_7| \neq 0$. Let $Z_1 = \mathcal{H}(p_4 p_6, p_5) \cap S$ and $Z_2 = \mathcal{H}(p_1 p_3, p_2) \cap S$. From Observation 2 we know $|Z_1| \leq 5$ and $|Z_2| \leq 5$.

Case 3.1: $|R_7| = 2$. If both $|Z_1|, |Z_2| \leq 4$, then $S_2 = ((R_1 \cup R_7) \cap S) \cup \{p_1, p_3, p_4, p_6\}$ forms a 6-hole, and the partition $S_1 = Z_1$, S_2 , and $S_3 = Z_3$ is an admissible partition of S . Next, assume that $|Z_1| = 5$ and $|Z_2| = 2$. As $|R_7| = 2$, $|R_6| = |R_8| = 0$ by Observation 2. Moreover, as $|Z_1| = 5$, $|R_9| \neq 0$. If $|R_9| = 2$, then the partition $S_1 = ((R_3 \cup R_9) \cap S) \cup \{p_1, p_2, p_4, p_5\}$, $S_2 = \mathcal{H}(p_2 p_4, p_3) \cap S$, and $S_3 = \mathcal{H}(p_1 p_5, p_6) \cap S$ is admissible. Otherwise, $|R_9| = 1$ and $|\overline{\mathcal{H}}(p_5 p_6, p_1) \cap S| = 3$. The admissibility of S then follows from Observation 2.

Case 3.2: $|R_7| = 1$. In fact, by symmetry and Case 3.1 we can assume that $|R_i| \leq 1$ for all $i \in \{1, 3, 5, 9, 11\}$. If $|Z_1| = |Z_2| = 4$, the partition $S_1 = Z_1, S_2 = ((R_1 \cup R_7) \cap S) \cup \{p_1, p_3, p_4, p_6\}, S_3 = Z_2$ is admissible for S . Therefore, assume that $|Z_1| = 5$ and $|Z_2| = 3$. If $|R_8| + |R_9| \leq 1$, then $|\overline{\mathcal{H}}(p_5 p_6, p_1) \cap S| \geq 3$ and admissibility follows from Observation 2. As $|R_9| \leq 1$ by assumption, this implies that $|R_8| = |R_9| = 1$. Therefore, $|R_{10}| = 0$.

Case 3.2.1: $|R_{11}| = |R_{12}| = 1$. In this case the partition of S given by $S_1 = ((R_{11} \cup R_5) \cap S) \cup \{p_2, p_3, p_5, p_6\}$, $S_2 = \mathcal{H}(p_2 p_6, p_1) \cap S$, and $S_3 = \mathcal{H}(p_3 p_5, p_4) \cap S$ is admissible.

Case 3.2.2: $|R_{11}| = 0$ and $|R_{12}| = 2$. Then $|R_1| = |R_2| = 0$.

Case 3.2.2.1: $|R_3| = 1$. Then the admissible partition of S is given by $S_1 = ((R_9 \cup R_3) \cap S) \cup \{p_1, p_2, p_4, p_5\}$, $S_2 = \mathcal{H}(p_1 p_5, p_6) \cap S$, and $S_3 = \mathcal{H}(p_2 p_4, p_3) \cap S$.
Case 3.2.2.2: $|R_3| = 0$ but $|R_5| = 1$. Then $S_1 = ((R_{11} \cup R_5) \cap S) \cup \{p_2, p_3, p_5, p_6\}$, $S_2 = \mathcal{H}(p_2 p_6, p_1) \cap S$, and $S_3 = \mathcal{H}(p_3 p_5, p_4) \cap S$ is admissible.
Case 3.2.2.3: $|R_3| = |R_5| = 0$ and $|R_4| = 2$. Observation 2 then implies that $|R_5| = |R_6| = 0$ and $|\overline{\mathcal{H}}(p_2 p_3, p_6) \cap R_7 \cap S| = 0$. Therefore, $S_2 = R_7 \cup R_8 \cup \{p_2, p_3, p_4\}$ forms a 5-hole, which together $S_1 = R_4 \cup \{p_1, p_6\}$, and $S_3 = R_9 \cup R_{12} \cup \{p_5\}$ forms an admissible partition.

□

4.4. $|CH(S)| = 8$. In this section we proof the admissibility of S when $|CH(S)| = 8$. As before assume that $L\{2, S\} = \{p_1, p_2, \dots, p_m\}$, where the vertices are taken in counter-clockwise order, and the indices are to be identified modulo m .

Lemma 4.3. S is admissible whenever $|CH(S)| = 8$.

Proof. If $|L\{2, S\}| = 3$ and none of the outer halfplanes induced by the three edges of the second convex layer contains more than two points of $\mathcal{V}(CH(S))$, then $|\mathcal{V}(CH(S))| \leq 2 \times 3 < 8$. Therefore, it suffices to assume that $4 \leq |L\{2, S\}| \leq 5$.

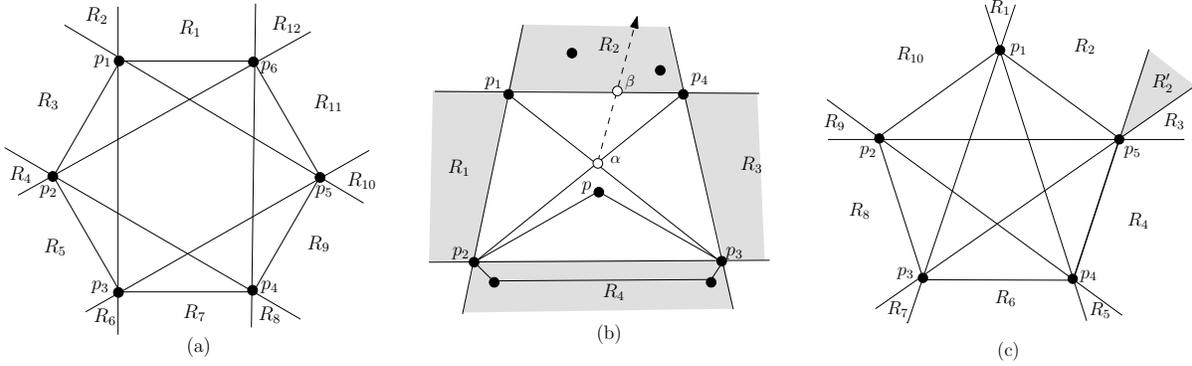


FIGURE 5. Illustration for the proof of Lemma 4.2 with $|L\{2, S\}| = 6$, and Illustration for the proof of Lemma 4.3: (b) $|L\{2, S\}| = 4$, (c) $|L\{2, S\}| = 5$.

Case 1: $|L\{2, S\}| = 4$. Let $L\{3, S\} = \{p\}$, and R_1, R_2, R_3, R_4 be the 4 shaded regions outside the second convex layer as shown in Figure 5(b). Note that if $|R_1| + |R_3| \leq 3$, then $S_2 = ((R_1 \cup R_3) \cap S) \cup \{p\}$ forms a convex quadrilateral or a 4-pseudo-triangle. Hence, $S_1 = \overline{\mathcal{H}}_c(p_1 p_4, p_2) \cap S$, S_2 , and $S_3 = \overline{\mathcal{H}}_c(p_2 p_3, p_1) \cap S$ is an admissible partition of S . Therefore, we may assume that $|R_1| + |R_3| \geq 4$ and $|R_2| + |R_4| \geq 4$. This implies that $|R_1| = |R_2| = |R_3| = |R_4| = 2$, as $\sum_{i=1}^4 |R_i| \leq 8$. Let α be the point of intersection of the diagonals of the quadrilateral $p_1 p_2 p_3 p_4$. W.l.o.g., assume that $p \in \mathcal{I}(p_2 p_3 \alpha)$. Then the points p, p_2, p_3 and along with two points in $R_4 \cap S$ form a 5-hole. The remaining 8 points can be partitioned into two disjoint convex regions with 4 points each, because there exists a point β on the line segment $p_1 p_4$ such that $|\text{Cone}(\beta \alpha p_2) \cap S| = 4$ and $|\text{Cone}(\beta \alpha p_3) \cap S| = 4$ (see Figure 5(b)).

Case 2: $|L\{2, S\}| = 5$. Consider the partition of the exterior of the second convex layer into regions R_i 's as shown in Figure 5(c). Observation 2 implies that S is admissible unless $|R_i| + |R_{i+1}| + |R_{i+2}| \leq 2$, for $i \in \{1, 3, 5, 7, 9\}$. Adding these 5 inequalities and using the

fact that $\sum_{i=1}^{10} |R_i| = 8$ we get, $|R_1| + |R_3| + |R_5| + |R_7| + |R_9| \leq 2$, that is, $|R_2| + |R_4| + |R_6| + |R_8| + |R_{10}| \geq 6$. Therefore, one of these 5 regions must contain exactly two points of $\mathcal{V}(CH(S))$. W.l.o.g., assume that $|R_2| = 2$. Let $Z_1 = \{\mathcal{H}(p_3p_5, p_4) \setminus R_7\} \cap \mathcal{V}(CH(S))$ and $Z_2 = \{\mathcal{H}(p_3p_1, p_2) \setminus R_7\} \cap \mathcal{V}(CH(S))$.

Case 2.1: $|R_7| \geq 1$. We have $|R_5| + |R_6| \leq 1$ and $|R_8| + |R_9| \leq 1$. Therefore, $|Z_1 \cup \{p_4\}| = a_1 \leq 4$ and $|Z_2 \cup \{p_2\}| = a_2 \leq 4$. Now, $|R_7| = 6 - (a_1 + a_2)$ and there exists a point $\alpha \in R_7 \setminus S$ such that both $Cone(\alpha p_3 p_1) \cap S$ and $Cone(\alpha p_3 p_5) \cap S$ contain 4 points and $\{p_1, p_3, p_5\} \cup (R_2 \cap \mathcal{V}(CH(S)))$ spans a 5-hole. Therefore, S is admissible.

Case 2.2: $|R_7| = 0$. Let $R'_2 = \overline{\mathcal{H}(p_4 p_5, p_1)} \cap R_2$. We know that $|Z_1| \leq 4$, $|Z_2| \leq 4$, and $|Z_1| + |Z_2| = 6$. If $|Z_1| = |Z_2| = 3$, the partition $S_1 = Z_1 \cup \{p_4\}$, $S_2 = Z_2 \cup \{p_2\}$, and $S_3 = (R_2 \cap S) \cup \{p_1, p_3, p_5\}$ is admissible. Otherwise, either $|Z_1| = 4$ or $|Z_2| = 4$. W.l.o.g. let $|Z_1| = 4$. This implies that $|R_6| = 2$, $|R_3| + |R_4| = 2$ and $|R'_2| = 0$. Let $s_i \in S$ be the first angular neighbor of $\overline{p_1 p_3}$ in R_6 . Then set $\{p_1, p_5, s_i\} \cup (R_2 \cap S)$ spans a 5-hole and the admissibility of S follows. □

4.5. $|CH(S)| = 9$. The proof of Theorem 3.1 can now be completed with the following simple observation:

Observation 7. S is admissible whenever $|CH(S)| \geq 9$.

Proof. Let $|CH(S)| = k \geq 9$. This implies that $|L\{2, S\}| = 13 - k \leq 4$, and so, there must exist an outer halfplane induced by an edge of $L\{2, S\}$ containing more than two points of S . The result now follows from Observation 2. □

5. CONCLUSIONS

In this paper we prove that every set of 13 points, in general position, can be partitioned into three disjoint regions each of which span an empty convex polygon or an empty pseudo-triangle. This proves that the pseudo-convex partition number $\psi(n) \leq \lceil \frac{3n}{13} \rceil$, thus answering a question posed by Aichholzer et al. [1].

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