

# Strong chromatic index of $k$ -degenerate graphs

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## Abstract

A *strong edge coloring* of a graph  $G$  is a proper edge coloring in which every color class is an induced matching. The *strong chromatic index*  $\chi'_s(G)$  of a graph  $G$  is the minimum number of colors in a strong edge coloring of  $G$ . In this note, we improve a result by Dębski et al. [Strong chromatic index of sparse graphs, arXiv:1301.1992v1] and show that the strong chromatic index of a  $k$ -degenerate graph  $G$  is at most  $(4k-2) \cdot \Delta(G) - 2k^2 + 1$ . As a direct consequence, the strong chromatic index of a 2-degenerate graph  $G$  is at most  $6\Delta(G) - 7$ , which improves the upper bound  $10\Delta(G) - 10$  by Chang and Narayanan [Strong chromatic index of 2-degenerate graphs, J. Graph Theory 73 (2013) (2) 119–126]. For a special subclass of 2-degenerate graphs, we obtain a better upper bound, namely if  $G$  is a graph such that all of its  $3^+$ -vertices induce a forest, then  $\chi'_s(G) \leq 4\Delta(G) - 3$ ; as a corollary, every minimally 2-connected graph  $G$  has strong chromatic index at most  $4\Delta(G) - 3$ . Moreover, all the results in this note are best possible in some sense.

## 1 Introduction

A *strong edge coloring* of a graph  $G$  is a proper edge coloring in which every color class is an induced matching. That is, an edge coloring is *strong* if for each edge  $uv$ , the color of  $uv$  is distinct from the colors of the edges (other than  $uv$ ) incident with  $N_G(u) \cup N_G(v)$ . The *strong chromatic index*  $\chi'_s(G)$  of a graph  $G$  is the minimum number of colors in a strong edge coloring of  $G$ . The *degree* of a vertex  $v$  in  $G$ , denoted by  $\deg(v)$ , is the number of incident edges of  $v$  in  $G$ . A vertex of degree  $k$ , at most  $k$  and at least  $k$  are called a  $k$ -vertex,  $k^-$ -vertex and  $k^+$ -vertex, respectively. The graph  $(\emptyset, \emptyset)$  is an *empty graph*, and  $(V, \emptyset)$  is an *edgeless graph*. We denote the minimum and maximum degrees of vertices in  $G$  by  $\delta(G)$  and  $\Delta(G)$ , respectively.

In 1985, Edrös and Nešetřil [5] constructed graphs with strong chromatic index  $\frac{5}{4}\Delta^2$  when  $\Delta$  is even,  $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$  when  $\Delta$  is odd. Inspired by their construction, they proposed the following strong edge coloring conjecture.

**Conjecture 1.** If  $G$  is a graph with maximum degree  $\Delta$ , then

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd. } \quad \square \end{cases}$$

A graph is  *$k$ -degenerate* if every subgraph has a vertex of degree at most  $k$ . Chang and Narayanan [3] showed the strong chromatic index of a 2-degenerate graph  $G$  is at most  $10\Delta(G) - 10$ . Recently, Luo and Yu [6] improved the upper bound to  $8\Delta(G) - 4$ . For general  $k$ -degenerate graphs, Dębski et al. [4] presented an upper bound  $(4k-1) \cdot \Delta(G) - k(2k+1) + 1$ . Very recently, Yu [7] obtained an improved upper bound  $(4k-2) \cdot \Delta(G) - 2k^2 + k + 1$ . In this note, we use the method developed in [4] and improve the upper bound to  $(4k-2) \cdot \Delta(G) - 2k^2 + 1$ . In particular, when  $G$  is a 2-degenerate graph, the strong chromatic index is at most  $6\Delta(G) - 7$ , which improves the upper bound  $10\Delta(G) - 10$  by Chang and Narayanan [3]. In addition, we show that if  $G$  is a graph such that all of its  $3^+$ -vertices induce a forest, then  $\chi'_s(G) \leq 4\Delta(G) - 3$ .

## 2 Results

**Lemma 1** (Chang and Narayanan [3]). If  $G$  is a  $k$ -degenerate graph with at least one edge, then there exists a vertex  $w$  such that at least  $\max\{1, \deg(w) - k\}$  of its neighbors are  $k^-$ -vertices.

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**Theorem 2.1.** If  $G$  is a  $k$ -degenerate graph with maximum degree  $\Delta$  and  $k \leq \Delta$ , then  $\gamma'_{k,s}(G) \leq (4k-2)\Delta - 2k^2 + 1$ .

**Proof.** By the definition of  $k$ -degenerate graph, every subgraph of  $G$  is also a  $k$ -degenerate graph. We want to obtain a sequence  $\Lambda_1, \dots, \Lambda_m$  of subsets of edges as follows. Let  $\Lambda_0 = \emptyset$ . Suppose that  $\Lambda_{i-1}$  is well-defined, let  $G_i$  be the graph  $G - (\Lambda_0 \cup \dots \cup \Lambda_{i-1})$ . Notice that  $G_i$  is an edgeless graph or a  $k$ -degenerate graph with at least one edge. Denote the degree of  $v$  in  $G_i$  by  $\deg_i(v)$ . If  $G_i$  has at least one edge, then we choose a vertex  $w_i$  of  $G_i$  as described in Lemma 1, and let

$$\Lambda_i = \{w_i v \mid w_i v \in E(G_i) \text{ and } \deg_i(v) \leq k\}.$$

Lemma 1 guarantees  $\Lambda_i \neq \emptyset$ , and this process terminates with a subset  $\Lambda_m$ . Note that the subgraph induced by  $\Lambda_i$  is a star with center  $w_i$ , so we call  $w_i$  the *center* of  $\Lambda_i$ .

**Claim 1.** If  $i \neq j$ , then  $\Lambda_i$  and  $\Lambda_j$  have distinct centers.

**Proof of the Claim.** Suppose to the contrary that there exists  $j < i$  such that  $\Lambda_i$  and  $\Lambda_j$  have the same center  $w$ . Let  $wv$  be an edge in  $\Lambda_i$ . From the construction, the vertex  $w$  is the center of  $\Lambda_j$ , thus  $\deg_{G_{j+1}}(w) \leq k$  and  $\deg_i(w) \leq \deg_{G_{j+1}}(w) \leq k$ . The fact  $wv \notin \Lambda_j$  implies that  $\deg_j(v) > k$ . Since  $wv \in \Lambda_i$ , it follows that  $\deg_i(v) \leq k$ , and then there exists  $j < t < i$  such that  $v$  is the center of  $\Lambda_t$ . But  $wv \notin \Lambda_t$ , which leads to a contradiction that  $\deg_{G_{j+1}}(w) \leq k < \deg_t(w)$ . This completes the proof of the claim.  $\square$

We color the edges from  $\Lambda_m$  to  $\Lambda_1$ , we remind the readers this is the reverse order of the ordinary sequence.

In the following, we want to give an algorithm to obtain a strong edge coloring of  $G$ . First, we can color the edges in  $\Lambda_m$  with distinct colors. Now, we consider the edge  $w_i v_i$ , where  $w_i v_i \in \Lambda_i$ . We want to assign a color to  $w_i v_i$  such that the resulting coloring is still a partial strong edge coloring. In order to guarantee the resulting coloring is a partial strong edge coloring in each step, any two edges which are incident with  $N_G(w_i) \cup N_G(v_i)$  must receive distinct colors.

Now, we compute the number of colored edges in  $G_i$  which are incident with  $N_G(w_i) \cup N_G(v_i)$ . Let  $X_i = N_G(w_i) \setminus N_{G_i}(w_i)$ , let  $x_i = |X_i|$ , let  $y_i = |\{v \mid w_i v \in E(G_i) \text{ and } \deg_{G_i}(v) \leq k\}|$ , and let  $z_i = |\{v \mid w_i v \in E(G_i) \text{ and } \deg_{G_i}(v) \geq k\}|$ . Suppose that  $x_i > 0$  and let  $w$  be an arbitrary vertex in  $X_i$ . By the claim, the edge  $w w_i$  is in some  $\Lambda_t$  with center  $w$ , thus  $\deg_G(w_i) \leq k$  and  $w$  is incident with at most  $k$  colored edges. Therefore, if  $x_i \neq 0$ , then the number of colored edges which are incident with  $N_G(w_i) \setminus \{v_i\}$  is at most

$$\begin{aligned} & (x_i + y_i - 1) \cdot k + z_i \cdot \Delta \\ &= (x_i + y_i - 1 + z_i) \cdot k + z_i \cdot (\Delta - k) \\ &\leq (k-1)k + (k-2)(\Delta - k), \quad \text{note that } x_i + y_i + z_i = \deg_G(w_i) \leq k \text{ and } z_i \leq k-2 \\ &= (k-2)\Delta + k; \end{aligned}$$

if  $x_i = 0$ , then the number of colored edges which are incident with  $N_G(w_i) \setminus \{v_i\}$  is at most

$$\begin{aligned} & (y_i - 1) \cdot k + z_i \cdot \Delta \\ &= (y_i - 1 + z_i) \cdot k + z_i \cdot (\Delta - k) \\ &\leq (\Delta - 1) \cdot k + k \cdot (\Delta - k), \quad \text{note that } z_i \leq k \\ &= 2k\Delta - k^2 - k. \end{aligned}$$

Let  $p_i = |N_G(v_i) \setminus N_{G_i}(v_i)|$ , and let  $q_i = |N_{G_i}(v_i)|$ . If  $\deg(v_i) > k$ , then  $p_i > 0$  and there exists some  $s$  with  $s < i$  such that  $v_i$  is the center of  $\Lambda_s$ . It follows that  $\deg_{G_{s+1}}(v_i) \leq k$  and for every edge  $u v_i$  in  $\Lambda_s$ , the vertex  $u$  is incident with at most  $k-1$  colored edges. Therefore, if  $\deg(v_i) > k$ , then the number of colored edges which are incident with  $N_G(v_i) \setminus \{w_i\}$  is at most

$$\begin{aligned} & (k-1) \cdot (\deg(v_i) - \deg_{G_{s+1}}(v_i)) + (\deg_{G_{s+1}}(v_i) - 1) \cdot \Delta \\ &= (k-1) \cdot \deg(v_i) + \deg_{G_{s+1}}(v_i) \cdot (\Delta - k + 1) - \Delta \\ &\leq (k-1)\Delta + k \cdot (\Delta - k + 1) - \Delta, \quad \text{note that } \deg_{G_{s+1}}(v_i) \leq k \\ &= 2(k-1)\Delta + k(1-k); \end{aligned}$$

if  $\deg(v_i) \leq k$ , then the number of colored edges which are incident with  $N_G(v_i) \setminus \{w_i\}$  is at most  $(k-1) \cdot \Delta$ .

Hence, the number of colored edges incident with  $N_G(w_i) \cup N_G(v_i)$  is at most

$$\begin{aligned} & \max\{(k-2)\Delta + k, 2k\Delta - k^2 - k\} + \max\{2(k-1)\Delta + k(1-k), (k-1)\Delta\} \\ &= (2k\Delta - k^2 - k) + 2(k-1)\Delta + k(1-k) \\ &= (4k-2)\Delta - 2k^2. \end{aligned}$$

Thus, there are at least one available color for  $w_iv_i$ . When all the edges are colored, we obtain a strong edge coloring of  $G$ .  $\square$

**Corollary 1.** If  $G$  is a 2-degenerate graph with maximum degree at least two, then  $\chi'_s(G) \leq 6\Delta(G) - 7$ .

**Remark 1.** The strong chromatic index of a 5-cycle is five, so it achieves the upper bound in [Corollary 1](#), thus the obtained upper bound is best possible in some sense.

Next, we investigate a class of graphs whose all  $3^+$ -vertices induce a forest.

**Lemma 2.** If  $G$  is a graph with at least one edge such that all the  $3^+$ -vertices induce a forest, then there exists a vertex  $w$  such that at least  $\max\{1, \deg_G(v) - 1\}$  of its neighbors are  $2^-$ -vertices.

**Proof.** Let  $A$  be the set of  $2^-$ -vertices. It is obvious that  $A \neq \emptyset$ , otherwise every vertex has degree at least three, and then  $G$  contains cycles. Now, consider the graph  $G - A$ . If  $G - A$  is an empty graph, then every nonisolated vertex satisfies the desired condition. So we may assume that the  $G - A$  has at least one vertex. Since the graph  $G - A$  is a forest, it follows that there exists at least one  $1^-$ -vertex in  $G - A$ , and every such vertex satisfies the desired condition.  $\square$

**Theorem 2.2.** If  $G$  is a graph such that all of its  $3^+$ -vertices induce a forest, then  $\chi'_s(G) \leq 4\Delta(G) - 3$ .

**Proof.** The proof is analogous to that in [Theorem 2.1](#), so we omit it.  $\square$

A 2-connected graph  $G$  is *minimally 2-connected* if  $G - e$  is not 2-connected for each edge  $e$  in  $G$ .

**Theorem 2.3** ([2]). The minimum degree of a minimally 2-connected graph is two, and the subgraph induced by all the  $3^+$ -vertices is a forest.

**Corollary 2.** If  $G$  is a minimally 2-connected graph, then  $\chi'_s(G) \leq 4\Delta(G) - 3$ .

**Remark 2.** Once again the strong chromatic index of a 5-cycle achieves the upper bound in [Theorem 2.2](#), so the upper bound is best possible in some sense.

A graph is *chordless* if every cycle is an induced cycle. It is easy to show that a 2-connected graph is chordless if and only if it is minimally 2-connected. Every chordless graph is a 2-degenerate graph. In [3], Chang and Narayanan proved that the strong chromatic index of a chordless graph is at most  $8\Delta(G) - 6$ , and Dębski et al. [4] improved the upper bound to  $4\Delta(G) - 3$ , but I doubt that their proofs are correct since they incorrectly used a lemma (see [3, Lemma 7]). Recently, Narayanan (private communication, Mar. 2014) has confirmed the mistake in the proof of the result  $8\Delta(G) - 6$ . For more detailed discussion on the strong chromatic index of chordless graphs, we refer the reader to [1].

**Remark 3.** All the proofs used the greedy algorithm, thus all the results are true for the list version of strong edge coloring.

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