# A SHARP REFINEMENT OF A RESULT OF ZVEROVICH-ZVEROVICH 

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#### Abstract

For a finite sequence of positive integers to be the degree sequence of a finite graph, Zverovich and Zverovich gave a sufficient condition involving only the length of the sequence, its maximal element and its minimal element. In this paper we give a sharp refinement of Zverovich-Zverovich's result.


## 1. Introduction

A finite sequence $\underline{d}=\left(d_{1}, \ldots, d_{n}\right)$ of positive integers is graphic if it occurs as the sequence of vertex degrees of a simple graph. The classic theorem of Erdős and Gallai Theorem gives a necessary and sufficient condition for a sequence to be graphic (see [6, 5, 7]). A theorem of Zverovich and Zverovich gives a sufficient condition involving only the length of the sequence, its maximal element and its minimal element. Their result can be stated in the following equivalent form.

Theorem 1 ([8, Theorem 6]). Suppose that $\underline{d}$ is a decreasing sequence of positive integers with even sum. Let a (resp. b) denote the maximal (resp. minimal) element of $\underline{d}$. Then $\underline{d}$ is graphic if

$$
\begin{equation*}
n b \geq \frac{(a+b+1)^{2}}{4} \tag{1}
\end{equation*}
$$

It is known that this result is not sharp (see [1]). A sharp bound in the case $b=1$ was given in [2]. The main aim of this paper is to prove the following result, which is sharp for all $a, b$ and $n$.

Theorem 2. Suppose that $\underline{d}$ is a decreasing sequence of positive integers with even sum. Let $a$ (resp. b) denote the maximal (resp. minimal) element of $\underline{d}$. Then $\underline{d}$ is graphic if

$$
n b \geq \begin{cases}\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor-1 & : \text { if } b \text { is odd, or } a+b \equiv 1 \quad(\bmod 4)  \tag{2}\\ \left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor & : \text { otherwise },\end{cases}
$$

where $\lfloor$.$\rfloor denotes the integer part. Moreover, for any triple ( a, b, n$ ) of positive integers with $b<a<n$ that fails (22), there is a nongraphic sequence of length $n$ having even sum with maximal element $a$ and minimal element $b$.

[^0]The paper is organised as follows. In Section [2 we examine condition (2) and rewrite it in a more convenient form. We then prove that condition (2) is sufficient in Section 3. The sharpness is shown in Section 5. To establish this we first prove the following result in Section 4, which may be of independent interest. Here, and in sequences throughout this paper, the superscripts indicate the number of repetitions of the entry.

Theorem 3. Consider natural numbers $b<a<n$ and suppose that $a s+b(n-s)$ is even. Then for $0<s<n$, the sequence $\left(a^{s}, b^{n-s}\right)$ is graphic if and only if $s^{2}-(1+a+b) s+n b \geq 0$.

Remark. The assumption $b<a<n$ is not restrictive. All sequences with $a \geq n$ are obviously nongraphic. For $a=b$, it follows from Theorem 2 that $\left(a^{n}\right)$ is graphic if and only if $a n$ is even and $a<n$.

Throughout the following, $\underline{d}=\left(d_{1}, \ldots, d_{n}\right)$ denotes a decreasing sequence with maximal element $d_{1}=a$ and minimal element $d_{n}=b$.

## 2. The hypothesis

We claim that the inequality (2) can be conveniently expressed according to the following four disjoint, exhaustive cases:
(I) If $a+b+1 \equiv 2 b n(\bmod 4)$, then $(a+b+1)^{2} \leq 4 b n$.
(II) If $a+b+1 \equiv 2 b n+2(\bmod 4)$, then $(a+b+\overline{1})^{2} \leq 4 b n+4$.
(III) If $a+b$ is even and $b n$ is even, then $(a+b+1)^{2} \leq 4 b n+1$.
(IV) If $n, a, b$ are all odd, then $(1+a+b)^{2} \leq 4 b n+5$.

First note that in cases (I) and (II), we have $\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor=\frac{(a+b+1)^{2}}{4}$, while in cases (III) and (IV), $\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor=\frac{(a+b+1)^{2}}{4}-\frac{1}{4}$.

Consider case (I). There are two subcases to consider here. First, if $b$ is even, then $a+b \equiv-1(\bmod 4)$, so (22) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor=\frac{(a+b+1)^{2}}{4}
$$

or equivalently $(a+b+1)^{2} \leq 4 b n$, as required. The other subcase of case (I) is where $b$ is odd. Here, (2) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor-1=\frac{(a+b+1)^{2}-4}{4}
$$

or equivalently $(a+b+1)^{2} \leq 4 b n+4$. Note that both sides of this inequality are multiples of 4 . We claim that equality is impossible here, and so the condition is equivalent to $(a+b+1)^{2} \leq$ $4 b n$. Indeed, if $(a+b+1)^{2}=4 b n+4$, then as $a+b+1 \equiv 2 b n(\bmod 4)$, we would have $4 b^{2} n^{2} \equiv 4 b n+4(\bmod 8)$. Hence $b^{2} n^{2} \equiv b n+1(\bmod 2)$. But this is impossible, as $x^{2} \equiv x$ $(\bmod 2)$ for all $x$.

Consider case (II). Here, if $b$ is even, $a+b \equiv 1(\bmod 4)$. So, regardless of whether $b$ is even or odd, (2) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor-1=\frac{(a+b+1)^{2}-4}{4}
$$

or equivalently $(a+b+1)^{2} \leq 4 b n+4$, as required.
Consider case (III). There are two subcases to consider here. First, if $b$ is even, then $a$ is necessarily even, so $a+b \not \equiv 1(\bmod 4)$, and hence (2) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor=\frac{(a+b+1)^{2}}{4}-\frac{1}{4}
$$

or equivalently $(a+b+1)^{2} \leq 4 b n+1$, as required. The other subcase of case (III) is where $b$ is odd. Here, (2) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor-1=\frac{(a+b+1)^{2}}{4}-\frac{1}{4}-1
$$

or equivalently $(a+b+1)^{2} \leq 4 b n+5$. We claim that equality is not possible and thus, as $(a+b+1)^{2} \equiv 1(\bmod 4)$, the condition is equivalent to $(a+b+1)^{2} \leq 4 b n+1$. Indeed, since $b n$ is even, $4 b n+5 \equiv 5(\bmod 8)$. But $a+b$ is even, say $a+b=2 k$, so we have $(a+b+1)^{2}=4\left(k^{2}+k\right)+1 \equiv 1(\bmod 8)$, as $k^{2}+k$ is even for all $k$. So $(a+b+1)^{2}=4 b n+5$ is impossible.

In case (IV), $b$ is odd, so (2) reads

$$
n b \geq\left\lfloor\frac{(a+b+1)^{2}}{4}\right\rfloor-1=\frac{(a+b+1)^{2}}{4}-\frac{1}{4}-1
$$

or equivalently $(a+b+1)^{2} \leq 4 b n+5$, as claimed.

## 3. Proof of sufficiency

Suppose that $\underline{d}$ has even sum and that inequality (22) holds. We consider the 4 cases (I) (IV) given in Section 2. The sufficiency in case (I) follows immediately from Theorem 1, For the other cases, we employ similar ideas to those of [8]. Let us first recall some terminology and results of [8]. A number $k$ is called a strong index, if $d_{k} \geq k$. Note that the set of strong indices is nonempty as $d_{1} \geq 1$, but not all indices are strong as $d_{n}=b<n$. The maximal strong index is denoted $k_{m}$. For $j \geq 0, n_{j}:=\#\left\{i: d_{i}=j\right\}$. By [ $\underline{8}$, Theorem 3], a sequence $\underline{d}$ is graphic if and only if it has an even sum and $r_{k} \leq k(n-1)$ for all strong indices $k$, where $r_{k}=\sum_{i=1}^{k}\left(d_{i}+i n_{k-i}\right)$.

Let $k$ be a strong index of $\underline{d}$. If $k \leq b$, then $n_{k-i}=0$ for all $1 \leq i \leq k$, so $r_{k}=\sum_{i=1}^{k} d_{i} \leq$ $k a \leq k(n-1)$. So we may suppose that $k>b$.

Lemma 1. We have

$$
\begin{equation*}
r_{k} \leq k(n-1)+k_{m}(a+b+1)-k_{m}^{2}-b n, \tag{3}
\end{equation*}
$$

with equality only possible when $k=k_{m}$ and $\underline{d}$ has the form $\underline{d}=\left(a^{k_{m}} b^{n-k_{m}}\right)$.
Proof. We have $\sum_{i=1}^{k} d_{i} \leq k a$, which becomes an equality only if $d_{1}=\cdots=d_{k}=a$. Moreover, $\sum_{i=1}^{k} i n_{k-i}=\sum_{j=0}^{k-1}(k-j) n_{j} \leq(k-b) \sum_{j=0}^{k-1} n_{j}$, with equality only possible when all the $n_{0}, n_{1}, \ldots, n_{k-1}$, but $n_{b}$ are zeros; that is, when $\left(d_{i} \leq k-1 \Rightarrow d_{i}=b\right)$. Furthermore,

$$
\sum_{j=0}^{k-1} n_{j}=\#\left\{i \in\{1,2, \ldots, n\}: d_{i} \leq k-1\right\} \leq \#\left\{i: n \geq i>k_{m}\right\}
$$

since for $i \leq k_{m}$ we have $d_{i} \geq d_{k_{m}} \geq k_{m} \geq k$. Thus $\sum_{j=0}^{k-1} n_{j} \leq n-k_{m}$, which becomes an equality only when $d_{k_{m}+1} \leq k-1$. Thus $\sum_{i=1}^{k} i n_{k-i} \leq(k-b)\left(n-k_{m}\right)$, with equality only possible when $d_{k_{m}+1}=\cdots=d_{n}=b$; indeed, if $d_{k_{m}+1} \leq k-1$ and $\left(d_{i} \leq k-1 \Rightarrow d_{i}=b\right)$, then $d_{k_{m}+1}=b$ and so $d_{k_{m}+1}=\cdots=d_{n}=b$. Thus

$$
\begin{equation*}
r_{k} \leq k a+(k-b)\left(n-k_{m}\right) \tag{4}
\end{equation*}
$$

with equality only possible when $\underline{d}=\left(a^{k}, d_{k+1}, \ldots, d_{k_{m}}, b^{n-k_{m}}\right)$. As $a \geq d_{k_{m}} \geq k_{m}$, we have $a+1-k_{m} \geq 1$. Thus, using $k \leq k_{m}$, inequality (4) gives

$$
\begin{aligned}
r_{k} \leq k a+(k-b)\left(n-k_{m}\right) & =k(n-1)+k\left(a+1-k_{m}\right)+b k_{m}-b n \\
& \leq k(n-1)+k_{m}\left(a+1-k_{m}\right)+b k_{m}-b n \\
& =k(n-1)+k_{m}(a+b+1)-k_{m}^{2}-b n
\end{aligned}
$$

as required, with equality only possible when $k=k_{m}$ and $\underline{d}=\left(a^{k_{m}}, b^{n-k_{m}}\right)$.
Now consider cases (II) - (IV) separately. In case (III), the maximal value of the righthand side of (3), regarded as a quadratic in $k_{m}$, is attained at $k_{m}=\frac{1}{2}(a+b+1 \pm 1)$ and is equal to $k(n-1)+\frac{1}{4}\left((a+b+1)^{2}-1\right)-b n$. Thus the inequality $(a+b+1)^{2} \leq 4 b n+1$ implies $r_{k} \leq k(n-1)$ and so the sequence $\underline{d}$ is graphic by [8, Theorem 3].

In case (II), the (unique) maximal value of the right-hand side of (3) for an integer $k_{m}$ is attained at $k_{m}=\frac{1}{2}(a+b+1)$ and is equal to $k(n-1)+\frac{1}{4}(a+b+1)^{2}-b n$, but the sum of the sequence $\left(a^{\frac{1}{2}(a+b+1)}, b^{n-\frac{1}{2}(a+b+1)}\right)$ is odd, so the inequality (3) becomes strict, hence $r_{k} \leq k(n-1)+\frac{1}{4}(a+b+1)^{2}-b n-1$. Thus the inequality $(a+b+1)^{2} \leq 4 b n+4$ implies $r_{k} \leq k(n-1)$ and therefore the sequence $\underline{d}$ is graphic by [ $\underline{8}$, Theorem 3].

In case (IV), $\underline{d}$ is not of the form $\left(a^{s}, b^{n-s}\right)$ since, as $a, b$ and $n$ are all odd, the sequences of the form $\left(a^{s}, b^{n-s}\right)$ have odd sum, contrary to our hypothesis. So by Lemma 1, the inequality (3) is not strict. As the maximum of the right hand side of (3) is attained for $k_{m}=\frac{1}{2}(a+b+1 \pm 1)$, we get $r_{k} \leq k(n-1)+\frac{1}{4}\left((a+b+1)^{2}-1\right)-b n-1$, which implies $r_{k} \leq k(n-1)$ whenever $(a+b+1)^{2} \leq 4 b n+5$.

## 4. Two-Element sequences

Proof of Theorem 图. We apply the Erdős-Gallai Theorem, which says that $\underline{d}$ is graphic if and only if its sum is even and for each integer $k$ with $1 \leq k \leq n$,

$$
\begin{equation*}
\sum_{i=1}^{k} d_{i} \leq k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}\right\} \tag{EG}
\end{equation*}
$$

For the sequence $\underline{d}=\left(a^{s}, b^{n-s}\right)$, we consider (EG) in 5 cases:
(i) If $k>s$ and $k \leq b$, then (EG) reads

$$
a s+b(k-s) \leq k(k-1)+(n-k) k=k(n-1)
$$

We have $a s+b(k-s)=b k+s(a-b)<b k+k(a-b)=k a \leq k(n-1)$, so (EG) holds in this case.
(ii) If $k \leq s$ and $k \leq b$, then (EG) reads

$$
a k \leq k(k-1)+(n-k) k=k(n-1),
$$

which is true as $a \leq n-1$.
(iii) If $k \leq s$ and $a<k$, then (EG) reads

$$
a k \leq k(k-1)+(s-k) a+(n-s) b .
$$

As $a \leq k-1$, we have $a k \leq k(k-1) \leq k(k-1)+(s-k) a+(n-s) b$, so (EG) holds in this case.
(iv) If $k>s$ and $k>b$, then (EG) reads $a s+b(k-s) \leq k(k-1)+(n-k) b$; that is

$$
\begin{equation*}
k^{2}-k(1+2 b)+n b+b s-a s \geq 0 \tag{5}
\end{equation*}
$$

(v) If $k \leq s$ and $b<k \leq a$, then (EG) reads

$$
a k \leq k(k-1)+(s-k) k+(n-s) b=(s-1) k+(n-s) b .
$$

This condition holds if $a \leq s-1$. If $a \geq s$, (EG) is $(a-s+1) k \leq(n-s) b$ and the most restrictive case occurs when $k=s$. Here the condition is

$$
\begin{equation*}
s^{2}-(1+a+b) s+n b \geq 0 \tag{6}
\end{equation*}
$$

From the above we see that $\left(a^{s}, b^{n-s}\right)$ is graphic if and only if (6) holds and (5) holds for all $k>s, k>b$.
Lemma 2. $s^{2}-(1+a+b) s+n b \geq 0$ if and only if $k^{2}-k(1+2 b)+n b+b s-a s \geq 0$ for all $k \in\{s, s+1, \ldots, n\}$.
Proof. Fix $n, a, b, s$ and let $\Delta_{k}=k^{2}-k(1+2 b)+n b+s b-a s$. So $\Delta_{s}=s^{2}-(1+a+b) s+n b$, and hence one direction in this lemma is trivial. For the other direction, note that $\Delta_{k}$ is quadratic in the integer $k$ and takes its minimum value at the integers $b$ and $b+1$. The minimum value of $\Delta_{k}$ is

$$
\Delta_{b}=b^{2}-b(1+2 b)+n b+b s-a s=b n-b^{2}-b-s(a-b)
$$

Suppose that $\Delta_{s} \geq 0$ and that $\Delta_{k}<0$ for some integer $k>s$. Then $s \leq b-1$ and $\Delta_{b}<0$, so $s(a-b)>b n-b^{2}-b$ and hence $(b-1)(a-b) \geq s(a-b)>b n-b^{2}-b$. Expanding gives so $a b-a+2 b>b n$. Hence, as $b<a$,

$$
b n<a b-a+2 b<a b+b=b(a+1)
$$

and so $n<a+1$. But this is impossible as $a<n$, by assumption.
This completes the proof of Theorem 3.

## 5. Proof of Necessity

Assume that (21) fails for the triple ( $a, b, n$ ), where $b<a<n$. We will exhibit a nongraphic sequence $\underline{d}$ of length $n$ having even sum with maximal element $a$ and minimal element $b$. We consider the same four cases (I) - (IV) given in Section 2. So our assumption is respectively:
(I) $a+b+1 \equiv 2 b n(\bmod 4)$, and $(a+b+1)^{2}>4 b n$.
(II) $a+b+1 \equiv 2 b n+2(\bmod 4)$, and $(a+b+1)^{2}>4 b n+4$.
(III) $a+b$ is even and $b n$ is even, and $(a+b+1)^{2}>4 b n+1$.
(IV) $n, a, b$ are all odd, and $(1+a+b)^{2}>4 b n+5$.

In cases (I) - (III), the proposed sequences have the form $\underline{d}=\left(a^{s}, b^{n-s}\right)$, where respectively:
(I) $s=\frac{a+b+1}{2}$;
(II) $s=\frac{a+b+3}{2}$;
(III) $s=\frac{a+b}{2}$.

In case (I), $s^{2}-(1+a+b) s+n b=-\frac{(1+a+b)^{2}}{4}+n b<0$ and so $\underline{d}$ is nongraphic by Theorem 3. Moreover, $\underline{d}$ has sum

$$
a s+b(n-s)=\frac{a(a+b+1)}{2}+\frac{b(2 n-(a+b+1))}{2}=\frac{(a-b)(a+b+1)+2 b n}{2} .
$$

In case $(\mathrm{I}), a+b$ is odd, so $a-b$ is odd and $a+b+1 \equiv 2 b n(\bmod 4)$, so

$$
(a-b)(a+b+1)+2 b n \equiv 2 b n+2 b n \equiv 0 \quad(\bmod 4) .
$$

Thus $\underline{d}$ has even sum. Cases (II) and (III) are treated in exactly the same manner.
In case (IV), $a, b, n$ are all odd. Here we consider the decreasing sequence $\underline{d}=\left(d_{1}, \ldots, d_{n}\right)=$ $\left(a^{\frac{a+b}{2}}, b+1, b^{\frac{2 n-(a+b)-2}{2}}\right)$. Let $s=\frac{a+b}{2}$. We will show that the sequence fails the $s$-th inequality of the Erdős-Gallai Theorem. By assumption, $(2 s+1)^{2}>4 n b+5$, so $n b<s^{2}+s-1$. As $n b$ is odd, this implies $n b \leq s^{2}+s-3$. Thus $b+1 \leq s^{2}+s-n b+b-2$. Therefore,

$$
\begin{aligned}
\sum_{i=1}^{s} d_{i}=a s & >a s-2=(2 s-b) s-2 \\
& =s(s-1)+\left[s^{2}+s-n b+b-2\right]+(n-s-1) b \\
& \geqslant s(s-1)+b+1+(n-s-1) b \\
& =s(s-1)+\sum_{i=s+1}^{n} \min \left\{s, d_{i}\right\}
\end{aligned}
$$

So (EG) fails for $k=s$. Finally, $\underline{d}$ has even sum since, as $a, b, n \equiv 1(\bmod 2)$,

$$
a s+(b+1)+(n-s-1) b \equiv s+0+s \equiv 0 \quad(\bmod 2) .
$$

This completes the proof of Theorem 2,

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[^0]:    1991 Mathematics Subject Classification. 05C07.
    Key words and phrases. graph, vertex degree, graphic sequence.

