Updown Categories: Generating Functions and Universal Covers

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Abstract

A poset can be regarded as a category in which there is at most one morphism between objects, and such that at most one of $\text{Hom}(c, c')$ and $\text{Hom}(c', c)$ is nonempty for $c \neq c'$. If we keep in place the latter axiom but allow for more than one morphism between objects, we have a sort of generalized poset in which there are multiplicities attached to the covering relations, and possibly nontrivial automorphism groups. We call such a category an "updown category." In this paper we give a precise definition of such categories and develop a theory for them. We also give a detailed account of ten examples, including updown categories of integer partitions, integer compositions, planar rooted trees, and rooted trees.

1   Introduction

Suppose we have a collection of combinatorial objects, naturally graded, so that any object of rank n can be built up in n steps from a single object in rank 0. Further, for any object p of rank n and q of rank $n + 1$, there are some number $u(p; q)$ of ways to build up p to make q, and $d(p; q)$ ways to break down q to get p. Here $u(p; q)$ and $d(p; q)$ are nonnegative integers, possibly unequal, though we require that $u(p; q) \neq 0$ if and only if $d(p; q) \neq 0$. For example (as in [6]) our collection might be the set of rooted trees, with $u(p; q)$ the number of vertices of the rooted tree p to which a new edge and terminal vertex can be added to get q, and $d(p; q)$ the number of terminal vertices (and incoming edges) that can be removed from q to get p.

We can obtain a natural definition of such a situation by modifying the categorical definition of a poset. A poset is usually thought of as a category with at most one morphism between objects, and at most one of the sets $\text{Hom}(p, q)$ and $\text{Hom}(p, q)$ nonempty

when $p \neq cq$. If we keep in place the second condition but permit $\mathrm{Hom}(p, q)$ to have more than one element, we allow for multiplicities (if $p \neq q$) and automorphisms (if $p = q$). If in addition the object set is graded, we call such a category (precisely defined in §2 below) an "updown category." For an updown category $\mathcal{C}$, there are nonnegative integers $u(p; q)$ and $d(p; q)$ for $p, q \in \mathrm{Ob}\,\mathcal{C}$ with q having rank one greater than p, such that

$$ u(p; q)|\mathrm{Aut}\,q| = d(p; q)|\mathrm{Aut}\,p|. $$

Then the set $\mathrm{Ob}\,\mathcal{C}$ has a natural graded poset structure, and the operators U and D on the free vector space $\Bbbk(\mathrm{Ob}\,\mathcal{C})$ defined by equations

$$ Up = \sum_{q \text{ covers } p} u(p; q)q \quad \text{and} \quad Dp = \sum_{p \text{ covers } q} d(q; p)q. $$

are adjoint for the inner product on $\Bbbk(\mathrm{Ob}\,\mathcal{C})$ given by $\langle p, q \rangle = |\mathrm{Aut}\,p|\delta_{p,q}$.

For any updown category $\mathcal{C}$, there are associated two generating functions, defined in §3: the object generating function and the morphism generating function. If $\mathcal{C}$ is a univalent updown category (i.e., $u(p; q) = d(p; q)$ for all $p, q \in \mathrm{Ob}\,\mathcal{C}$), then the former is the rank-generating function of the graded poset $\mathrm{Ob}\,\mathcal{C}$. Computation of these generating functions is facilitated if $\mathcal{C}$ is evenly up-covered (i.e., $\sum_{q \text{ covers } p} u(p, q)$ depends only on the grade $|p|$ of p) or evenly down-covered ($\sum_{p \text{ covers } q} d(q; p)$ only depends on $|p|$).

Univalent updown categories admit a natural definition of universal covers. In [5] the author developed a theory of universal covers for weighted-relation posets, i.e., ranked posets in which each covering relation has a single number $n(x, y)$ assigned to it. The universal cover of a weighted-relation poset P is the "unfolding" of P into a usually much larger weighted-relation poset $\widetilde{P}$, so that the Hasse diagram of $\widetilde{P}$ is a tree and all covering relations of $\widetilde{P}$ have multiplicity 1. Although $\widetilde{P}$ had a natural description in each of the seven examples considered in [5], the general construction of $\widetilde{P}$ given in [5, Theorem 3.3] was somewhat unsatisfactory since it involved many arbitrary choices. In §4 we show that univalent updown categories are essentially "categorified" weighted-relation posets and give a functorial definition of universal covers for them (Theorem 4.3 below). We also give a functorial description of two univalent updown categories $\mathcal{C}^{\uparrow}$ and $\mathcal{C}^{\downarrow}$ associated with an updown category $\mathcal{C}$.

In §5 we offer ten examples, which encompass all those given in [5]. These include updown categories whose objects are the subsets of a finite set, monomials, necklaces, integer partitions, integer compositions, planar rooted trees, and rooted trees. For each example we compute the object and morphism generating functions and describe the associated covering spaces.

2 Updown categories

We begin by defining an updown category.

Definition 2.1. An updown category is a small category $\mathcal{C}$ with a rank functor $|\cdot| : \mathcal{C} \to \mathbb{N}$ (where $\mathbb{N}$ is the ordered set of natural numbers regarded as a category) such that

A1. Each rank $\mathcal{C}_n = \{p \in \mathrm{Ob}\,\mathcal{C} : |p| = n\}$ is finite.

A2. The zeroth rank $\mathcal{C}_0$ consists of a single object $\hat{0}$, and $\mathrm{Hom}(\hat{0}, p)$ is nonempty for all objects p of $\mathcal{C}$.

A3. For objects p, p' of $\mathcal{C}$, $\mathrm{Hom}(p, p')$ is always finite, and $\mathrm{Hom}(p, p') = \emptyset$ unless $|p| < |p'|$ or $p = p'$. In the latter case, $\mathrm{Hom}(p, p)$ is a group, denoted $\mathrm{Aut}(p)$.

A4. Any morphism $p \to p'$, where $|p'| = |p| + k$, factors as a composition $p = p_0 \to p_1 \to \cdots \to p_k = p'$, where $|p_{i+1}| = |p_i| + 1$;

A5. If $|p'| = |p| + 1$, the actions of $\mathrm{Aut}(p)$ and $\mathrm{Aut}(p')$ on $\mathrm{Hom}(p, p')$ (by precomposition and postcomposition respectively) are free.

Given an updown category, we can define the multiplicities mentioned in the introduction as follows.

Definition 2.2. For any two objects p, p' of an updown category $\mathcal{C}$ with $|p'| = |p| + 1$, define
$$u(p; p') = |\mathrm{Hom}(p, p')/\mathrm{Aut}(p')| = \frac{|\mathrm{Hom}(p, p')|}{|\mathrm{Aut}(p')|}$$
and
$$d(p; p') = |\mathrm{Hom}(p, p')/\mathrm{Aut}(p)| = \frac{|\mathrm{Hom}(p, p')|}{|\mathrm{Aut}(p)|}.$$

It follows immediately from these definitions that
$$u(p; p')|\mathrm{Aut}(p')| = d(p; p')|\mathrm{Aut}(p)|. \tag{1}$$

We note two extreme cases. First, suppose $\mathcal{C}_n$ is empty for all $n > 0$. Then $\mathcal{C}$ is essentially the finite group $\mathrm{Aut}\,\hat{0}$. Second, suppose that every set $\mathrm{Hom}(p, p')$ has at most one element. Then $\mathcal{C}$ is a graded poset with least element $\hat{0}$.

Two important special types of updown categories are defined as follows.

Definition 2.3. An updown category $\mathcal{C}$ is univalent if $\operatorname{Aut}(p)$ is trivial for all $p \in \operatorname{Ob}\mathcal{C}$. An updown category $\mathcal{C}$ is simple if $\operatorname{Hom}(c, c')$ has at most one element for all $c, c' \in \operatorname{Ob}\mathcal{C}$, and the factorization in A4 is unique, i.e., for $|c'| > |c|$ any $f \in \operatorname{Hom}(c, c')$ has a unique factorization into morphisms between adjacent ranks.

Of course simple implies univalent, but not conversely. A univalent updown category is the "categorification" of a weighted-relation poset in the sense of [5]; see §4 below for details.

If $\mathcal{C}$ and $\mathcal{D}$ are updown categories, their product $\mathcal{C} \times \mathcal{D}$ is the usual one, i.e. $\operatorname{Ob}(\mathcal{C} \times \mathcal{D}) = \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{D}$ and

$$\operatorname{Hom}_{\mathcal{C}\times\mathcal{D}}((c, d), (c', d')) = \operatorname{Hom}_{\mathcal{C}}(c, c') \times \operatorname{Hom}_{\mathcal{D}}(d, d').$$

The rank is defined on $\mathcal{C} \times \mathcal{D}$ by $|(c, d)| = |c| + |d|$. We have the following result.

Proposition 2.1. *If $\mathcal{C}$ and $\mathcal{D}$ are updown categories, then so is their product $\mathcal{C} \times \mathcal{D}$.*

Proof. Since

$$(\mathcal{C} \times \mathcal{D})_n = \coprod_{i+j=n} \mathcal{C}_i \times \mathcal{D}_j,$$

axiom A1 is clear; and evidently $\hat{0} = (\hat{0}_{\mathcal{C}}, \hat{0}_{\mathcal{D}})$ satisfies A2. Checking A3 is routine, and for A4 we can combine factorizations

$$c = c_0 \to c_1 \to \cdots \to c_k = c' \quad \text{and} \quad d = d_0 \to d_1 \to \cdots \to d_l = d'$$

into

$$(c, d) \to (c_1, d) \to \cdots \to (c', d) \to (c', d_1) \to \cdots \to (c', d').$$

Finally, for A5 note that, e.g.,

$$\operatorname{Hom}((c, d), (c', d)) \cong \operatorname{Hom}(c, c') \times \operatorname{Aut}(d),$$

and the action of $\operatorname{Aut}(c, d) \cong \operatorname{Aut}(c) \times \operatorname{Aut}(d)$ on this set is free if and only if the action of $\operatorname{Aut}(c)$ on $\operatorname{Hom}(c, c')$ is free. $\square$

We note that the product of two univalent updown categories is univalent, but the product of simple updown categories need not be simple: see Example 1 in §6 below.

We now define a morphism of updown categories.

Definition 2.4. Let $\mathcal{C}, \mathcal{D}$ be updown categories. A morphism from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ with $|F(p)| = |p|$ for all $p \in \operatorname{Ob}\mathcal{C}$, and such that, for any $p, q \in \operatorname{Ob}\mathcal{C}$ with $|q| = |p| + 1$, the induced maps

$$\operatorname{Aut}(p) \to \operatorname{Aut}(F(p)),$$

$$\coprod_{\{q':F(q')=F(q)\}} \operatorname{Hom}(p, q')/\operatorname{Aut}(p) \to \operatorname{Hom}(F(p), F(q))/\operatorname{Aut}(F(p)),$$

and

$$\coprod_{\{q':F(q')=F(q)\}} \operatorname{Hom}(p, q')/\operatorname{Aut}(q') \to \operatorname{Hom}(F(p), F(q))/\operatorname{Aut}(F(q))$$

are injective.

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We have the following result.

Proposition 2.2. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a morphism of updown categories. If $\mathcal{D}$ is univalent, then so is $\mathcal{C}$; if $\mathcal{D}$ is simple, then $\mathcal{C}$ is also simple and F is injective as a function on object sets.*

Proof. It follows immediately from Definition 2.4 that $\mathcal{C}$ must be univalent when $\mathcal{D}$ is. Now suppose $\mathcal{D}$ is simple. Then $\mathcal{C}$ is univalent, and it follows from Definition 2.4 that the induced function

$$\coprod_{\{q':F(q')=F(q)\}} \mathrm{Hom}(p, q') \to \mathrm{Hom}(F(p), F(q))$$

is injective when $|q| = |p| + 1$: but $\mathrm{Hom}(F(p), F(q))$ is (at most) a one-element set, so F must be injective on object sets and $\mathrm{Hom}(p, q)$ can have at most one object. But then unique factorization of morphisms in $\mathcal{C}$ follows from that in $\mathcal{D}$, so $\mathcal{C}$ is simple. $\square$

There is a morphism of updown categories $\mathcal{C} \to \mathcal{C} \times \mathcal{D}$ given by sending $c \in \mathrm{Ob}\,\mathcal{C}$ to $(c, \hat{0}_{\mathcal{D}})$ whenever $\mathcal{C}$ and $\mathcal{D}$ are updown categories; similarly there is a morphism $\mathcal{D} \to \mathcal{C} \times \mathcal{D}$. We denote the n-fold cartesian power of $\mathcal{C}$ by $\mathcal{C}^n$.

Let $\Bbbk$ be a field of characteristic 0, $\Bbbk(\mathrm{Ob}\,\mathcal{C})$ the free vector space on $\mathrm{Ob}\,\mathcal{C}$ over $\Bbbk$. We now define "up" and "down" operators on $\Bbbk(\mathrm{Ob}\,\mathcal{C})$.

Definition 2.5. For an updown category $\mathcal{C}$, let $U, D : \Bbbk(\mathrm{Ob}\,\mathcal{C}) \to \Bbbk(\mathrm{Ob}\,\mathcal{C})$ be the the linear operators given by

$$Up = \sum_{|p'|=|p|+1} u(p; p')p'$$

and

$$Dp = \begin{cases} \sum_{|p'|=|p|-1} d(p'; p)p', & |p| > 0, \\ 0, & p = \hat{0}, \end{cases}$$

for all $p \in \mathrm{Ob}\,\mathcal{C}$.

Theorem 2.1. *The operators U and D are adjoint with respect to the inner product on $\Bbbk(\mathrm{Ob}\,\mathcal{C})$ defined by*

$$\langle p, p' \rangle = \begin{cases} |\mathrm{Aut}(p)|, & \text{if } p' = p, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $\langle Up, p' \rangle = \langle p, Dp' \rangle = 0$ unless $|p'| = |p| + 1$, it suffices to consider that case. Then

$$\langle Up, p' \rangle = u(p; p')\langle p', p' \rangle = u(p; p')|\mathrm{Aut}(p')|$$

while

$$\langle p, Dp' \rangle = d(p; p')\langle p, p \rangle = d(p; p')|\mathrm{Aut}(p)|,$$

and the two agree by equation (1). $\square$

Now we extend the definitions of $u(p; p')$ and $d(p; p')$ to any pair $p, p' \in \text{Ob}\,\mathcal{C}$ by setting $u(p; p') = d(p; p') = 0$ if $\text{Hom}(p, p') = \emptyset$ and

$$u(p; p') = \frac{\langle U^{|p'|-|p|}(p), p' \rangle}{|\operatorname{Aut}(p')|}, \quad d(p; p') = \frac{\langle U^{|p'|-|p|}(p), p' \rangle}{|\operatorname{Aut}(p)|} \tag{2}$$

otherwise. It is immediate that equation (1) still holds, and that

$$U^k(p) = \sum_{|p'|=|p|+k} u(p; p') p'$$

and

$$D^k(p) = \sum_{|p'|=|p|-k} d(p; p') p'$$

for any $p \in \text{Ob}\,\mathcal{C}$. (However, it is no longer true that $u(p; q)$ and $d(p; q)$ have any simple relation to $|\operatorname{Hom}(p, q)|$ when $|q| - |p| > 1$.) An important special case of the extended equation (1) is

$$\frac{d(\hat{0}; p)}{u(\hat{0}; p)} = \frac{|\operatorname{Aut}(p)|}{|\operatorname{Aut}\hat{0}|} \tag{3}$$

for any object p of $\mathcal{C}$. If $\operatorname{Aut}\hat{0}$ is trivial, equation (3) gives the order of the automorphism group of $p \in \text{Ob}\,\mathcal{C}$ as a ratio of multiplicities (cf. Proposition 2.6 of [6]). We also have the following result.

Theorem 2.2. *If $|p| \le k \le |q|$, then*

$$u(p; q) = \sum_{|p'|=k} u(p; p') u(p'; q),$$

and similarly for u replaced by d.

Proof. Using equation (2), we can write $u(p, q)$ as

$$\frac{\langle U^{|q|-|p|}p, q \rangle}{|\operatorname{Aut}(q)|} = \frac{1}{|\operatorname{Aut}(q)|} \langle U^{k-|p|} U^{|q|-k}(p), q \rangle = \frac{1}{|\operatorname{Aut}(q)|} \sum_{|p'|=k} u(p; p') \langle U^{k-|p|} p', q \rangle$$

$$= \frac{1}{|\operatorname{Aut}(q)|} \sum_{|p'|=k} u(p; p') u(p'; q) |\operatorname{Aut}(q)| = \sum_{|p'|=k} u(p; p') u(p'; q),$$

and the proof for d is similar. $\qquad\square$

Definition 2.6. For an updown category $\mathcal{C}$, define the induced partial order on $\text{Ob}\,\mathcal{C}$ by setting $p \preceq q$ if and only if $\operatorname{Hom}(p, q) \ne \emptyset$.

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It follows from Theorem 2.2 that $p \preceq q \iff u(p; q) \neq 0 \iff d(p; q) \neq 0$. Henceforth we write $p \lhd q$ if q covers p in the induced partial order. Of course different updown categories can have the same induced poset: see Examples 5 and 6 of §5 below. If the updown category $\mathcal{C}$ was a poset to start with (thought of as a category in the usual way), then all the weights $u(p; q) = d(p; q)$ assigned to the covering relations in $(\text{Ob}\,\mathcal{C}, \preceq)$ are 1. We call such an updown category unital. Evidently simple $\implies$ unital $\implies$ univalent.

In the univalent case, equation (3) is trivial since $u(p; q) = d(p; q)$ for all p and q. Nevertheless, we have the following interpretation of the multiplicity in this case.

Theorem 2.3. *Let $\mathcal{C}$ be a univalent updown category. For $p, q \in \text{Ob}\,\mathcal{C}$ with $|q| - |p| = n > 0$, $u(p; q) = d(p; q)$ is the number of distinct strings $(h_1, \ldots, h_n)$ such that each h_i is a morphism between adjacent ranks and $h_n h_{n-1} \cdots h_1$ is a morphism from p to q.*

Proof. We use induction on n. The result is immediate if $n = 1$, since in a univalent updown category

$$u(p; q) = d(p; q) = |\text{Hom}(p, q)|$$

when $|q| = |p| + 1$. Now if $N(p, q)$ denotes the number of strings $(h_1, \ldots, h_n)$ as in the statement of the proposition, it is evident that, for $|q| > |p| + 1$,

$$N(p, q) = \sum_{r \lhd q} N(p, r) N(r, q).$$

But then the inductive step follows from Theorem 2.2. $\qquad\square$

3 Even Covering and Generating Functions

In this section we introduce the even covering properties, which are satisfied in many of the examples of updown categories given in §6 below. We also define several generating functions associated with any updown category.

Definition 3.1. Let $\mathcal{C}$ be an updown category. Then $\mathcal{C}$ is evenly up-covered if there is a sequence of numbers $u_0, u_1, \ldots$ so that, for any $p \in \mathcal{C}_n$,

$$\sum_{q \rhd p} u(p; q) = u_n.$$

Dually, $\mathcal{C}$ is evenly down-covered if there is a sequence of numbers $d_1, d_2, \ldots$ so that, for any $p \in \mathcal{C}_n$,

$$\sum_{q \lhd p} d(q; p) = d_n.$$

We note that any simple updown category is evenly down-covered with $d_n = 1$ for all n. Another special case occurs often enough that we make the following definition.

Definition 3.2. We call $\mathcal{C}$ factorial if it is evenly down-covered with

$$\sum_{q \lhd p} d(q; p) = |p|$$

for all $p \in \mathrm{Ob}\,\mathcal{C}$.

If $\mathcal{C}$ is evenly up-covered, then by induction using Theorem 2.2 it follows that

$$\sum_{|c|=n} u(\hat{0}; c) = u_0 u_1 \cdots u_{n-1} \tag{4}$$

for $c \in \mathcal{C}_n$. On the other hand, if $\mathcal{C}$ is evenly down-covered then one has $D^n c = d_n d_{n-1} \cdots d_1 \hat{0}$ for $c \in \mathcal{C}_n$, and so

$$d(\hat{0}; c) = \frac{\langle \hat{0}, D^n c \rangle}{|\operatorname{Aut} \hat{0}|} = d_n d_{n-1} \cdots d_1 \tag{5}$$

for such c. In particular, $d(\hat{0}; c) = |c|!$ for all $c \in \mathrm{Ob}\,\mathcal{C}$ if $\mathcal{C}$ is factorial.

Although the even covering properties are not generally preserved under products, we do have the following result.

Theorem 3.1. *If $\mathcal{C}$ and $\mathcal{D}$ are factorial updown categories, then so is $\mathcal{C} \times \mathcal{D}$.*

Proof. Any object covered by $(c, d) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$ must have the form (c', d) with $c' \lhd c$, or (c, d'), with $d' \lhd d$. Thus

$$\sum_{p \lhd (c,d)} d(p; (c, d)) = \sum_{c' \lhd c} d((c', d); (c, d)) + \sum_{d' \lhd d} d((c, d'); (c, d))$$

$$= \sum_{c' \lhd c} d(c'; c) + \sum_{d' \lhd d} d(d'; d) = |c| + |d| = |(c, d)|.$$

$\square$

Neither of the two even covering properties implies the other. Examples 8 and 10 of §5 below are evenly up-covered but not evenly down-covered, and it is easy to construct simple updown categories that are not evenly up-covered. For simple updown categories that are evenly up-covered, we have the following result.

Theorem 3.2. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are simple updown categories that are both evenly up-covered with the same sequence $\{u_n\}$. Then $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as updown categories.*

Proof. It suffices to give a functor $F : \mathcal{C} \to \mathcal{D}$ that is bijective on the object sets such that $F(c) \lhd F(c')$ for all $c, c' \in \mathrm{Ob}\,\mathcal{C}$. We proceed by induction on rank. For rank 0 we set $F(\hat{0}_{\mathcal{C}}) = \hat{0}_{\mathcal{D}}$. Suppose F has been defined through rank n. For each $p \in \mathcal{C}_n$, choose a bijection ϕ_p from $C^+(p) = \{p' \in \mathcal{C}_{n+1} | p \lhd p'\}$ to $C^+(F(p))$ (which is possible since both sets have u_n elements). Then for $q \in \mathcal{C}_{n+1}$, set $F(q) = \phi_p(q)$, where p is the unique element of $\mathcal{C}_n$ that q covers. $\square$

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Now we turn to generating functions.

Definition 3.3. Let $\mathcal{C}$ be an updown category. The object generating function of $\mathcal{C}$ is

$$O_{\mathcal{C}}(t) = \sum_{p \in \mathrm{Ob}\,\mathcal{C}} \frac{t^{|p|}}{|\operatorname{Aut}(p)|} = \sum_{n \geq 0} \sum_{p \in \mathcal{C}_n} \frac{t^n}{|\operatorname{Aut}(p)|},$$

and the morphism generating function of $\mathcal{C}$ is

$$M_{\mathcal{C}}(t) = \sum_{p,q \in \mathrm{Ob}\,\mathcal{C},\ p \lhd q} \frac{u(p;q)t^{|p|+|q|}}{|\operatorname{Aut}(p)|} = \sum_{n \geq 0} \sum_{p \in \mathcal{C}_n} \sum_{q \in \mathcal{C}_{n+1}} \frac{u(p;q)t^{2n+1}}{|\operatorname{Aut}(p)|}. \tag{6}$$

Both $O_{\mathcal{C}}(t)$ and $M_{\mathcal{C}}(t)$ are elements of the formal power series ring $\mathbb{Q}[[t]]$. If $\mathcal{C}$ is univalent, then

$$O_{\mathcal{C}}(t) = \sum_{n \geq 0} |\mathcal{C}_n| t^n$$

and

$$M_{\mathcal{C}}(t) = \sum_{n \geq 0} \sum_{p \in \mathcal{C}_n} \sum_{q \in \mathcal{C}_{n+1}} |\operatorname{Hom}(p,q)| t^{2n+1}$$

are elements of $\mathbb{Z}[[t]]$. In view of equation (1), the morphism generating function can be written

$$M_{\mathcal{C}}(t) = \sum_{p,q \in \mathrm{Ob}\,\mathcal{C},\ p \lhd q} \frac{d(p;q)t^{|p|+|q|}}{|\operatorname{Aut}(q)|}. \tag{7}$$

Definition 3.4. For an updown category $\mathcal{C}$, the formal series of $\mathcal{C}$ is

$$S_{\mathcal{C}}(t) = \sum_{p \in \mathrm{Ob}\,\mathcal{C}} \frac{p\,t^{|p|}}{|\operatorname{Aut}(p)|} \in \Bbbk(\mathrm{Ob}\,\mathcal{C})[[t]].$$

These definitions are related by the following result.

Theorem 3.3. *If the inner product $\langle,\rangle$ of Theorem 2.1 is extended to $\Bbbk(\mathrm{Ob}\,\mathcal{C})[[t]]$, then*

$$\langle S_{\mathcal{C}}(t), S_{\mathcal{C}}(t) \rangle = O_{\mathcal{C}}(t^2) \tag{8}$$

and

$$\langle U S_{\mathcal{C}}(t), S_{\mathcal{C}}(t) \rangle = \langle S_{\mathcal{C}}(t), D S_{\mathcal{C}}(t) \rangle = M_{\mathcal{C}}(t). \tag{9}$$

Proof. Immediate from Theorem 2.1 and the definitions. $\qquad\square$

The generating functions of a product can be obtained from those of its factors as follows.

Corollary 3.1. *For updown categories $\mathcal{C}$ and $\mathcal{D}$,*

$$O_{\mathcal{C} \times \mathcal{D}}(t) = O_{\mathcal{C}}(t) O_{\mathcal{D}}(t) \tag{10}$$

and

$$M_{\mathcal{C} \times \mathcal{D}}(t) = M_{\mathcal{C}}(t) O_{\mathcal{D}}(t^2) + O_{\mathcal{C}}(t^2) M_{\mathcal{D}}(t). \tag{11}$$

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Proof. We have $S_{\mathcal{C}\times\mathcal{D}}(t) = S_{\mathcal{C}}(t) \otimes S_{\mathcal{D}}(t)$ under the evident identification of $\Bbbk(\mathrm{Ob}(\mathcal{C}\times\mathcal{D}))$ with $\Bbbk(\mathrm{Ob}\,\mathcal{C}) \otimes \Bbbk(\mathrm{Ob}\,\mathcal{D})$. Hence

$$\langle S_{\mathcal{C}\times\mathcal{D}}(t), S_{\mathcal{C}\times\mathcal{D}}(t)\rangle = \langle S_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t), S_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t)\rangle = \langle S_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle\langle S_{\mathcal{D}}(t), S_{\mathcal{D}}(t)\rangle,$$

and equation (10) follows using equation (8). Similarly, we have

$$\langle US_{\mathcal{C}\times\mathcal{D}}(t), S_{\mathcal{C}\times\mathcal{D}}(t)\rangle = \langle U(S_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t)), S_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t)\rangle =$$
$$\langle US_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t) + S_{\mathcal{C}}(t)\otimes US_{\mathcal{D}}(t), S_{\mathcal{C}}(t)\otimes S_{\mathcal{D}}(t)\rangle =$$
$$\langle US_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle\langle S_{\mathcal{D}}(t), S_{\mathcal{D}}(t)\rangle + \langle S_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle\langle US_{\mathcal{D}}(t), S_{\mathcal{D}}(t)\rangle$$

from which equation (11) follows via equation (9). $\qquad\square$

Remark. It follows from the preceding result that

$$O_{\mathcal{C}^n}(t) = (O_{\mathcal{C}}(t))^n \quad \text{and} \quad M_{\mathcal{C}^n}(t) = nM_{\mathcal{C}}(t)(O_{\mathcal{C}}(t^2))^{n-1}$$

where $\mathcal{C}^n$ is the n-fold product of $\mathcal{C}$.

If $\mathcal{C}$ is evenly up-covered or evenly down-covered, there is a direct relation between the object and morphism generating functions.

Theorem 3.4. *Let $\mathcal{C}$ be an updown category with*

$$O_{\mathcal{C}}(t) = \sum_{n\geq 0} a_n t^n.$$

1. *If $\mathcal{C}$ is evenly up-covered, then*

$$M_{\mathcal{C}}(t) = \sum_{n\geq 0} a_n u_n t^{2n+1}.$$

2. *If $\mathcal{C}$ is evenly down-covered, then*

$$M_{\mathcal{C}}(t) = \sum_{n\geq 1} a_n d_n t^{2n-1}.$$

Proof. Immediate from equations (6) and (7) respectively. $\qquad\square$

Remark. Two consequences of the second part are: (i) if $\mathcal{C}$ is simple, then $O_{\mathcal{C}}(t^2) = 1 + tM_{\mathcal{C}}(t)$; and (ii) if $\mathcal{C}$ is factorial, then $M_{\mathcal{C}}(t) = tO'_{\mathcal{C}}(t^2)$.

If the updown category $\mathcal{C}$ is both evenly up-covered and evenly down-covered, the preceding result gives two expressions for $M_{\mathcal{C}}(t)$ which must agree. This gives us the following result.

Corollary 3.2. *Suppose the updown category $\mathcal{C}$ is both evenly up-covered (with sequence $\{u_n\}$) and evenly down-covered (with sequence $\{d_n\}$). Then $a_n u_n = a_{n+1}d_{n+1}$ for all $n \geq 0$, where $O_{\mathcal{C}}(t) = \sum_{n\geq 0} a_n t^n$. In particular, if $\mathcal{C}$ is evenly up-covered and factorial, then $a_0 = |\operatorname{Aut}\hat{0}|^{-1}$ and*

$$a_n = \frac{u_0 u_1 \cdots u_{n-1}}{n!\,|\operatorname{Aut}\hat{0}|}, \quad n \geq 1.$$

4 Univalent Updown Categories, Weighted-relation Posets, and Universal Covers

Let $\mathfrak{U}$ be the category of updown categories, $\mathfrak{UU}$ the full subcategory of univalent updown categories. For a functor F between univalent updown categories $\mathcal{C}$, $\mathcal{D}$, Definition 2.4 reduces to the requirement that F preserve rank and that the induced function

$$\coprod_{\{q':F(q')=F(q)\}} \text{Hom}(p,q') \to \text{Hom}(F(p),F(q)) \tag{12}$$

be injective whenever $p,q \in \text{Ob}\,\mathcal{C}$ with $|q| = |p| + 1$.

The notion of a weighted-relation poset was defined in [5]. This consists of a ranked poset

$$P = \bigcup_{n \geq 0} P_n$$

with a least element $\hat{0} \in P_0$, together with nonnegative integers $n(x,y)$ for each $x,y \in P$ so that $n(x,y) = 0$ unless $x \preceq y$, and

$$n(x,y) = \sum_{|z|=k} n(x,z)n(z,y) \tag{13}$$

whenever $|x| \leq k \leq |y|$. A morphism of weighted-relation posets P,Q is a rank-preserving map $f : P \to Q$ such that

$$n(f(t),f(s)) \geq \sum_{s' \in f^{-1}(f(s))} n(t,s') \tag{14}$$

for any $s,t \in P$ with $|s| = |t| + 1$. Let $\mathfrak{W}$ be the category of weighted-relation posets.

Given an updown category $\mathcal{C}$, it follows from Theorem 2.2 that the weight functions $n(x,y) = u(x;y)$ and $n(x,y) = d(x;y)$ on the poset $\text{Ob}\,\mathcal{C}$ (with the partial order defined by Definition 2.6) both satisfy equation (13). So we have two weighted-relation posets based on $\text{Ob}\,\mathcal{C}$ corresponding to these two sets of weights. In fact, we can describe them functorially.

If $\mathcal{C}$ is an updown category, we can form a univalent updown category $\mathcal{C}^{\uparrow}$ with $\text{Ob}\,\mathcal{C}^{\uparrow} = \text{Ob}\,\mathcal{C}$, and with $\text{Hom}_{\mathcal{C}^{\uparrow}}(p,p')$ defined as follows. We declare $\text{Hom}_{\mathcal{C}^{\uparrow}}(p,p) = \text{Aut}_{\mathcal{C}^{\uparrow}}(p)$ trivial for all p, and for $|p'| > |p|$ define $\text{Hom}_{\mathcal{C}^{\uparrow}}(p,p')$ as the set of equivalence classes in $\text{Hom}_{\mathcal{C}}(p,p')$ under the relation $f_n f_{n-1} \cdots f_1 \sim \alpha_n f_n \cdots \alpha_1 f_1$, where each f_i is a morphism between adjacent ranks and $\alpha_i \in \text{Aut}(\text{trg}\,f_i)$. It is routine to check that $\mathcal{C}^{\uparrow}$ satisfies the axioms of an updown category, and for $p,p' \in \text{Ob}\,\mathcal{C}$ with $|p'| = |p| + 1$ the multiplicity is

$$|\text{Hom}_{\mathcal{C}^{\uparrow}}(p,p')| = |\text{Hom}_{\mathcal{C}}(p,p')/\text{Aut}_{\mathcal{C}}(p')| = u(p;p').$$

Of course $\mathcal{C}^{\uparrow}$ coincides with $\mathcal{C}$ if $\mathcal{C}$ is univalent.

Similarly, for any updown category $\mathcal{C}$ there is a univalent updown category $\mathcal{C}^{\downarrow}$ with $\text{Ob}\,\mathcal{C}^{\downarrow} = \text{Ob}\,\mathcal{C}$, trivial automorphisms, and $\text{Hom}_{\mathcal{C}^{\downarrow}}(p,p')$ defined as the set of equivalence

classes in $\mathrm{Hom}_{\mathcal{C}}(p, p')$ under the relation $f \sim f_n \beta_n f_{n-1} \cdots f_1 \beta_1$ for $f = f_n f_{n-1} \cdots f_1$ a factorization of $f \in \mathrm{Hom}_{\mathcal{C}}(p, p')$ into morphisms between adjacent ranks and $\beta_i \in \mathrm{Aut}(\mathrm{src}\, f_i)$. Then

$$|\mathrm{Hom}_{\mathcal{C}^\downarrow}(p, p')| = |\mathrm{Hom}_{\mathcal{C}}(p, p') / \mathrm{Aut}_{\mathcal{C}}(p)| = d(p; p')$$

for $p, p' \in \mathrm{Ob}\,\mathcal{C}$ with $|p'| = |p| + 1$. We have the following result.

Theorem 4.1. *There are two functors $\mathfrak{U} \to \mathfrak{UU}$, taking an updown category $\mathcal{C}$ to $\mathcal{C}^\uparrow$ and $\mathcal{C}^\downarrow$ respectively.*

Proof. We first consider the "up" functor. For a morphism $F : \mathcal{C} \to \mathcal{D}$ of updown categories, there is an induced functor $F^\uparrow : \mathcal{C}^\uparrow \to \mathcal{D}^\uparrow$ of univalent updown categories: $F^\uparrow(p) = F(p)$ for $p \in \mathrm{Ob}\,\mathcal{C}$, and $F^\uparrow$ sends the equivalence class $[f]$, where $f \in \mathrm{Hom}_{\mathcal{C}}(p, q)$, to the equivalence class $[F(f)] \in \mathrm{Hom}_{\mathcal{D}^\uparrow}(F(p), F(q))$. Now Definition 2.4 requires that F preserve rank and that the induced function

$$\coprod_{\{q' : F(q') = F(q)\}} \mathrm{Hom}_{\mathcal{C}}(p, q') / \mathrm{Aut}_{\mathcal{C}}(p') \to \mathrm{Hom}_{\mathcal{D}}(F(p), F(q)) / \mathrm{Aut}_{\mathcal{D}}(F(q))$$

be injective for all $p, q \in \mathrm{Ob}\,\mathcal{C}$ with $|q| = |p| + 1$. This is exactly the statement that the induced functor $F^\uparrow$ is a morphism of univalent updown categories. The proof for the "down" functor is similar. $\square$

Note that the functors of the preceding result respect products, e.g., $(\mathcal{C} \times \mathcal{D})^\uparrow$ can be naturally identified with $\mathcal{C}^\uparrow \times \mathcal{D}^\uparrow$. Note also that $\mathcal{C}^\uparrow$ is evenly up-covered if $\mathcal{C}$ is, and $\mathcal{C}^\downarrow$ is evenly-down covered if $\mathcal{C}$ is. Now we pass from univalent updown categories to weighted-relation posets.

Theorem 4.2. *There is a functor $Wrp : \mathfrak{UU} \to \mathfrak{W}$, sending a univalent updown category $\mathcal{C}$ to the set $\mathrm{Ob}\,\mathcal{C}$ with the partial order of Definition 2.6 and the weight function $n(x, y) = u(x; y) = d(x; y)$.*

Proof. The only thing to check is the morphisms. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a morphism of $\mathfrak{UU}$. Then F defines a function on the object sets, and the function (12) is injective. Hence

$$\sum_{\{q' : F(q') = F(q)\}} |\mathrm{Hom}(p, q')| \leq |\mathrm{Hom}(F(p), F(q))|$$

and so (since, e.g., $n(p, q') = |\mathrm{Hom}(p, q')|$), inequality (14) holds and F induces a morphism of weighted-relation posets. $\square$

As defined in [5], a morphism $f : P \to Q$ of weighted-relation posets is a covering map if f is surjective and the inequality (14) is an equality. A universal cover $\widetilde{P}$ of P is a cover $\widetilde{P} \to P$ such that, if $P' \to P$ is any other cover, then there is a covering map $\widetilde{P} \to P'$ so that the composition $\widetilde{P} \to P' \to P$ is the cover $\widetilde{P} \to P$. In [5] such a universal cover was constructed for any weighted-relation poset P.

In fact, the construction of [5] can be made considerably simpler and more natural if we work instead with univalent updown categories. We first categorify the definition of covering map.

Definition 4.1. A morphism $\pi : \mathcal{C}' \to \mathcal{C}$ of univalent updown categories is a covering map if π is surjective on the object sets and the induced function

$$\coprod_{\{q' : \pi(q') = \pi(q)\}} \mathrm{Hom}(p, q') \to \mathrm{Hom}(\pi(p), \pi(q)) \tag{15}$$

is a bijection for all $p, q \in \mathrm{Ob}\,\mathcal{C}'$ with $|q| = |p| + 1$. A covering map $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ is universal if for any other covering map $\phi : \mathcal{C}' \to \mathcal{C}$ there is a covering map $\psi : \widetilde{\mathcal{C}} \to \mathcal{C}'$ with $\pi = \phi\psi$.

Then we have the following result.

Theorem 4.3. *Every univalent updown category $\mathcal{C}$ has a universal cover $\widetilde{\mathcal{C}}$.*

Proof. We define $\widetilde{\mathcal{C}}$ to be the category whose rank-n objects are strings $(f_1, f_2, \ldots, f_n)$ of morphisms $f_i \in \mathrm{Hom}(c_{i-1}, c_i)$, where $c_i \in \mathcal{C}_i$, and whose morphisms are just inclusions of strings. It is straightforward to verify that $\widetilde{\mathcal{C}}$ is a univalent updown category (with $\hat{0}_{\widetilde{\mathcal{C}}}$ the empty string). Define the functor $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ by sending the empty string to $\hat{0} \in \mathrm{Ob}\,\mathcal{C}$, the nonempty string $(f_1, \ldots, f_n)$ of $\widetilde{\mathcal{C}}$ to the target of f_n in $\mathrm{Ob}\,\mathcal{C}$, and the inclusion $(f_1, \ldots, f_j) \subset (f_1, \ldots, f_n)$ to the morphism $f_n f_{n-1} \cdots f_{j+1} \in \mathrm{Hom}(c_j, c_n)$. That the induced function (15) is a bijection is a tautology.

Now let $P : \mathcal{C}' \to \mathcal{C}$ be another cover of $\mathcal{C}$. To define a covering map $F : \widetilde{\mathcal{C}} \to \mathcal{C}'$ with $\pi = PF$, we proceed by induction on rank. Start by sending the empty string in $\widetilde{\mathcal{C}}_0$ to the element $\hat{0}$ of $\mathcal{C}'$. Now suppose F is defined through rank $n - 1$, and consider a rank-n object $(f_1, \ldots, f_n)$ of $\widetilde{\mathcal{C}}$. Let $c_n = \pi(f_1, \ldots, f_n)$. By the induction hypothesis we have $c'_{n-1} = F(f_1, \ldots, f_{n-1}) \in \mathrm{Ob}\,\mathcal{C}'$, and $c_{n-1} = P(c'_{n-1})$ is the target of f_{n-1}, hence the source of f_n. Since

$$P : \coprod_{\{c' : p(c') = c_n\}} \mathrm{Hom}(c'_{n-1}, c') \to \mathrm{Hom}(c_{n-1}, c_n)$$

is a bijection, there is a unique morphism g of $\mathcal{C}'$ with $\mathrm{src}(g) = c'_{n-1}$ sent to $f_n : c_{n-1} \to c_n$. We define $F(f_1, \ldots, f_n)$ to be $\mathrm{trg}(g)$, and the image of the inclusion of $(f_1, \ldots, f_{n-1})$ in $(f_1, \ldots, f_n)$ to be g. This actually defines the functor F through rank n, since by the induction hypothesis F assigns to the inclusion of any proper substring $(f_1, \ldots, f_k)$ in $(f_1, \ldots, f_{n-1})$ a morphism h from $F(f_1, \ldots, f_k)$ to c'_{n-1} in $\mathcal{C}'$; then F sends the inclusion of $(f_1, \ldots, f_k)$ in $(f_1, \ldots, f_n)$ to gh. $\qquad\square$

Remark. Thinking of the functor $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ as a function on the object sets, the number of elements of $\widetilde{\mathcal{C}}_n$ which π sends to $p \in \mathcal{C}_n$ is

$$|\pi^{-1}(p)| = u(\hat{0}; p) = d(\hat{0}; p) = \frac{\langle \hat{0}, D^n p \rangle}{|\mathrm{Aut}\,\hat{0}|}, \tag{16}$$

as follows from Theorem 2.3. In particular, if $\mathcal{C}$ is factorial then $|\pi^{-1}(p)| = n!$ and $|\widetilde{\mathcal{C}}_n| = n!|\mathcal{C}_n|$. Also, it follows from equation (16) that

$$|\widetilde{\mathcal{C}}_n| = \sum_{p \in \mathcal{C}_n} u(\hat{0}; p). \tag{17}$$

If $\mathcal{C}$ is evenly up-covered, this equation and equation (4) imply that

$$|\widetilde{\mathcal{C}}_n| = u_0 u_1 \cdots u_{n-1}. \tag{18}$$

The construction of $\widetilde{\mathcal{C}}$ in the the proof of Theorem 4.3 is functorial: given a morphism $F : \mathcal{C} \to \mathcal{D}$ of univalent updown categories, we have a morphism $\widetilde{F} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$ given by

$$\widetilde{F}(f_1, f_2, \ldots, f_n) = (F(f_1), F(f_2), \ldots, F(f_n)).$$

Also, the updown category $\widetilde{\mathcal{C}}$ is evidently simple. Thus, if $\mathfrak{SU}$ is the full subcategory of simple updown categories in $\mathfrak{U}$, then there is a functor $\mathfrak{UU} \to \mathfrak{SU}$ taking $\mathcal{C}$ to $\widetilde{\mathcal{C}}$. In fact, we have the following result.

Proposition 4.1. *The functor $\mathfrak{UU} \to \mathfrak{SU}$ taking $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ is right adjoint to the inclusion functor $\mathfrak{SU} \to \mathfrak{UU}$.*

Proof. It suffices to show that

$$\mathrm{Hom}_{\mathfrak{UU}}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathfrak{SU}}(\mathcal{C}, \widetilde{\mathcal{D}})$$

for any simple updown category $\mathcal{C}$ and univalent updown category $\mathcal{D}$. A morphism $F : \mathcal{C} \to \mathcal{D}$ of univalent updown categories gives rise to $\widetilde{F} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$, and since $\mathcal{C}$ is simple there is a natural identification $\mathcal{C} \cong \widetilde{\mathcal{C}}$, giving us a morphism $\mathcal{C} \to \widetilde{\mathcal{D}}$. To go back the other way, just compose with the covering map $\pi : \widetilde{\mathcal{D}} \to \mathcal{D}$. $\square$

The universal cover functor $\mathfrak{UU} \to \mathfrak{SU}$ does not respect products: in fact, $\widetilde{\mathcal{C}} \times \widetilde{\mathcal{D}}$ is generally not simple. (This does not contradict the preceding result, because our product is not a categorical product in $\mathfrak{UU}$.) We do have the following result.

Proposition 4.2. *If $\mathcal{C}$ and $\mathcal{D}$ are univalent updown categories, then the number of rank-n objects in $\widetilde{\mathcal{C} \times \mathcal{D}}$ is*

$$\sum_{k=0}^{n} \binom{n}{k} |\widetilde{\mathcal{C}}_k| |\widetilde{\mathcal{D}}_{n-k}|.$$

Proof. Using the generating functions of the preceding section, equation (17) can be written

$$O_{\widetilde{\mathcal{C}}}(t^2) = \langle (1 - tU)^{-1} \hat{0}, S_{\mathcal{C}}(t) \rangle.$$

Then

$$O_{\widetilde{\mathcal{C}\times\mathcal{D}}}(t^2) = \langle (1-tU)^{-1}(\hat{0}_{\mathcal{C}} \otimes \hat{0}_{\mathcal{D}}), S_{\mathcal{C}}(t) \otimes S_{\mathcal{D}}(t) \rangle$$

$$= \sum_{n \geq 0} t^n \sum_{k=0}^{n} \binom{n}{k} \langle U^k \hat{0}_{\mathcal{C}} \otimes U^{n-k}\hat{0}_{\mathcal{D}}, S_{\mathcal{C}}(t) \otimes S_{\mathcal{D}}(t) \rangle$$

$$= \sum_{n \geq 0} t^n \sum_{k=0}^{n} \binom{n}{k} \langle U^k \hat{0}_{\mathcal{C}}, S_{\mathcal{C}}(t) \rangle \langle U^{n-k}\hat{0}_{\mathcal{D}}, S_{\mathcal{D}}(t) \rangle$$

$$= \sum_{n \geq 0} t^n \sum_{k=0}^{n} \binom{n}{k} t^k |\widetilde{\mathcal{C}}_k| t^{n-k} |\widetilde{\mathcal{D}}_{n-k}|,$$

from which the conclusion follows. $\qquad\square$

5 Examples

In this section we present ten examples of updown categories. Many of the associated weighted-relation posets appear in the last section of [5]. For the convenience of the reader we have included a cross-reference to [5] at the beginning of each example where it applies.

Example 1. (Subsets of a finite set; [5, Ex. 1], [13, Ex. 2.5(b)], [3, Ex. 6.2.6].) First, let $\mathcal{A}$ be an updown category such that $\mathcal{A}_0 = \{\hat{0}\}$, $\mathcal{A}_1 = \{\hat{1}\}$, $\mathcal{A}_n = \emptyset$ for $n \neq 0, 1$, and $\mathrm{Hom}(\hat{0}, \hat{1})$ has a single element. The groups $\mathrm{Aut}(\hat{0})$ and $\mathrm{Aut}(\hat{1})$ are trivial since they act freely on the one-element set $\mathrm{Hom}(\hat{0}, \hat{1})$. The object and morphism generating functions are evidently

$$O_{\mathcal{A}}(t) = 1 + t \quad \text{and} \quad M_{\mathcal{A}}(t) = t.$$

Evidently $\mathcal{A}$ is simple and factorial.

Now let $\mathcal{B} = \mathcal{A}^n$. Since $\mathcal{A}$ is factorial, $\mathcal{B}$ is factorial by Theorem 3.1. Objects of $\mathcal{B}$ can be identified with subsets of $\{1, 2, \ldots, n\}$: an n-tuple $(c_1, \ldots, c_n)$ corresponds to the set $\{i : c_i = \hat{1}\}$. The induced partial order is inclusion of sets, and in fact $\mathcal{B}$ is unital (but not simple for $n \geq 2$). In [5] it is shown that the universal cover $\widetilde{\mathcal{B}}$ is the simple updown category whose rank-m elements are linearly ordered m-element subsets of $\{1, \ldots, n\}$, and whose morphisms are inclusions of initial segments. This makes it obvious that $|\pi^{-1}(b)| = m!$ for all $b \in \mathcal{B}_m$, which also follows from equation (16).

From the remark following Corollary 3.1, the generating functions are

$$O_{\mathcal{B}}(t) = (1+t)^n \quad \text{and} \quad M_{\mathcal{B}}(t) = nt(1+t^2)^{n-1}.$$

Example 2. (Monomials; [5, Ex. 2], [4, Ex. 2.2.1].) Let $\mathcal{S}$ be the category with $\mathcal{S}_n = \{[n]\}$, where $[n] = \{1, 2, \ldots, n\}$ (and $[0] = \emptyset$), and let $\mathrm{Hom}([m], [n])$ be the set of injective functions from $[m]$ to $[n]$. Then the axioms are easily seen to hold, with $\mathrm{Aut}[n] = \Sigma_n$, the symmetric group on n letters. Since $\mathrm{Hom}([n], [n+1])$ has $(n+1)!$ elements, we have

$u([n];[n+1]) = 1$ and $d([n];[n+1]) = n+1$ (so $\mathcal{S}$ is factorial). The generating functions are

$$O_{\mathcal{S}}(t) = \sum_{n\geq 0} \frac{t^n}{n!} = e^t \quad \text{and} \quad M_{\mathcal{S}}(t) = \sum_{n\geq 0} \frac{t^{2n+1}}{n!} = te^{t^2}. \tag{19}$$

Now let $\mathcal{M} = \mathcal{S}^n$. Objects of $\mathcal{M}$ can be identified with monomials in n commuting indeterminates $t_1, \ldots, t_n$. The automorphism group of $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ is $\Sigma_{i_1} \times \Sigma_{i_2} \times \cdots \times \Sigma_{i_n}$, and a monomial u precedes a monomial v in the induced partial order if u is a factor of v. By Theorem 3.1 $\mathcal{M}$ is factorial, so

$$d(1; t_1^{i_1} \cdots t_n^{i_n}) = (i_1 + \cdots + i_n)!$$

by equation (5). Hence by equation (3)

$$u(1; t_1^{i_1} \cdots t_n^{i_n}) = \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!}.$$

Then it follows (using equation (17)) that

$$|\widetilde{\mathcal{M}^{\uparrow}}_m| = \sum_{i_1 + \cdots + i_n = m} \binom{m}{i_1 \ i_2 \ \cdots \ i_n} = n^m$$

and

$$|\widetilde{\mathcal{M}^{\downarrow}}_m| = \sum_{i_1 + \cdots + i_n = m} m! = n(n+1)\cdots(n+m-1).$$

The weighted-relation poset $Wrp(\mathcal{M}^{\uparrow})$ appears in [5], where it is shown that the universal cover $\widetilde{\mathcal{M}^{\uparrow}}$ can be identified with the simple updown category whose objects are monomials in n noncommuting indeterminates $T_1, \ldots, T_n$, and whose morphisms are inclusions as left factors; the covering map $\pi : \widetilde{\mathcal{M}^{\uparrow}} \to \mathcal{M}^{\uparrow}$ sends T_i to t_i (e.g., $\pi^{-1}(t_1^2 t_2) = \{T_1^2 T_2, T_1 T_2 T_1, T_2 T_1^2\}$).

A similar description of $\widetilde{\mathcal{M}^{\downarrow}}$ can be obtained by reworking the construction of Theorem 4.3 as follows. Objects in $\widetilde{\mathcal{M}^{\downarrow}}$ are those monomials in the noncommuting indeterminates $\{T_{ij} : 1 \leq i \leq n, j \geq 1\}$ such that (a) no indeterminate is repeated; and (b) if T_{ij} occurs, then so does T_{ik} for $k < j$. The covering map $\pi : \widetilde{\mathcal{M}^{\downarrow}} \to \mathcal{M}^{\downarrow}$ sends T_{ij} to t_i (e.g., $\pi^{-1}(t_1^2 t_2) = \{T_{11}T_{12}T_{21}, T_{12}T_{11}T_{21}, T_{11}T_{21}T_{12}, T_{12}T_{21}T_{11}, T_{21}T_{11}T_{12}, T_{21}T_{12}T_{11}\}$). For any object w of $\widetilde{\mathcal{M}^{\downarrow}}$, there are n permutations $\sigma_1, \sigma_2 \ldots, \sigma_n$ that can be extracted from the second subscripts: e.g., for $T_{13}T_{21}T_{11}T_{12}$ the permutations are $\sigma_1 = 312$ and $\sigma_2 = 1$. The partial order on objects of $\widetilde{\mathcal{M}^{\downarrow}}$ is given by having the monomial wT_{ij} cover w' if w' has the same sequence of first subscripts as w, and wT_{ij} has the same associated permutations as w' except that σ_i for wT_{ij} covers σ_i for w' in the sense of the preceding example. For example, $T_{13}T_{21}T_{11}T_{12}$ generates the order ideal $T_{13}T_{21}T_{11}T_{12} \rhd T_{12}T_{21}T_{11} \rhd T_{11}T_{21} \rhd T_{11} \rhd 1$.

By equations (19) and the remark following Corollary 3.1, the generating functions are

$$O_{\mathcal{M}}(t) = e^{nt} \quad \text{and} \quad M_{\mathcal{M}}(t) = nte^{nt^2}.$$

Example 3. Let $\mathcal{G}$ be the category whose objects are isomorphism classes of finite graphs. Then $\mathcal{G}$ is graded by the number of vertices, with $\hat{0}$ the empty graph. A morphism from H to G is an injective function $f : v(H) \to v(G)$ on the vertex sets such that $f(v_1)$ and $f(v_2)$ are connected in G if and only if v_1 and v_2 are connected in H. If $G \rhd H$, then there is a vertex v of G so that $G - \{v\}$ is isomorphic to H. Evidently $\mathcal{G}$ is factorial, since for any $G \in \mathcal{G}_n$

$$\sum_{|H|=n-1} d(H;G) = n.$$

But $\mathcal{G}$ is also uniformly up-covered, any G covering $H \in \mathcal{G}_n$ can be obtained from H by adjoining a new vertex and edges between that vertex and some subset of the n vertices of $\mathfrak{H}$: thus

$$\sum_{|G|=n+1} u(H;G) = 2^n.$$

It follows from Corollary 3.2 that

$$O_\mathcal{G}(t) = \sum_{n \geq 0} \frac{2^{\binom{n}{2}}}{n!} t^n,$$

and thus from Theorem 3.4 that

$$M_\mathcal{G}(t) = \sum_{n \geq 1} \frac{2^{\binom{n}{2}}}{(n-1)!} t^{2n-1}.$$

Objects of the universal cover $\widetilde{\mathcal{G}}^{\uparrow}$ can be identified with graphs whose vertices are labelled by the positive integers; morphisms of $\mathcal{G}^{\uparrow}$ preserve labels. From equation (18) follows $|\widetilde{\mathcal{G}}_n^{\uparrow}| = 2^{\binom{n}{2}}$. On the other hand, an element $(\emptyset, G_1, G_2, \ldots, G_n)$ of $\widetilde{\mathcal{G}}_n^{\downarrow}$ can be specified by giving a bijection

$$f : \{1, 2, \ldots, n\} \to v(G_n)$$

such that each G_i is the full subgraph of G_n on the vertices $\{f(1), \ldots, f(i)\}$. This makes it evident that $|\pi^{-1}(G_n)| = n!$, in accordance with the remark following Theorem 4.3.

Example 4. (Necklaces; [5, Ex. 3].) For a fixed positive integer c, let $\mathcal{N}_m$ be the set of m-bead necklaces with beads of c possible colors. More precisely, a rank-m object of $\mathcal{N}$ is an equivalence class of functions $f : \mathbf{Z}/m\mathbf{Z} \to [c]$, where f is equivalent to g if there is some n so that $f(a+n) = g(a)$ for all $a \in \mathbf{Z}/m\mathbf{Z}$. Thus, for $c = 2$ the equivalence class

$$\{(1,1,2,2), (2,1,1,2), (2,2,1,1), (1,2,2,1)\} \quad \text{represents the necklace} \quad \raisebox{-0.5ex}{\includegraphics[width=1em]{necklace}} \, .$$

A morphism from the equivalence class of f in $\mathcal{N}_m$ to the equivalence class of g in $\mathcal{N}_n$ is an injective function $h : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ with $f(a) = gh(a)$ for all $a \in \mathbf{Z}/m\mathbf{Z}$, and such that h preserves the cyclic order, i.e., if we pick representatives of the $h(i)$ in $\mathbf{Z}$

with $0 \leq h(i) \leq n-1$, then some cyclic permutation of $(h(0), h(1), \ldots, h(m-1))$ is an increasing sequence. Informally, $u(p; q)$ is the number of ways to insert a bead into necklace p to get necklace q, and $d(p; q)$ is the number of different beads of q that can be deleted to give p.

Note that $\mathcal{N}$ is factorial (there are m different beads that can be removed from $p \in \mathcal{N}_m$) and also evenly up-covered with $u_m = mc$ for $m \geq 1$ (in a necklace with $m \geq 1$ beads there are m places that a bead of c possible colors can be inserted); of course $u_0 = c$. Thus, by Corollary 3.2

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ \frac{c^n}{n}, & \text{if } n \geq 1; \end{cases}$$

and so $O_{\mathcal{N}}(t) = 1 - \log(1 - ct)$. Again using the fact that $\mathcal{N}$ is factorial (and Theorem 3.4), we have

$$M_{\mathcal{N}}(t) = \frac{ct}{1 - ct^2}.$$

We have $|\widetilde{\mathcal{N}}_m^{\uparrow}| = (m-1)! c^m$ by equation (18): cf. the discussion in [5], where the same result is obtained by identifying elements of rank $\widetilde{\mathcal{N}}_m^{\uparrow}$ with necklaces of m beads in c colors in which the beads are labelled by $1, 2, \ldots, m$. On the other hand, an element of $\widetilde{\mathcal{N}}_m^{\downarrow}$ can be regarded as an equivalence class of pairs (f, σ), where $f : \mathbf{Z}/m\mathbf{Z} \to [c]$ and σ is a permutation of $\{0, 1, \ldots, m-1\}$. The equivalence relation is that $(f, \sigma) \sim (g, \tau)$ if $f \neq g$ and there is some $0 \leq n \leq m-1$ with $g(x) = f(x+n)$ and $\tau(x) = \sigma(x+n)$ for all $x \in \mathbf{Z}/m\mathbf{Z}$. Evidently there are $m!$ such equivalence classes for a given $[f] \in \mathcal{N}_m$, in accord with the factoriality of $\mathcal{N}$.

Example 5. (Integer partitions with unit weights; [5, Ex. 5], [12], [4, Ex. 1.6.8].) Let $\mathcal{Y}$ be the category with Ob $\mathcal{Y}$ the set of integer partitions, i.e., finite sequences $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The rank of a partition is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$; we write $\ell(\lambda)$ for the length (number of parts) of λ. The set of morphisms $\mathrm{Hom}(\lambda, \mu)$ contains a single element if and only if $\lambda_i \leq \mu_i$ for all i. Then $\mathcal{Y}$ is evidently unital but not simple.

Since $\mathcal{Y}_n$ is the set of partitions of n, the object generating function

$$O_{\mathcal{Y}}(t) = \sum_{n \geq 0} |\mathcal{Y}_n| t^n = \frac{1}{(1-t)(1-t^2)(1-t^3) \cdots}$$

is familiar. The morphism generating function is

$$M_{\mathcal{Y}}(t) = \sum_{n \geq 0} |\{(\lambda, \mu) : \lambda \in \mathcal{Y}_n, \ \lambda \lhd \mu\}| t^{2n+1}$$

since $\mathcal{Y}$ is unital. Using the case $k = 1$ of [12, Theorem 3.2], it follows that

$$M_{\mathcal{Y}}(t) = \frac{t}{1-t^2} O_{\mathcal{Y}}(t^2) = \frac{t}{(1-t^2)^2 (1-t^4)(1-t^6) \cdots}.$$

In [5] it is shown that the universal cover $\widetilde{\mathcal{Y}}$ is the poset of standard Young tableaux, so $u(\hat{0}; \lambda) = d(\hat{0}; \lambda)$ is the number of standard Young tableaux of shape λ.

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Example 6. Let $\mathcal{K}$ be the category with $\mathrm{Ob}\,\mathcal{K}$ the set of integer partitions, and $\mathrm{Hom}(\lambda,\mu)$ defined as follows. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$, always written in decreasing order. Then a morphism from λ to μ is an injective function $f : [n] \to [m]$ such that $\lambda_i \leq \mu_j$ whenever $f(i) = j$.

The partial order induced on $\mathrm{Ob}\,\mathcal{K} = \mathrm{Ob}\,\mathcal{Y}$ is the same as that of the preceding example: the difference is that we now have nontrivial automorphism groups and weights on covering relations. The automorphism group of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the subgroup of Σ_k consisting of those permutations σ such that $\lambda_i = \lambda_j$ whenever $\sigma(i) = j$. If we let $m_i(\lambda)$ be the number of parts of λ of size i, this means that

$$|\mathrm{Aut}(\lambda)| = m_1(\lambda)!\, m_2(\lambda)! \cdots .$$

For partitions λ, μ with $|\mu| = |\lambda| + 1$, $\mathrm{Hom}(\lambda, \mu)$ is nonempty exactly when (i) μ comes from λ by adding a part of size 1; or (ii) μ comes by replacing a size-k part of λ by a part of size $k + 1$. In case (i) we put $u(\lambda; \mu) = 1$ and $d(\lambda; \mu) = m_1(\mu)$, while in case (ii) we put $u(\lambda; \mu) = m_k(\lambda)$ and $d(\lambda; \mu) = m_{k+1}(\mu)$. The weights $d(\lambda; \mu)$ appear implicitly in [9] and explicitly in [8], where they are referred to as "Kingman's branching": see especially Figure 4 of [8].

The object generating function can be computed as follows:

$$O_{\mathcal{K}}(t) = \sum_{\lambda \in \mathrm{Ob}\,\mathcal{K}} \frac{t^{|\lambda|}}{|\mathrm{Aut}(\lambda)|} = \sum_{m_1, m_2, \cdots \geq 0} \frac{t^{m_1 + 2m_2 + \cdots}}{m_1!\, m_2! \cdots} =$$

$$\left(\sum_{m_1 \geq 0} \frac{t^{m_1}}{m_1!} \right) \left(\sum_{m_2 \geq 0} \frac{t^{2m_2}}{m_2!} \right) \cdots = \exp(t + t^2 + \cdots) = \exp\left(\frac{t}{1 - t} \right).$$

To find the morphism generating function

$$M_{\mathcal{K}}(t) = \sum_{\lambda \in \mathrm{Ob}\,\mathcal{K}} \frac{t^{2|\lambda|+1}}{|\mathrm{Aut}(\lambda)|} \sum_{\lambda \lhd \mu} u(\lambda; \mu) \tag{20}$$

we first observe that

$$\sum_{\lambda \lhd \mu} u(\lambda; \mu) = 1 + \ell(\lambda) = 1 + m_1(\lambda) + m_2(\lambda) + \cdots,$$

since (using the description of $u(\lambda; \mu)$ above) this is the number of ways to obtain a partition covering λ: we can increase by one any of the $\ell(\lambda)$ parts of λ, or add a new part of size 1. Thus equation (20) is

$$M_{\mathcal{K}}(t) = \sum_{m_1, m_2, \cdots \geq 0} \frac{t^{1 + 2m_1 + 4m_2 + \cdots}}{m_1!\, m_2! \cdots} (1 + m_1 + m_2 + \cdots)$$

$$= (t + t^3 + t^5 + \cdots) O_{\mathcal{K}}(t^2) = \frac{t}{1 - t^2} \exp\left(\frac{t^2}{1 - t^2} \right).$$

The universal cover $\widetilde{\mathcal{K}}^\uparrow$ can be described in terms of set partitions: elements of $\widetilde{\mathcal{K}}_n^\uparrow$ are partitions of the set $[n]$, with $\pi : \widetilde{\mathcal{K}}^\uparrow \to \mathcal{K}^\uparrow$ sending each partition to the integer partition of n given by its block sizes. Thus $|\widetilde{\mathcal{K}}_n^\uparrow|$ is the nth Bell number [11, A000110]. We can identify set partitions with the construction of Theorem 4.3 as follows. For convenience we write a set partition as $(P_1, \dots, P_k)$ with $|P_1| \geq |P_2| \geq \cdots \geq |P_k|$ and, if $|P_i| = |P_j|$ for $i < j$, then $\max P_i < \max P_j$. Assign the unique partition of $[1]$ to the morphism from $\hat{0}$ to (1), and suppose inductively that we have assigned an ordered partition $P = (P_1, \dots, P_k)$ of $[n]$ to the chain $(h_1, \dots, h_n)$ of morphisms between adjacent ranks of $\mathcal{K}^\uparrow$ from $\hat{0}$ to $\mathrm{trg}(h_n) = (\lambda_1, \dots, \lambda_k) \in \mathrm{Ob}\,\mathcal{K}_n^\uparrow$ so that $\lambda_i = |P_i|$. Let $f \in \mathrm{Hom}_{\mathcal{K}}(\lambda, \mu)$ be a representative of the equivalence class $h_{n+1} \in \mathrm{Hom}_{\mathcal{K}^\uparrow}(\lambda, \mu)$, where $|\mu| = n+1$. If μ has length $k+1$, assign $(P_1, \dots, P_k, \{n+1\})$ to the chain $(h_1, \dots, h_n, h_{n+1})$. Otherwise, μ has length k and there is a unique $i \in [k]$ such that $\lambda_i < \mu_{f(i)}$: in this case, assign to $(h_1, \dots, h_{n+1})$ the rearrangement of $(P_1', \dots, P_k')$, where

$$
P_j' = \begin{cases} P_j \cup \{n+1\}, & \text{if } j = i, \\ P_j, & \text{otherwise,} \end{cases}
$$

so that P_i' immediately follows P_m', where $m = \max\{j < i : |P_j'| \geq |P_i'|\}$. Evidently the set partition assigned to $(h_1, \dots, h_{n+1})$ projects to μ in either case.

Rank-n objects of the universal cover $\widetilde{\mathcal{K}}^\downarrow$ can be described as sequences $s = (a_1, \dots, a_n)$ such that $m_1(s) \geq m_2(s) \geq \cdots$, where $m_i(s)$ is the number of occurrences of i in s; the covering map sends s to $(m_1(s), m_2(s), \dots)$. See [11, A005651]. As in the preceding paragraph, we can proceed inductively to identify these objects with the construction of Theorem 4.3. Start by assigning $s = (1)$ to the morphism from $\hat{0}$ to (1). Suppose now we have assigned $s = (a_1, \dots, a_n)$ to a chain of morphisms $(h_1, \dots, h_n)$ between adjacent ranks of $\mathcal{K}^\downarrow$ from $\hat{0}$ to $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{K}_n^\downarrow$ so that $m_i(s) = \lambda_i$ for $1 \leq i \leq k$, and let $h_{n+1} \in \mathrm{Hom}_{\mathcal{K}^\downarrow}(\lambda, \mu)$ where $|\mu| = n+1$. Now a representative $f \in \mathrm{Hom}_{\mathcal{K}}(\lambda, \mu)$ of h_{n+1} must be "almost an automorphism" exchanging parts of equal size with just one exception: there is a unique $i \in [\ell(\mu)]$ such that either i is not in the image of f (in which case $\mu_i = 1$), or else $\lambda_{f^{-1}(i)} < \mu_i$ (in which case $\mu_i = \lambda_{f^{-1}(i)} + 1$). Let $S = \{j > i : \lambda_{f^{-1}(j)} = \mu_i\}$: note that S is independent of the choice of f. Now define a permutation σ of $[\ell(\mu)]$ as follows. If $S = \emptyset$, let σ be the identity; otherwise, if $S = \{i+1, \dots, l\}$, let $\sigma(a) = a+1$ for $i \leq a \leq l-1$, $\sigma(l) = i$, and $\sigma(a) = a$ for $a \notin \{i, \dots, l\}$. We then assign the sequence $s' = (\sigma(a_1), \dots, \sigma(a_k), i)$ to the chain $(h_1, \dots, h_n, h_{n+1})$. If $i \notin \mathrm{im}\, f$, then $\mu_j = 1$ for $j \geq i$ and either $i = \ell(\mu) = k+1$ (if S is empty) or $l = \ell(\mu) = k+1$ (if it isn't): either way μ differs by λ by having 1 inserted in the ith position, and s' projects to μ. If $\mu_i = \lambda_{f^{-1}(i)} + 1$, then we must have $\lambda_j = \mu_j$ for $j < i$, and μ differs from λ in having a part of size $\mu_i - 1$ increased by 1. If S is empty, $\lambda_{f^{-1}(i)} = \lambda_i$ and $\mu_i = m_i(s') = m_i(s) + 1 = \lambda_i + 1$. Otherwise, $\mu_i = m_i(s') = m_l(s) + 1 = \lambda_{f^{-1}(l)}$ and $m_{j+1}(s') = m_j(s)$ for $i \leq j \leq l-1$. Either way, s' again projects to μ.

Example 7. (Integer compositions; [5, Ex. 6].) Let $\mathcal{C}_n$ be the set of integer compositions of n, i.e. sequences $I = (i_1, \dots, i_p)$ of positive integers with $a_1 + \cdots + a_m = n$; as

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with partitions we write $\ell(I)$ for the length of I. A morphism from $(i_1, \ldots, i_p) \in \mathcal{C}_n$ to $(j_1, \ldots, j_q) \in \mathcal{C}_m$ is an order-preserving injective function $f : [p] \to [q]$ such that $i_a \leq j_{f(a)}$ for all $a \in [p]$. Then $\mathcal{C}$ is a univalent updown category (but not simple).

The object generating function is

$$O_{\mathcal{C}}(t) = \sum_{n \geq 0} |\mathcal{C}_n| t^n = 1 + \sum_{n \geq 1} 2^{n-1} t^n = \frac{1-t}{1-2t}.$$

Now for any composition I,

$$\sum_{I \lhd J} u(I; J) = \ell(I) + \ell(I) + 1 = 2\ell(I) + 1$$

since we can get a composition covering I either by increasing each of its $\ell(I)$ parts, or by inserting a part of size 1 into one of $\ell(I) + 1$ possible positions. Thus, the morphism generating function is

$$M_{\mathcal{C}}(t) = \sum_{n \geq 0} t^{2n+1} \sum_{k=1}^{n} |\mathcal{C}_{n,k}|(2k+1),$$

where $\mathcal{C}_{n,k}$ is the set of compositions of n with k parts. Evidently $|\mathcal{C}_{n,k}| = \binom{n-1}{k-1}$, so

$$M_{\mathcal{C}}(t) = \sum_{n \geq 0} t^{2n+1} \sum_{k=1}^{n} \binom{n-1}{k-1}(2k+1) = \sum_{n \geq 0}(n+2)2^{n-1}t^{2n+1} = \frac{t - t^3}{(1 - 2t^2)^2}.$$

The universal cover $\widetilde{\mathcal{C}}$ is constructed in [5] using Cayley permutations as defined in [10]: a Cayley permutation of rank n is a length-n sequence $s = (a_1, \ldots, a_n)$ of positive integers such that any positive integer $i < j$ appears in s whenever j does. See [11, A00679]. The covering map $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ sends a sequence s to the composition $(m_1(s), m_2(s), \ldots)$. To relate this to the construction of Theorem 4.3, we again proceed inductively. Send the morphism from $\hat{0}$ to (1) to the Cayley permutation (1), and suppose we have assigned to a chain $(h_1, h_2, \ldots, h_n)$ of morphisms between consecutive ranks of $\mathcal{C}$ from $\hat{0}$ to $I = (i_1, \ldots, i_k) \in \mathcal{C}_n$ a Cayley permutation $s = (a_1, \ldots, a_n)$ that projects to I: note that $\max\{a_1, \ldots, a_n\} = k$. Now let $h_{n+1} \in \mathrm{Hom}(I, J)$ with $J \in \mathcal{C}_{n+1}$. Then either $\ell(J) = k$ and h_{n+1} is the identity function on $[k]$, or $\ell(J) = k + 1$. In the first case, there is exactly one position q where J differs from I: assign to $(h_1, \ldots, h_{n+1})$ the Cayley permutation $s' = (a_1, \ldots, a_n, q)$. Then $m_q(s') = m_q(s) + 1 = i_q + 1$ and $m_i(s') = m_i(s)$ for $i \neq q$, so s' projects to J. In the second case, there is exactly one element $q \in [k+1]$ that h_{n+1} misses: assign $s' = (h_{n+1}(a_1), \ldots, h_{n+1}(a_n), q)$ to $(h_1, \ldots, h_{n+1})$. Then $\pi(s') = (m_1(s'), m_2(s'), \ldots)$ differs from I only in having an additional 1 inserted in the qth place, and so must be J.

Example 8. (Planar rooted trees; [5, Ex. 4].) Let $\mathcal{P}_n$ consist of functions $f : [2n] \to \{-1, 1\}$ so that the partial sums $S_i = f(1) + \cdots + f(i)$ have the properties that $S_i \geq 0$ for all $1 \leq i \leq 2n$, and $S_{2n} = 0$. We declare $\mathrm{Aut}(f)$ to be trivial for all objects f of $\mathcal{P}$, and define a morphism from $f \in \mathcal{P}_n$ to $g \in \mathcal{P}_{n+1}$ to be an injective, order-preserving

function $h : [2n] \to [2n + 2]$ such that the two values of $[2n + 2]$ not in the image of h are consecutive, and $f(i) = gh(i)$ for $1 \le i \le 2n$. Then $\mathcal{P}$ is a univalent updown category. Using the well-known identification of balanced bracket arrangements with planar rooted trees, e.g.

$$(1, 1, -1, 1, 1, -1, -1, -1) \quad \text{is identified with} \quad \text{(tree diagram)},$$

we can think of $\mathcal{P}$ as the updown category of planar rooted trees; the rank is the count of non-root vertices. (The empty bracket arrangement $\emptyset$ is identified with the tree $\bullet$ consisting of the root vertex.) In view of the well-known enumeration of planar rooted trees by Catalan numbers, the object generating function is simply

$$O_{\mathcal{P}}(t) = \sum_{n \ge 0} |\mathcal{P}_n| t^n = \sum_{n \ge 0} \frac{1}{n+1} \binom{2n}{n} t^n = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

Since there are $2n + 1$ possibilities for order-preserving injections $[2n] \to [2n + 2]$ that miss two consecutive values, $\mathcal{P}$ is evenly up-covered with $u_n = 2n + 1$ and by Theorem 3.4 the morphism generating function is

$$M_{\mathcal{P}}(t) = \sum_{n \ge 0} \frac{2n + 1}{n + 1} \binom{2n}{n} t^{2n+1} = \sum_{n \ge 0} \binom{2n + 1}{n + 1} t^{2n+1} = \frac{1 - \sqrt{1 - 4t^2}}{2t\sqrt{1 - 4t^2}}.$$

From equation (18) we have $|\widetilde{\mathcal{P}}_n| = (2n - 1)!!$ for $n \ge 1$. In [5] the universal cover of $Wrp(\mathcal{P})$ is described as the weighted-relation poset whose rank-n elements are permutations $(a_1, a_2, \ldots, a_{2n})$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that, if $a_i > a_j$ with $i < j$, then there is some $k < j$, $k \ne i$, such that $a_k = a_i$. (The covering map sends a sequence $s = (a_1, \ldots, a_{2n})$ to a sequence of 1's and -1's by sending the first occurrence of i in s to 1 and the second to -1.) This construction can be identified with $\widetilde{\mathcal{P}}$ as described in Theorem 4.3 in an obvious way. For example, the morphism from $\emptyset$ to $(1, 1, -1, 1, -1, -1)$ given by the composition $h_3 h_2 h_1$, where $h_1 = \emptyset$, $h_2 = \{(1, 1), (2, 4)\}$ and $h_3 = \{(1, 1), (2, 2), (3, 3), (4, 6)\}$, can be coded by the sequence $(1, 2, 2, 3, 3, 1)$.

Example 9. (Rooted trees; [5, Ex. 7].) Let $\mathcal{T}_n$ consist of partially ordered sets P such that (i) P has $n + 1$ elements; (ii) P has a greatest element; and (iii) for any $v \in P$, the set of elements of P exceeding v forms a chain. The Hasse diagram of such a poset P is a tree with the greatest element (the root vertex) at the top. A morphism of $\mathcal{T}$ from $P \in \mathcal{T}_m$ to $Q \in \mathcal{T}_n$ is an injective order-preserving function $f : P \to Q$ that sends the root of P to the root of Q, and which preserves covering relations (i.e., if $v \lhd w$ in the partial order on P, then $f(v) \lhd f(w)$ in the partial order on Q). Then $\mathcal{T}$ is an updown category.

The updown category $\mathcal{T}$ was studied extensively in [6], though without using the categorical language. To see that the construction of the preceding paragraph gives the same multiplicities as in [6], consider a morphism from $P \in \mathcal{T}_n$ to $Q \in \mathcal{T}_{n+1}$. Any such morphism misses only some terminal vertex $v \in Q$, so we can think of it as identifying P with $Q - \{v\}$. Elements of

$$\text{Hom}(P, Q) / \text{Aut}(Q)$$

amount to different choices for the parent of v in Q, i.e., different choices for terminal vertices of P to which a new edge and vertex can be attached to form Q: this is $n(P; Q)$ as defined in [6]. On the other hand, elements of

$$\mathrm{Hom}(P, Q)/\mathrm{Aut}(P)$$

amount to different choices of v, and thus to different choices for an edge of Q that when cut leaves P: this is $m(P; Q)$ as defined in [6].

The object generating function

$$O_{\mathcal{T}}(t) = \sum_{n \geq 0} t^n \sum_{P \in \mathcal{T}_n} \frac{1}{|\mathrm{Aut}(P)|}$$

can be evaluated using a result of [2]. First, we note from [1] (cf. the discussion in [5]) that

$$u(\bullet; P) = n(\bullet; P) = \frac{(|P| + 1)!}{P! |\mathrm{Aut}(P)|},$$

where $P!$ is the "tree factorial," i.e., the product

$$\prod_{v \text{ is a vertex of } P} (|P_v| + 1)!$$

where P_v is the subtree of P having v as its root. Thus

$$O_{\mathcal{T}}(t) = \sum_{n \geq 0} t^n \sum_{P \in \mathcal{T}_n} \frac{u(\bullet; P) P!}{(n + 1)!}.$$

From §5.3 of [2] we have

$$\sum_{P \in \mathcal{T}_n} u(\bullet; P) P! = (n + 1)^n,$$

so

$$O_{\mathcal{T}}(t) = \sum_{n \geq 0} \frac{(n + 1)^n}{(n + 1)!} t^n.$$

(We note that $tO_{\mathcal{T}}(t)$ is the functional inverse of te^{-t}: see [14, §5.3].) Now $P \in \mathcal{T}_n$ has a total of $n + 1$ vertices to which new edges can be added, so $\mathcal{T}$ is evenly up-covered with $u_n = n + 1$ and by Theorem 3.4 the morphism generating function is

$$M_{\mathcal{T}}(t) = \sum_{n \geq 0} \frac{(n + 1)^n}{n!} t^{2n+1}.$$

In [5] the weighted-relation poset $Wrp(\mathcal{T}^{\uparrow})$ is discussed, and it is shown that rank-n objects of the universal cover $\widetilde{\mathcal{T}}^{\uparrow}$ can be described as permutations of $[n]$. (The partial order on permutations in $\widetilde{\mathcal{T}}^{\uparrow}$ is as follows: a permutation τ of $[n+1]$ covers the permutation $\tau \iota_{\tau^{-1}(n+1)}^n$ of $[n]$, where ι_m^n is the order-preserving injection from $[n]$ to $[n + 1]$ that misses

m.) On the other hand, objects of $\widetilde{\mathcal{T}}_n^{\downarrow}$ can be thought of as pairs (P, f), where $P \in \mathcal{T}_n$ and

$$f : \{0, 1, 2, \ldots, n\} \to P$$

is a bijection such that $f(i)$ exceeds $f(j)$ (in the partial order on P) whenever $i > j$. (Cf. the remark following [6, Prop. 2.5].)

Example 10. (Binary rooted trees) Let $\mathcal{B}_n$ be the set of rooted binary trees with $n + 1$ terminal vertices (leaves). That is, an element of $\mathcal{B}_n$ is a rooted tree in which each vertex has two daughters or none (in which case it is a leaf). Any $P \in \mathcal{B}_n$ defines a metric on its set $L(P)$ of leaves: the distance $\delta(p, q)$ from leaf p to leaf q is the number of non-terminal vertices contained in the unique shortest path from p to q. An automorphism of $P \in \mathcal{B}_n$ is a bijection f on $L(P)$ such that $\delta(f(p), f(q)) = \delta(p, q)$ for all p, q. A morphism from $P \in \mathcal{B}_n$ to $Q \in \mathcal{B}_{n+1}$ is an injection $f : L(P) \to L(Q)$ such that $\delta(f(p_1), f(p_2)) \geq \delta(p_1, p_2)$ for all $p_1, p_2 \in L(P)$, and the only $r \in L(Q)$ with $r \notin \operatorname{im} f$ is distance 1 from some $s \in \operatorname{im} f$. For example, if

$$P = \bigwedge \quad \text{and} \quad Q = \bigwedge$$
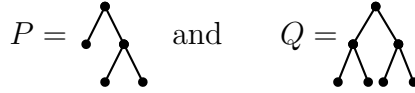
then $|\operatorname{Aut}(P)| = 2$, $|\operatorname{Aut}(Q)| = 8$, and $|\operatorname{Hom}(P, Q)| = 8$: hence $u(P; Q) = 1$ and $d(P; Q) = 4$. (If we call a pair of leaves distance 1 apart together with their common parent a "bud", then $u(P; Q)$ is the number of leaves of P that can be replaced by a bud to get Q, and $d(P; Q)$ is twice the number of buds of Q that can be replaced by a leaf to get P.)

Let $R \in \mathcal{B}_n$, and let T be a particular realization of R in the plane, i.e., a planar binary rooted tree. Since T has n non-terminal vertices, the group $G = \mathbf{Z}_2^n$ (where $\mathbf{Z}_2$ is the group of order 2) acts on T by rotations around each such vertex: the isotropy group of T is $\operatorname{Aut} R$. Then the number of distinct planar binary rooted trees T that can represent R is

$$|G/\operatorname{Aut} R| = \frac{2^n}{|\operatorname{Aut} R|}.$$

Now there are C_n distinct planar binary rooted trees with n non-terminal vertices, where C_n is the nth Catalan number, so

$$\sum_{P \in \mathcal{B}_n} \frac{2^n}{|\operatorname{Aut} R|} = C_n$$

and the object generating function is

$$O_{\mathcal{B}}(t) = \sum_{R \in \operatorname{Ob} \mathcal{B}} \frac{t^{|R|}}{|\operatorname{Aut} R|} = \sum_{n \geq 0} \frac{C_n}{2^n} t^n = \frac{1 - \sqrt{1 - 2t}}{t}.$$

Since $\mathcal{B}$ is evenly up-covered with $u_n = n + 1$, by Theorem 3.4 the morphism generating function is

$$M_{\mathcal{B}}(t) = \sum_{n \geq 0} \frac{C_n(n+1)}{2^n} t^{2n+1} = \frac{t}{\sqrt{1 - 2t^2}}.$$

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Since $\mathcal{B}$ is evenly up-covered, equation (18) implies $|\widetilde{\mathcal{B}}_n^\uparrow| = n!$, and by Theorem 3.2 $\widetilde{\mathcal{B}}^\uparrow$ must be isomorphic to $\widetilde{\mathcal{T}}^\uparrow$. In fact, there is a natural way to associate a permutation of $[n]$ to any $c \in \widetilde{\mathcal{B}}_n^\uparrow$. Given $(c_0, c_1, \ldots, c_n) \in \widetilde{\mathcal{B}}_n^\uparrow$, there is a corresponding planar binary root tree with labelled non-terminal vertices: a node gets label i if $c_{i-1} \to c_i$ involves adding a bud at that node. Put another set of labels $0, 1, \ldots, n$ on the leaves, running left to right. For example, the two elements of $\widetilde{\mathcal{B}}_2^\uparrow$ are

$$\text{and}$$
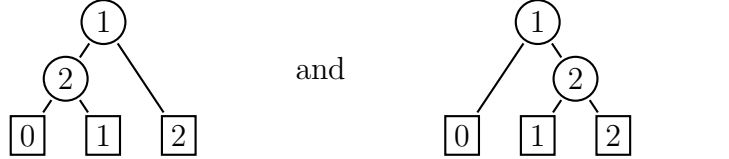
Now define a permutation of $[n]$ by sending $i \in [n]$ to the label on the last common ancestor of the leaves labelled $i-1$ and i. For example, our two labelled trees above correspond respectively to the permutation exchanging 1 and 2, and to the identity permutation.

An element $U \in \widetilde{\mathcal{B}}_n^\downarrow$ with $\pi(U) = V \in \mathcal{B}_n^\downarrow$ can be thought of as V equipped with an appropriate set of labels on its edges so that exactly one of each pair of edges coming out of a non-terminal vertex carries a label. More precisely, let $\hat{V}$ be the set of non-terminal vertices of V: then $U \in \pi^{-1}(V)$ can be identified with a pair of functions (g, h), where $g : [n] \to \hat{V}$ is a bijection such that $g(v) > g(w)$ in U when $v > w$, and $h : \hat{V} \to \{L, R\}$ (so that there are 2^n possiblities for h). In this way one sees, e.g., that for the binary trees P, Q above one has $|\pi^{-1}(P)| = 1 \cdot 2^2 = 4$ and $|\pi^{-1}(Q)| = 2 \cdot 2^3 = 16$.

A summary of our examples (U=univalent, UC=evenly up-covered, F=factorial):

#	Description	Object g.f.	Morphism g.f.	U	UC	F
1	Subsets of $[n]$	$(1+t)^n$	$nt(1+t^2)^{n-1}$	yes	yes	yes
2	Monomials	e^{nt}	nte^{nt^2}	no	yes	yes
3	Finite graphs	$\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{t^n}{n!}$	$\sum_{n\geq 1} 2^{\binom{n}{2}} \frac{t^{2n-1}}{(n-1)!}$	no	yes	yes
4	Necklaces	$1 - \log(1 - ct)$	$\frac{ct}{1-ct^2}$	no	yes	yes
5	Partitions	$\prod_{n=1}^\infty \frac{1}{1-t^n}$	$\frac{t}{1-t^2}\prod_{n=1}^\infty \frac{1}{1-t^{2n}}$	yes	no	no
6	Partitions	$\exp(\frac{t}{1-t})$	$\frac{t}{1-t^2}\exp(\frac{t^2}{1-t^2})$	no	no	no
7	Compositions	$\frac{1-t}{1-2t}$	$\frac{t(1-t^2)}{(1-2t^2)^2}$	yes	no	no
8	Planar rtd. trees	$\frac{1-\sqrt{1-4t}}{2t}$	$\frac{1-\sqrt{1-4t^2}}{2t\sqrt{1-4t^2}}$	yes	yes	no
9	Rooted trees	$\sum_{n\geq 0} \frac{(n+1)^n}{(n+1)!}t^n$	$\sum_{n\geq 0} \frac{(n+1)^n}{n!}t^{2n+1}$	no	yes	no
10	Binary rtd. trees	$\frac{1-\sqrt{1-2t}}{t}$	$\frac{t}{\sqrt{1-2t^2}}$	no	yes	no

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