# On the number of edges in a graph with no $(k+1)$-connected subgraphs 

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#### Abstract

Mader proved that for $k \geq 2$ and $n \geq 2 k$, every $n$-vertex graph with no $(k+1)$-connected subgraphs has at most $\left(1+\frac{1}{\sqrt{2}}\right) k(n-k)$ edges. He also conjectured that for $n$ large with respect to $k$, every such graph has at most $\frac{3}{2}\left(k-\frac{1}{3}\right)(n-k)$ edges. Yuster improved Mader's upper bound to $\frac{193}{120} k(n-k)$ for $n \geq \frac{9 k}{4}$. In this note, we make the next step towards Mader's Conjecture: we improve Yuster's bound to $\frac{19}{12} k(n-k)$ for $n \geq \frac{5 k}{2}$.

Mathematics Subject Classification: 05C35, 05C40 Key words and phrases: Average degree, connectivity, $k$-connected subgraphs.


## 1 Introduction

All graphs considered here are finite, undirected, and simple. For a graph $G, V(G)$ and $E(G)$ denote its vertex set and edge set respectively. If $U \subseteq V(G)$, then $G[U]$ denotes the induced subgraph of $G$ whose vertex set is $U$, and $G-U:=G[V(G) \backslash U]$. For $v \in V(G), N(v):=\{u \in V(G): u v \in E(G)\}$ denotes the neighborhood of $v$ in $G$.

Let $k \in \mathbb{N}$. Recall that a graph $G$ is $(k+1)$-connected if, for every set $S \subset V(G)$ of size $k$, the graph $G[V(G) \backslash S]$ is connected and contains at least two vertices (so $|V(G)| \geq k+2$ ). Mader [1] posed the following question:

What is the maximum possible number of edges in an $n$-vertex graph that does not contain a $(k+1)$-connected subgraph?

It is easy to see that for $k=1$ the answer is $n-1$ : every tree on $n$ vertices contains $n-1$ edges and no 2 -connected subgraphs, whereas every graph on $n$ vertices with at least $n$ edges contains a cycle, and cycles are 2-connected. Thus for the rest of the note we will assume $k \geq 2$.

The following construction due to Mader [2] gives an example of a graph with no ( $k+1$ )connected subgraphs and a large number of edges. Fix $k$ and $n$, and suppose that $n=k q+r$, where $1 \leq r \leq k$. The graph $G_{n, k}$ has vertex set $\bigcup_{i=0}^{q} V_{i}$, where the sets $V_{0}, \ldots, V_{q}$ are pairwise disjoint and satisfy the following contitions.

[^0]1. $\left|V_{0}\right|=\ldots=\left|V_{q-1}\right|=k$, while $\left|V_{q}\right|=r$.
2. $V_{0}$ is an independent set in $G_{n, k}$.
3. For $1 \leq i \leq q, V_{i}$ is a clique in $G_{n, k}$.
4. Every vertex in $V_{0}$ is adjacent to every vertex in $\bigcup_{i=1}^{q} V_{i}$.
5. $G_{n, k}$ has no other edges.

Note that $V_{0}$ is a separating set of size $k$ and every component of $G_{n, k}-V_{0}$ has at most $k$ vertices. It follows that $G_{n, k}$ has no $(k+1)$-connected subgraphs. A direct calculation shows that $G_{n, k}$ has at most $\frac{3}{2}\left(k-\frac{1}{3}\right)(n-k)$ edges, where the equality holds if $n$ is a multiple of $k$. Mader [2] conjectured that this example is, in fact, best possible.

Conjecture 1 (Mader [2]). Let $k \geq 2$. Then for $n$ sufficiently large, the number of edges in an $n$-vertex graph without a $(k+1)$-connected subgraph cannot exceed $\frac{3}{2}\left(k-\frac{1}{3}\right)(n-k)$.

Mader himself proved Conjecture $\mathbb{1}$ for $k \leq 6$. Moreover, he showed that for all $k$, the weaker version of the conjecture, where the coefficient $\frac{3}{2}$ is replaced by $1+\frac{1}{\sqrt{2}}$, holds. Yuster 4 improved this result by showing that the coefficient can be taken to be $\frac{193}{120}$.
Theorem 2 (Yuster [4]). Let $k \geq 2$ and $n \geq \frac{9 k}{4}$. Then every $n$-vertex graph $G$ with $|E(G)|>$ $\frac{193}{120} k(n-k)$ contains a $(k+1)$-connected subgraph.

Here we improve Yuster's bound, obtaining the value $\frac{19}{12}$ for the coefficient.
It turns out that for this problem, computations work out nicer if we "normalize" vertex and edge counts by assigning a weight $\frac{1}{k}$ to each vertex and a weight $\frac{1}{k^{2}}$ to each edge in a graph. Using this terminology, we can restate Conjecture 1 in the following way.

Conjecture 1. Let $k \geq 2$. Then for $\gamma$ sufficiently large, every graph $G$ with $\frac{1}{k}|V(G)|=\gamma$ and $\frac{1}{k^{2}}|E(G)|>\frac{3}{2}(\gamma-1)$ contains a $(k+1)$-connected subgraph.

Our main result in these terms is as follows.
Theorem 3. Let $k \geq 2$. Then every graph $G$ with $\frac{1}{k}|V(G)|=\gamma \geq \frac{5}{2}$ and $\frac{1}{k^{2}}|E(G)|>\frac{19}{12}(\gamma-1)$ contains a $(k+1)$-connected subgraph.

We follow the ideas of Mader and Yuster: Use induction on the number of vertices for graphs with at least $\frac{5}{2} k$ vertices. The hardest part is to prove the case when after deleting a separating set of size $k$, exactly one of the components of the remaining graph has fewer than $\frac{3}{2} k$ vertices, since the induction assuption does not hold for $n<\frac{5}{2} k$. New ideas in the proof are in Lemmas 8, (9, and 10 below.

## 2 Proof of Theorem 3

We want to derive a linear in $(n-k)$ bound on the number of edges in a graph that does not contain $(k+1)$-connected subgraphs. But the bound becomes linear only for graphs with large number of vertices; while for small graphs the dependency is quadratic in $n-k$. The main difficulties we encounter are around the transition between the quadratic and linear regimes. To deal with small $n$, we use the following lemma due to Matula [3], whose bound is asymptotically exact for $n<2 k$.

Lemma 4 (Matula [3]). Let $k \geq 2$. Then every graph $G$ with $|V(G)|=n \geq k+1$ and $|E(G)|>$ $\binom{n}{2}-\frac{1}{3}\left((n-k)^{2}-1\right)$ contains a $(k+1)$-connected subgraph.

We will use the following "normalized" version of this lemma.
Lemma 4. Let $k \geq 2$. Then every graph $G$ with $\frac{1}{k}|V(G)|=\gamma>1$, and

$$
\begin{equation*}
\frac{1}{k^{2}}|E(G)|>\frac{1}{6}\left(\gamma^{2}+4 \gamma-2\right) \tag{1}
\end{equation*}
$$

contains a $(k+1)$-connected subgraph.
Proof. Indeed, (1) yields

$$
\begin{aligned}
|E(G)| & >\frac{k^{2}}{6}\left(\gamma^{2}+4 \gamma-2\right) \\
& =\binom{\gamma k}{2}-\frac{1}{3}\left((\gamma k-k)^{2}-1\right)+\frac{\gamma k}{2}-\frac{1}{3} \\
& >\binom{\gamma k}{2}-\frac{1}{3}\left((\gamma k-k)^{2}-1\right)
\end{aligned}
$$

and we are done by original Matula's lemma.
From now on, fix a graph $G$ with $\frac{1}{k}|V(G)|=\gamma \geq \frac{5}{2}$ and $\frac{1}{k^{2}}|E(G)|>\frac{19}{12}(\gamma-1)$, and suppose for contradiction that $G$ does not contain a $(k+1)$-connected subgraph. Choose $G$ to have the least possible number of vertices (so we can apply induction hypothesis for subgraphs of $G$ ). Since $G$ itself is not $(k+1)$-connected, it contains a separating set $S \subset V(G)$ of size $k$. Let $A \subset V(G) \backslash S$ be such that $G[A]$ is a smallest connected component of $G-S$, and let $B:=V(G) \backslash(S \cup A)$. Let $\alpha:=\frac{1}{k}|A|$ and $\beta:=\frac{1}{k}|B|$.

We start by showing that the graph $G$ cannot be too small, using Matula's Lemma.
Lemma 5. $\gamma>3$.
Proof. Suppose that $\gamma \leq 3$. Then, by Lemma 4 ,

$$
\begin{equation*}
0 \leq \frac{1}{k^{2}}|E(G)|-\frac{19}{12}(\gamma-1) \leq \frac{1}{6}\left(\gamma^{2}+4 \gamma-2\right)-\frac{19}{12}(\gamma-1)=\frac{1}{12}\left(2 \gamma^{2}-11 \gamma+15\right) \tag{2}
\end{equation*}
$$

The function $g(\gamma)=2 \gamma^{2}-11 \gamma+15$ on the right-hand side of (2) is convex in $\gamma$. Hence it is maximized on the boundary of the interval $\left[\frac{5}{2} ; 3\right]$. But it is easy to check that $g\left(\frac{5}{2}\right)=g(3)=0$, hence it is nonpositive on the whole interval. Therefore, $\gamma>3$.

All the edges in $G$ either belong to the graph $G[S \cup B]$, or are incident to the vertices in $A$. The number of edges in $G[S \cup B]$ can be bounded either using Matula's lemma (which is efficient for $\beta \leq \frac{3}{2}$ ) or using the induction hypothesis (which can be applied if $\beta>\frac{3}{2}$ ). Hence the difficulty is in bounding the number of edges incident to the vertices in $A$.

The first step is to show that $A$ cannot be too large, because otherwise we can use induction.
Lemma 6. $\alpha<\frac{3}{2}$.

Proof. If $\alpha \geq \frac{3}{2}$, then we can apply the induction hypothesis both for $G[S \cup A]$ and for $G[S \cup B]$, and thus obtain

$$
\frac{1}{k^{2}}|E(G)| \leq \frac{19}{12} \alpha+\frac{19}{12} \beta=\frac{19}{12}(\alpha+\beta)=\frac{19}{12}(\gamma-1) .
$$

The next lemma shows that $A$ cannot be too small either, since otherwise the total number of edges between the vertices in $A$ and the vertices in $S \cup A$ is small.

Lemma 7. $\alpha>1$.
Proof. Suppose that $\alpha \leq 1$. Then $\beta>1$, since $\alpha+\beta+1=\gamma>3$. If $\beta \geq \frac{3}{2}$, then using the induction hypothesis for $G[S \cup B]$, we get

$$
\frac{1}{k^{2}}|E(G)| \leq \frac{1}{2} \alpha^{2}+\alpha+\frac{19}{12} \beta \leq \frac{3}{2} \alpha+\frac{19}{12} \beta<\frac{19}{12}(\alpha+\beta)=\frac{19}{12}(\gamma-1) .
$$

Thus $\beta<\frac{3}{2}$. Therefore, $\alpha>\frac{1}{2}$. In this case, applying Lemma 4 to $G[S \cup B]$ reduces the problem to proving the inequality

$$
\frac{1}{2} \alpha^{2}+\alpha+\frac{1}{6}\left((\beta+1)^{2}+4(\beta+1)-2\right) \leq \frac{19}{12}(\alpha+\beta)
$$

which is equivalent to

$$
\begin{equation*}
6 \alpha^{2}+2 \beta^{2}-7 \alpha-7 \beta+6 \leq 0 . \tag{3}
\end{equation*}
$$

For $\alpha$ fixed, the left-hand side of (3) is monotone decreasing in $\beta$ when $\beta<\frac{7}{4}$, so its maximum is attained at $\beta=1$. Thus (3) will hold if the function $g_{1}(\alpha)=6 \alpha^{2}-7 \alpha+1$ is nonpositive. Since $g_{1}(\alpha)$ is a convex function, its maximum on the interval $\left[\frac{1}{2} ; 1\right]$ is attained at one of the boundary points. We have

$$
g_{1}\left(\frac{1}{2}\right)=6 \cdot\left(\frac{1}{2}\right)^{2}-7 \cdot \frac{1}{2}+1=-1<0, \quad \text { and } \quad g_{1}(1)=6 \cdot 1^{2}-7 \cdot 1+1=0
$$

So we know that $1<\alpha<\frac{3}{2}$. How can we bound the number of edges incident to the vertices in $A$ ? The tricks of Lemma 7 and of Lemma 4 are not sufficient here. The idea is to combine them by applying Lemma 团 only to the graph $G\left[A \cup\left(S \backslash S^{\prime}\right)\right.$ ], where $S^{\prime}$ is a subset of $S$ with relatively few edges between $A$ and $S^{\prime}$. To obtain such set $S^{\prime}$, we will use Lemma $\quad$ below, which asserts that there are many vertices in $S$ that have not too many neighbors in $A$.

Lemma 8. Let $S_{1}:=\left\{v \in S: \frac{1}{k}|N(v) \cap A| \leq \frac{1}{2}(\alpha+1)\right\}$. Then $\frac{1}{k}\left|S_{1}\right|>\frac{1}{3}$.
Proof. Suppose that $\frac{1}{k}\left|S_{1}\right|=\sigma \leq \frac{1}{3}$. Let $G_{1}:=G\left[A \cup\left(S \backslash S_{1}\right)\right]$. Since $G_{1}$ is not $(k+1)$-connected, it has a separating set $T \subset V\left(G_{1}\right)$ of size $k$. Let $X$ and $Y$ form a partition of $V\left(G_{1}\right) \backslash T$ and be separated by $T$ in $G_{1}$. Without loss of generality assume that $|X \cap A| \geq|Y \cap A|$. Then

$$
\frac{1}{k}|X \cap A| \geq \frac{1}{2} \cdot \frac{1}{k}|A \backslash T| \geq \frac{1}{2}(\alpha-1)
$$

Hence if $v \in Y \cap S$, then

$$
\frac{1}{k}|N(v) \cap A| \leq \frac{1}{k}(|A|-|X \cap A|) \leq \frac{1}{2}(\alpha+1)
$$

which means that $v \in S_{1}$. But that is impossible, since $S_{1} \cap V\left(G_{1}\right)=\emptyset$. Thus $Y \cap S=\emptyset$, i.e. $Y \subset A$. In particular, since $|X \cap A| \geq|Y \cap A|=|Y|$, we have $\frac{1}{k}|Y| \leq \frac{1}{2} \alpha$. Then

$$
\frac{1}{k}|V(G) \backslash Y|=\alpha+\beta+1-\frac{1}{k}|Y| \geq \frac{1}{2} \alpha+\beta+1 \geq \frac{5}{2},
$$

so the induction hypothesis holds for $G-Y$, and

$$
\frac{1}{k^{2}}|E(G-Y)| \leq \frac{19}{12}\left(\frac{1}{k}|V(G-Y)|-1\right) .
$$

Hence we are done if

$$
\frac{1}{k^{2}}(|E(G)|-|E(G-Y)|) \leq \frac{19}{12} \cdot \frac{1}{k}|Y|,
$$

so assume that that is not the case. Let $\mu:=\frac{1}{k}|Y|$. Then

$$
\frac{1}{2} \mu^{2}+\mu(1+\sigma)>\frac{19}{12} \mu,
$$

so

$$
\mu>\frac{7}{6}-2 \sigma .
$$

We consider two cases.
Case 1: $X \cap S \neq \emptyset$. Let $v \in X \cap S$. Then $v$ has more than $k \cdot \frac{1}{2}(\alpha+1)$ neighbors in $A$, none of which belong to $Y$. Hence $\mu<\frac{1}{2}(\alpha-1)$, and so

$$
\frac{1}{2}(\alpha-1)>\frac{7}{6}-2 \sigma .
$$

Therefore,

$$
\alpha>\frac{10}{3}-4 \sigma \geq \frac{10}{3}-4 \cdot \frac{1}{3}=2 ;
$$

a contradiction.
Case 2: $X \cap S=\emptyset$. Then $S \backslash S_{1} \subset T$, and the set $T \cap A$ separates $X$ and $Y$ in $G[A]$ and satisfies $\frac{1}{k}|T \cap A|=\frac{1}{k}(|T|-|T \cap S|)=1-(1-\sigma)=\sigma$. Note that since $|Y| \leq|X|$, we have

$$
\frac{7}{6}-2 \sigma<\mu \leq \frac{1}{2}(\alpha-\sigma)
$$

so

$$
\sigma>\frac{7}{9}-\frac{1}{3} \alpha>\frac{7}{9}-\frac{1}{3} \cdot \frac{3}{2}=\frac{5}{18} .
$$

Now observe that

$$
\frac{1}{k^{2}}|E(G[A])| \leq \frac{1}{2} \alpha^{2}-\mu(\alpha-\sigma-\mu) .
$$

Since $\frac{7}{6}-2 \sigma<\mu \leq \frac{1}{2}(\alpha-\sigma)$, the latter expression is less than

$$
\frac{1}{2} \alpha^{2}-\left(\frac{7}{6}-2 \sigma\right) \cdot\left(\alpha+\sigma-\frac{7}{6}\right) .
$$

Hence $\frac{1}{k^{2}}(|E(G)|-|E(G-A)|)$ is less than

$$
\frac{1}{2}(\alpha+1) \sigma+\alpha(1-\sigma)+\frac{1}{2} \alpha^{2}-\left(\frac{7}{6}-2 \sigma\right) \cdot\left(\alpha+\sigma-\frac{7}{6}\right) .
$$

CASE 2.1. $\beta \leq \frac{3}{2}$. Then, after adding Matula's estimate for the number of edges in $G[S \cup B]$ and subtracting $\frac{19}{12}(\alpha+\beta)$, it is enough to prove that the following quantity is nonpositive:

$$
\begin{aligned}
& \frac{1}{2}(\alpha+1) \sigma+\alpha(1-\sigma)+\frac{1}{2} \alpha^{2}-\left(\frac{7}{6}-2 \sigma\right) \cdot\left(\alpha+\sigma-\frac{7}{6}\right) \\
+ & \frac{1}{6}\left((\beta+1)^{2}+4(\beta+1)-2\right)-\frac{19}{12}(\alpha+\beta),
\end{aligned}
$$

which in equal to

$$
\frac{1}{36}\left(18 \alpha^{2}+54 \alpha \sigma-63 \alpha+6 \beta^{2}-21 \beta+72 \sigma^{2}-108 \sigma+67\right)
$$

Note that for $\alpha$ and $\sigma$ fixed, the last expression is monotone decreasing in $\beta$ (recall that $\beta \leq \frac{3}{2}$, while the minimum is attained at $\beta=\frac{7}{4}$ ), so its maximum is attained when $\beta=\alpha$, where it turns into

$$
\varphi_{1}(\alpha, \sigma)=\frac{1}{36}\left(24 \alpha^{2}+54 \alpha \sigma-84 \alpha+72 \sigma^{2}-108 \sigma+67\right)
$$

Since $\varphi_{1}(\alpha, \sigma)$ is convex in both $\alpha$ and $\sigma$, it attains its maximum at some point $\left(\alpha_{0}, \sigma_{0}\right)$, where $\alpha_{0} \in\left\{1, \frac{3}{2}\right\}$ and $\sigma_{0} \in\left\{\frac{5}{18}, \frac{1}{3}\right\}$. It remains to check the four possibilities:

$$
\begin{gathered}
\varphi_{1}\left(1, \frac{5}{18}\right)=-\frac{11}{162}<0, \quad \varphi_{1}\left(1, \frac{1}{3}\right)=-\frac{1}{12}<0 \\
\varphi_{1}\left(\frac{3}{2}, \frac{5}{18}\right)=-\frac{125}{648}<0, \quad \text { and } \quad \varphi_{1}\left(\frac{3}{2}, \frac{1}{3}\right)=-\frac{1}{6}<0 .
\end{gathered}
$$

CASE 2.2. $\beta>\frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that the function

$$
\varphi_{2}(\alpha, \sigma)=\frac{1}{2}(\alpha+1) \sigma+\alpha(1-\sigma)+\frac{1}{2} \alpha^{2}-\left(\frac{7}{6}-2 \sigma\right) \cdot\left(\alpha+\sigma-\frac{7}{6}\right)-\frac{19}{12} \alpha
$$

is nonpositive. Again, we only have to check the boundary values:

$$
\begin{gathered}
\varphi_{2}\left(1, \frac{5}{18}\right)=-\frac{49}{324}<0, \quad \varphi_{2}\left(1, \frac{1}{3}\right)=-\frac{1}{6}<0, \\
\varphi_{2}\left(\frac{3}{2}, \frac{5}{18}\right)=-\frac{125}{648}<0, \quad \text { and } \quad \varphi_{2}\left(\frac{3}{2}, \frac{1}{3}\right)=-\frac{1}{6}<0 .
\end{gathered}
$$

This finishes the proof.
Now we can simply try to use as the set $S^{\prime}$ the set $S_{1}$ itself. This choice indeed gives a good bound if $A$ is large, as the next lemma shows.

Lemma 9. $\alpha<\frac{4}{3}$.

Proof. Suppose that $\alpha \geq \frac{4}{3}$. Recall that $\sigma=\left|S_{1}\right|>\frac{1}{3}$. Using Lemma 团 for $G\left[A \cup\left(S \backslash S_{1}\right)\right]$, we get that

$$
\begin{aligned}
& \frac{1}{k^{2}}(|E(G)|-|E(G-A)|) \\
\leq & \frac{1}{6}\left((\alpha+1-\sigma)^{2}+4(\alpha+1-\sigma)-2\right)+\frac{1}{2}(\alpha+1) \sigma \\
= & \frac{1}{6}\left(\alpha^{2}+\alpha \sigma+6 \alpha+\sigma^{2}-3 \sigma+3\right) .
\end{aligned}
$$

Case 1: $\beta \leq \frac{3}{2}$. Adding Matula's estimate for $G[B \cup S]$ and subtracting $\frac{19}{12}(\alpha+\beta)$, we get

$$
\begin{aligned}
& \frac{1}{k^{2}}|E(G)|-\frac{19}{12}(\alpha+\beta) \\
& \quad \leq \frac{1}{6}\left(\alpha^{2}+\alpha \sigma+6 \alpha+\sigma^{2}-3 \sigma+3\right)+\frac{1}{6}\left((\beta+1)^{2}+4(\beta+1)-2\right)-\frac{19}{12}(\alpha+\beta) \\
& \quad=\frac{1}{12}\left(2 \alpha^{2}+2 \alpha \sigma-7 \alpha+2 \beta^{2}-7 \beta+2 \sigma^{2}-6 \sigma+12\right) .
\end{aligned}
$$

Again, the maximum is attained when $\beta=\alpha$, so we should consider the expression

$$
\varphi_{3}(\alpha, \sigma)=\frac{1}{6}\left(2 \alpha^{2}+\alpha \sigma-7 \alpha+\sigma^{2}-3 \sigma+6\right) .
$$

It is convex in both $\alpha$ and $\sigma$, so again it is enough to check the boundary points:

$$
\begin{gathered}
\varphi_{3}\left(\frac{4}{3}, \frac{1}{3}\right)=-\frac{1}{27}<0, \quad \varphi_{3}\left(\frac{4}{3}, 1\right)=-\frac{2}{27}<0, \\
\varphi_{3}\left(\frac{3}{2}, \frac{1}{3}\right)=-\frac{7}{108}<0, \quad \text { and } \quad \varphi_{3}\left(\frac{3}{2}, 1\right)=-\frac{1}{12}<0 .
\end{gathered}
$$

CASE 2: $\beta>\frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that the function

$$
\varphi_{4}(\alpha, \sigma)=\frac{1}{6}\left(\alpha^{2}+\alpha \sigma+6 \alpha+\sigma^{2}-3 \sigma+3\right)-\frac{19}{12} \alpha=\frac{1}{12}\left(2 \alpha^{2}+2 \alpha \sigma-7 \alpha+2 \sigma^{2}-6 \sigma+6\right)
$$

is nonpositive. The function is convex in both $\alpha$ and $\sigma$, so we check the boundary points:

$$
\begin{gathered}
\varphi_{4}\left(\frac{4}{3}, \frac{1}{3}\right)=-\frac{1}{18}<0, \quad \varphi_{4}\left(\frac{4}{3}, 1\right)=-\frac{5}{54}<0, \\
\varphi_{4}\left(\frac{3}{2}, \frac{1}{3}\right)=-\frac{7}{108}<0, \quad \text { and } \quad \varphi_{4}\left(\frac{3}{2}, 1\right)=-\frac{1}{12}<0 .
\end{gathered}
$$

This finishes the proof.
The next lemma is the final piece of the jigsaw. It shows that if $A$ is small, we can still obtain the desired bound if we take the set $S^{\prime}$ to be slightly bigger than $S_{1}$.

Lemma 10. $\alpha>\frac{4}{3}$.

Proof. Suppose that $\alpha \leq \frac{4}{3}$. Then $1-2(\alpha-1) \geq \frac{1}{3}$. Let $S^{\prime}$ be a subset of $S$ with $\frac{1}{k}\left|S^{\prime}\right|=1-2(\alpha-1)$ such that $\frac{1}{k}\left|S^{\prime} \cap S_{1}\right| \geq \frac{1}{3}$. Observe that the normalized number of edges between $A$ and $S^{\prime}$ is at most

$$
\frac{1}{k^{2}}|A| \cdot\left|S^{\prime}\right|-\frac{1}{2}(\alpha-1) \cdot \frac{1}{3}
$$

by the definition of $S_{1}$. Hence, using Lemma for $G\left[A \cup\left(S \backslash S^{\prime}\right)\right]$, we get that

$$
\begin{aligned}
\frac{1}{k^{2}}(|E(G)|-|E(G-A)|) & \leq \frac{1}{6}\left((3 \alpha-2)^{2}+4(3 \alpha-2)-2\right)+\alpha(3-2 \alpha)-\frac{1}{6}(\alpha-1) \\
& =\frac{1}{6}\left(-3 \alpha^{2}+17 \alpha-5\right)
\end{aligned}
$$

Case 1: $\beta \leq \frac{3}{2}$. Adding Matula's estimate for $G[B \cup S]$ and subtracting $\frac{19}{12}(\alpha+\beta)$, we get

$$
\begin{aligned}
\frac{1}{k^{2}}|E(G)|-\frac{19}{12}(\alpha+\beta) & \leq \frac{1}{6}\left(-3 \alpha^{2}+17 \alpha-5\right)+\frac{1}{6}\left((\beta+1)^{2}+4(\beta+1)-2\right)-\frac{19}{12}(\alpha+\beta) \\
& =\frac{1}{12}\left(-6 \alpha^{2}+15 \alpha+2 \beta^{2}-7 \beta-4\right)
\end{aligned}
$$

Since $\alpha \leq \beta \leq \frac{3}{2}$, the maximum is attained when $\beta=\alpha$, in which case the last expression turns into $-\frac{1}{3}(\alpha-1)^{2} \leq 0$.

CASE 2: $\beta>\frac{3}{2}$. Then, instead of using Matula's bound for $G[S \cup B]$, we can apply the induction hypothesis, so it is enough to prove that

$$
\begin{equation*}
\frac{1}{6}\left(-3 \alpha^{2}+17 \alpha-5\right)-\frac{19}{12} \alpha=-\frac{1}{12}\left(6 \alpha^{2}-15 \alpha+10\right) \leq 0 \tag{4}
\end{equation*}
$$

Since the discriminant of the quadratic $6 \alpha^{2}-15 \alpha+10$ is negative, (4) holds for all $\alpha$, and we are done.

Since Lemmas 9 and 10 contradict each other, we conclude that such graph $G$ does not exist. This completes the proof of the theorem.

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