Split Graphs and Nordhaus-Gaddum Graphs

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Abstract

A graph G is an NG-graph if $\chi(G) + \chi(\overline{G}) = |V(G)| + 1$. We characterize NG-graphs solely from degree sequences leading to a linear-time recognition algorithm. We also explore the connections between NG-graphs and split graphs. There are three types of NG-graphs and split graphs can also be divided naturally into two categories, balanced and unbalanced. We characterize each of these five classes by degree sequence. We construct bijections between classes of NG-graphs and balanced and unbalanced split graphs which, together with the known formula for the number of split graphs on n vertices, allows us to compute the sizes of each of these classes. Finally, we provide a bijection between unbalanced split graphs on n vertices and split graphs on n-1 or fewer vertices providing evidence for our conjecture that the rapid growth in the number of split graphs comes from the balanced split graphs.

Keywords: Nordhaus-Gaddum theorem, NG-graphs, split graphs, pseudo-split graphs, degree sequence characterization, bijection, counting

1 Introduction

For a graph G, the number of vertices in a largest clique in G is denoted by $\omega(G)$ and the number of vertices in a largest stable set (independent set) in G is denoted by $\alpha(G)$. We denote the complement of G by \overline{G} and the graph induced in G by $X \in V(G)$ by G[X]. We write nbh(x) to denote the set of vertices adjacent to vertex x. For a set of graphs, \mathcal{C} , we denote by \mathcal{C}_n the set of graphs in \mathcal{C} with n vertices.

A well-known theorem by Nordhaus and Gaddum [14] states that the following is true for any graph G:

$$2\sqrt{|V(G)|} \leq \chi(G) + \chi(\bar{G}) \leq |V(G)| + 1.$$

We call G a Nordhaus-Gaddum graph or NG-graph if G satisfies the maximum value of this inequality; i.e., $\chi(G) + \chi(\bar{G}) = |V(G)| + 1$. Finck [9] and Starr and Turner [16] provide two different characterizations of NG-graphs. More recently, Collins and Trenk [7] define the ABC-partition of a graph and characterize NG-graphs in terms of this partition.

Definition 1. For a graph G, the ABC-partition of V(G) (or of G) is

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A_G = \{ v \in V(G) : deg(v) = \chi(G) - 1 \}
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$$B_G = \{ v \in V(G) : deg(v) > \chi(G) - 1 \}$$

$$B_G = \{ v \in V(G) : deg(v) > \chi(G) - 1 \}$$

$$C_G = \{ v \in V(G) : deg(v) < \chi(G) - 1 \}.$$

When it is unambiguous, we write $A = A_G$, $B = B_G$, $C = C_G$.

Theorem 2. (Collins and Trenk [7]) A graph G is an NG-graph if and only if its ABC-partition satisfies

- (i) $A \neq \emptyset$ and G[A] is a clique, a stable set, or a 5-cycle
- (ii) G[B] is a clique
- (iii) G[C] is a stable set
- (iv) $uv \in E(G)$ for all $u \in A$, $v \in B$
- (v) $uw \notin E(G)$ for all $u \in A$, $w \in C$.

By (i) of Theorem 2, there are three possible forms of an NG-graph. (See Figure 1.)

Definition 3. We say that G is an NG-1 graph if G[A] is a clique, an NG-2 graph if G[A] is a stable set, and an NG-3 graph if G[A] is a 5-cycle. We also let \mathcal{NG} -1 be the set of NG-1 graphs and likewise define the sets \mathcal{NG} -2 and \mathcal{NG} -3.

The characterization in Theorem 2 not only provides a clear description of NGgraphs but it also lends itself to an $O(|V(G)|^3)$ -time recognition algorithm for NGgraphs [7]. More importantly for this article, it shows that NG-graphs are related to split graphs and pseudo-split graphs.

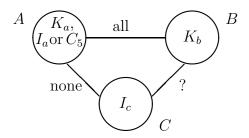


Figure 1: The forms of an NG-graph

A split graph is a graph G whose vertex set can be partitioned as $V(G) = K \cup S$, where K induces a clique and S induces a stable set in G. A detailed introduction to this class appears in [11]. Split graphs are a well-known class of perfect graphs, and thus $\chi(G) = \omega(G)$ for split graphs. Split graphs also have elegant characterization theorems. Földes and Hammer [10] give a forbidden subgraph characterization of split graphs as those graphs with no induced $2K_2$, C_4 or C_5 . Split graphs also have a degree sequence characterization due to Hammer and Simeone [12] which we present in Theorem 14. This latter characterization implies that split graphs can be recognized in linear time.

Blázsik et al. [3] consider the class of graphs that do not contain C_4 and $2K_2$ as induced subgraphs, later referred to as pseudo-split graphs [13]. They show that like split graphs, pseudo-split graphs can be defined in terms of vertex sets partitions. In particular, a graph G is a pseudo-split graph if and only if V(G) can be partitioned into three parts so that (i) first part is either empty or induces a 5-cycle, the second part a clique, the third part a stable set and (ii) whenever the first part is a 5-cycle, every vertex in the first part is adjacent to every vertex in the second part but there are no edges between the first part and the third part. Interestingly, Blázsik et al. also note that pseudo-split graphs are almost extremal in terms of the Nordhaus-Gaddum inequality because for any such graph G, $\chi(G) + \chi(\bar{G}) \geq |V(G)|$. In the process of proving this result, they show that if G contains an induced 5-cycle then $\chi(G) + \chi(\bar{G}) \geq |V(G)| + 1$ and thus G is an NG-graph.

The next result follows from Theorem 2 and the characterization of pseudo-split graphs discussed above. In particular, a graph is an NG-3 graph if and only if it is a pseudo-split graph containing an induced 5-cycle.

Remark 4. A graph is a pseudo-split graph if and only if it is a split graph or an NG-3 graph.

Proposition 5. Let G be an NG-graph. Then G is a split graph if and only if $G \in (\mathcal{NG}-1 \cup \mathcal{NG}-2)$.

Proof. By definition, NG-3 graphs contain an induced C_5 , hence are not split graphs. Now suppose $G \in (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)$ and let its ABC-partition be $V(G) = A \cup B \cup C$. If $G \in \mathcal{NG}\text{-}1$, let $K = A \cup B$ and S = C and otherwise, $G \in \mathcal{NG}\text{-}2$, and we let K = B and $S = A \cup C$. In either case, using Theorem 2, we get a partition of V(G) into a clique K and a stable set S, so G is a split graph. \square

Not all split graphs are NG-graphs. Indeed, the relationship between these classes, as well as the results in Remark 4 and Proposition 5, are shown in Figure 2. We will define the classes of balanced and unbalanced split graphs in the next section.

Building on the work of Blázsik et al. [3], Maffray and Preissmann [13] present a degree sequence characterization for NG-3 graphs. They combine this with the similar characterization for split graphs to get a linear-time recognition algorithm for pseudo-split graphs. We will discuss similar algorithms for NG-graphs in Section 3.

Finally, we note that Theorem 2 is very much related to the notion of graph decomposition that was studied systematically by Tyshchevich [17]. A graph is said to be decomposable if its vertex set can be partitioned into three parts A, B and C so that $A \neq \emptyset$ and $B \cup C \neq \emptyset$ and conditions (ii) to (v) of Theorem 2 are satisfied. Thus, every NG-graph is decomposable except when it is a single vertex or a 5-cycle. Chvátal and Hammer [5], Blázik et al. [3] and Barrus [1, 2] also characterized various graph classes in terms of their decompositions.

Our main objective is to explore the connections between NG-graphs and split graphs. In Section 2, we study split graphs through the lens of NG-graphs. In particular, we determine which split graphs are NG-graphs and show how their ABC-partitions relate to their clique-stable set partitions. In Section 3, we do the opposite and consider NG-graphs through the lens of split graphs. We provide degree characterizations for NG-1 and NG-2 graphs that are quite similar to the one for split graphs. We also present a degree characterization of NG-3 graphs that is equivalent to the one in [13]. These results show that, like split graphs and pseudo-split graphs, NG-graphs have a linear-time recognition algorithm. In Section 4 we present various kinds of bijections including those between subclasses of NG-graphs and subclasses of split graphs. Finally, in Section 5, we take advantage of these bijections and present formulas for the number of graphs on n vertices in each of the graph classes we studied. Our work comparing NG-graphs to split graphs leads to a theorem only about split graphs: that for $n \geq 1$, the number of unbalanced split graphs on n vertices is equal to the number of split graphs on n = 1 or fewer vertices.

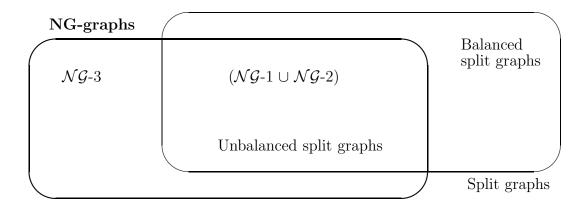


Figure 2: A partition of the class of pseudo-split graphs into NG-graphs and split graphs.

2 Split Graphs and NG-graphs

In this section we consider the set S of split graphs and discuss connections to NG-graphs. A KS-partition of a split graph G is a partition of the vertex set as $V(G) = K \cup S$ where K is a clique and S is a stable set. Just as it is helpful to characterize NG-graphs into the classes \mathcal{NG} -1, \mathcal{NG} -2 and \mathcal{NG} -3 based on their ABC-partition, it is also useful to categorize split graphs based on their KS-partitions.

Definition 6. A split graph G is balanced if it has a KS-partition satisfying $|K| = \omega(G)$ and $|S| = \alpha(G)$ and unbalanced otherwise. We denote the set of balanced split graphs by \mathcal{B} and the set of unbalanced split graphs by \mathcal{U} . A KS-partition is S-max if $|S| = \alpha(G)$ and K-max if $|K| = \omega(G)$.

Unlike ABC-partitions, KS-partitions of a split graph are not always unique. The terms balanced and unbalanced in Definition 6 refer to a split graph G while the terms K-max and S-max refer to a particular KS-partition of G. The next theorem follows from the work of Hammer and Simeone [12] and appears in [11]. We include a proof for completeness.

Theorem 7. (Hammer and Simeone [12]) For any KS-partition of a split graph G, exactly one of the following holds:

(i)
$$|K| = \omega(G)$$
 and $|S| = \alpha(G)$. (balanced)
(ii) $|K| = \omega(G) - 1$ and $|S| = \alpha(G)$. (unbalanced, S-max)
(iii) $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$. (unbalanced, K-max)

Moreover, in (ii) there exists $s \in S$ so that $K \cup \{s\}$ is complete and in (iii) there exists $k \in K$ so that $S \cup \{k\}$ is a stable set.

Proof. Partition the vertex set of G as $V(G) = K \cup S$ where K is a clique and S is a stable set. If both K and S are maximum size then $|K| = \omega(G)$ and $|S| = \alpha(G)$, resulting in case (i). If K is not maximum size, then $\omega(G) = |K| + 1$ because only one vertex of S can be part of a clique. In this case, there must exist $s \in S$ adjacent to each vertex in K. Then no vertex of K can be added to S to make a larger stable set, and at most one vertex of K can be in any stable set, so S is maximum size. Thus $|S| = \alpha(G)$, resulting in case (ii), and moreover, $K \cup \{s\}$ is complete. Similarly, if S is not maximum size, the result is case (iii).

In Proposition 5, we showed which NG-graphs are split graphs. In the next theorem we show which split graphs are NG-graphs.

Theorem 8. The following are equivalent for a split graph G.

- (1) G is an NG-graph.
- (2) $G \in (\mathcal{NG}-1 \cup \mathcal{NG}-2)$.
- (3) G is unbalanced.

Proof. $(1) \Longrightarrow (2)$. Follows directly from Proposition 5.

- (2) \Longrightarrow (3). For a contradiction, assume G is a balanced split graph and fix a a KS-partition with $|K| = \omega(G)$ and $|S| = \alpha(G)$. Since split graphs are perfect, $\chi(G) = \omega(G)$ and $\chi(\overline{G}) = \alpha(G)$. Thus $\chi(G) + \chi(\overline{G}) = \omega(G) + \alpha(G) = |K| + |S| = |V(G)| \neq |V(G)| + 1$ and G is not an NG-graph.
- (3) \Longrightarrow (1). Let G be an unbalanced split graph and fix a KS-partition of G. First consider the case in which the KS-partition is K-max, thus $|K| = \omega(G)$ and by Theorem 7, $|S| = \alpha(G) 1$. Again, since split graphs are perfect, so $\chi(G) = \omega(G)$ and $\chi(\overline{G}) = \alpha(G)$. Then, $\chi(G) + \chi(\overline{G}) = \omega(G) + \alpha(G) = |K| + |S| + 1 = |V(G)| + 1$ and G is an NG-graph. The proof for a S-max KS-partition is similar.

The next remark follows from Proposition 5 and Theorem 8. The Venn diagram in Figure 2 shows the relationships we have proven about NG-graphs and split graphs.

Remark 9. $\mathcal{U} = (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)$, and consequently, pseudo-split graphs are either NG-graphs or balanced split graphs.

Our knowledge of NG-graphs allows us to refine the Hammer/Simeone conditions in Theorem 7. In particular, we can characterize all KS-partitions of a split graph and for split graphs with more than one KS-partition (unbalanced) it is precisely the vertices in A_G that can be moved between K and S.

Theorem 10. Suppose G is an unbalanced split graph and let $V(G) = A \cup B \cup C$ be its ABC-partition. The KS-partitions of G can be characterized as follows.

- If $G \in \mathcal{NG}$ -1, the partitions are $K = A \cup B$, S = C (unique K-max), $K = (A \cup B) \{a\}$, $S = C \cup \{a\}$ for any $a \in A$ (S-max).
- If $G \in \mathcal{NG}$ -2, the partitions are K = B, $S = A \cup C$ (unique S-max) $K = B \cup \{a\}$, $S = (A \cup C) \{a\}$ for any $a \in A$ (K-max).

Proof. Since G is an unbalanced split graph, we know that G is also an NG-graph by Theorem 8. Indeed, by Proposition 5, it is an NG-1 graph or an NG-2 graph. We will give the argument in the case in which G is an NG-1 graph, that is, G[A] is a clique. The case in which G is an NG-2 graph is similar.

Let $K = A \cup B$ and S = C. We claim that $|K| = \omega(G)$, that is, this is a K-max partition of G. At most one vertex of stable set C can be in any clique of G but no vertex of C can be added to K because vertices of C are not adjacent to vertices in A and $A \neq \emptyset$. Thus $|K| = \omega(G)$ as desired and by Theorem 7, $|S| = \alpha(G) - 1$.

We next show that this is the only K-max partition of G. As before, at most one vertex of C can be in a clique, so if there were a different K-max partition, we would have to move one vertex $c \in C$ from S to the clique K and remove one vertex from $K = A \cup B$ to S. Since vertices in C are not adjacent to vertices in A, we must remove all vertices in A from K in order to retain a clique, thus |A| = 1. But in this case, the vertex C and the same neighbor set (namely, all vertices in C) and thus the same degree, violating Definition 1.

Next we characterize the S-max partitions. Take any $a \in A$ and let $K' = (A \cup B) - \{a\}$ and $S' = C \cup \{a\}$. Thus $V(G) = K' \cup S'$. We know K' is a clique and S' is a stable set by Theorem 2. Furthermore, $|S'| = |S| + 1 = \alpha(G)$ so the partition $V(G) = K' \cup S'$ is S-max for each $a \in A$. Finally, we show these are the *only* S-max partitions of G. Since $A \cup B$ is a clique in G, at most one vertex of $A \cup B$ can be in any stable set. If a vertex $b \in B$ were added to S to form a larger stable set, then

b would have the same degree (namely |A| + |B| - 1) as each vertex in A, violating Definition 1.

The next corollary follows directly from Theorem 10 and allows us to conclude that there are exactly (|A|+1) KS-partitions of an unbalanced labeled split graph. In contrast, there is a unique KS-partition of a balanced split graph as shown in Proposition 12.

Corollary 11. Suppose G is an unbalanced split graph and let $V(G) = A \cup B \cup C$ be its ABC-partition. Then if we consider G to be a labelled graph, it has either a unique K-max partition and exactly |A| distinct S-max partitions or a unique S-max partition and exactly |A| distinct K-max partitions.

Proposition 12. Each balanced split graph has a unique KS-partition.

Proof. Let G be a balanced split graph and fix a KS-partition of G with $|K| = \omega(G)$ and $|S| = \alpha(G)$. For a contradiction, suppose K'S' is a different KS-partition of G. Since K and S are already of maximum size and at most one vertex of S can be in a clique and at most one vertex of K can be in a stable set, we know $K' = K \cup \{y\} - \{x\}$ and $S' = S \cup \{x\} - \{y\}$ for some $x \in K$ and $y \in S$. Now K and K' are cliques but $K \cup \{y\}$ has $\omega(G) + 1$ vertices and is not a clique, so $xy \notin E(G)$. But then x is not adjacent to any vertex in S, so $S \cup \{x\}$ is a stable set of size $\alpha(G) + 1$, a contradiction.

We use the term "unlabeled graphs" in the remainder of the paper to refer to isomorphism classes of graphs as in [18].

Corollary 13. Each unlabeled unbalanced split graph has exactly two KS-partitions, one is K-max and the other is S-max.

Proof. Let G be an unbalanced split graph. Fix a KS-partition of G. If it is S-max, then we can move a vertex from S to K, as in the proof of Theorem 7, to obtain a different KS-partition of G which is K-max. Similarly, if it is K-max, we can move a vertex from K to S to obtain a different KS-partition of G which is S-max.

Now suppose there are two non-isomorphic K-max partitions of G, K_1S_1 , K_2S_2 . Thus $|K_1| = |K_2| = \omega(G)$, but $K_1 \neq K_2$. Since at most one vertex of $S_1 = V(G) - K_1$ can be in a clique, there exists $x \in S_1$ and $y \in K_1$ such that $K_2 = K_1 + \{x\} - \{y\}$. Thus, $N(x) = K_1 - \{x, y\} = N(y)$. Then there is an automorphism of G obtained by switching vertices x and y and leaving the remaining vertices unchanged. This contradicts our assumption that K_1S_1 and K_2S_2 are not isomorphic. The proof for two non-isomorphic S-max partitions is similar.

3 Degree sequence characterizations

We begin with the Hammer and Simeone result showing that split graphs can be recognized solely from their degree sequences. The proof serves as a foundation for the proofs of Theorems 17 and 22.

Theorem 14. (Hammer and Simeone, [12]) Let G = (V, E) be a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ and let $m = max\{i : d_i \ge i-1\}$. Then G is a split graph if and only if

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i.$$
 (1)

Furthermore, if equality holds in (1), then $\omega(G) = m$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where $deg(v_i) = d_i$ for each i. Since $d_1 \geq 0$, the value m is well-defined. If $G = K_n$ then m = n, G is a split graph, equality (1) holds and $\omega(G) = m$ as desired. Thus we may assume G is not complete, thus $d_n < n - 1$ and $m \leq n - 1$.

First we assume that G satisfies the equality (1) in Theorem 14. We let $K = \{v_1, v_2, \dots v_m\}$ and $S = \{v_{m+1}, v_{m+2}, \dots v_n\}$ and will show that K is a clique and S is a stable set in G. Partition $E(G) = E_1 \cup E_2 \cup E_3$ where $E_1 = E(G[K])$,

$$E_2 = \{xy : x \in K, y \in S\}, \text{ and } E_3 = E(G[S]). \text{ Then } \sum_{i=1}^m d_i = 2|E_1| + |E_2| \text{ and }$$

$$\sum_{i=m+1}^{n} d_i = 2|E_3| + |E_2|. \text{ Using equality (1) we get } 2|E_1| + |E_2| = m(m-1) + 2|E_3| + |E_2|$$

or equivalently, $|E_1| = \frac{m(m-1)}{2} + |E_3|$. Thus $|E_1| \ge \frac{m(m-1)}{2}$. But E_1 is the edge set of the graph G[K] with m vertices, thus $|E_1| \le \frac{m(m-1)}{2}$ and we conclude $|E_1| = \frac{m(m-1)}{2}$ and $|E_3| = 0$. Thus K is a clique and S a stable set as desired.

Conversely, let G be a split graph and by Theorem 7 we can fix a K-max partition of G. As before, let E_1 be the edge set of G[K], E_2 be the set of edges with one endpoint in K and the other in S, and E_3 be the edge set of G[S]. Since K is a clique, $deg(v) \geq |K| - 1$ for each $v \in K$ and since S is a stable set and K is a maximum size clique, $deg(v) \leq |K| - 1$ for each $v \in S$. Thus m = k and without loss

of generality,
$$K = \{v_1, v_2, \dots, v_{|K|}\}$$
 and $S = \{v_{|K|+1}, v_{|K|+2}, \dots, v_n\}$. Now $\sum_{i=1}^m d_i = 1$

$$2|E_1| + |E_2| = m(m-1) + |E_2|$$
 and $\sum_{i=m+1}^n d_i = 2|E_3| + |E_2| = |E_2|$ since K is a clique

and S a stable set. Thus equality holds in (1).

Remark 15. In the proof of Theorem 14, the partition $K = \{v_1, v_2, \dots, v_m\}$, $S = \{v_{m+1}, v_{m+2}, \dots, v_n\}$ is a K-max partition of G.

We will see in Examples 18 and 19 that index m is either the first or last index i for which $d_i = m - 1$. The next theorem shows that this is true for all split graphs.

Theorem 16. Let G = (V, E) be a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ and $m = max\{i : d_i \ge i - 1\}$. If $d_{m-1} = d_m = d_{m+1}$ then G is not a split graph.

Proof. For a contradiction, assume G is a split graph. If $d_m > m-1$ then by definition of m we have $d_{m+1} \leq m-1$ and we contradict $d_m = d_{m+1}$. Otherwise, $d_m = m-1$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $deg(v_i) = d_i$ for each i. As in the proof of Theorem 14, vertices v_1, v_2, \ldots, v_m form a maximum clique K and vertices $v_{m+1}, v_{m+2}, \ldots, v_n$ form a stable set S. Since S is a stable set, $nbh(v_{m+1}) \subseteq K$ and $|nbh(v_{m+1})| = d_{m+1} = m-1$ so v_{m+1} must be adjacent to m-1 vertices of K. Thus v_{m+1} is adjacent to at least one of v_m, v_{m-1} . But vertices v_m and v_{m-1} each have degree m-1 and each already has m-1 neighbors in K, a contradiction. \square

In Theorems 17 and 22, we characterize the graphs in \mathcal{NG} -1, \mathcal{NG} -2, and \mathcal{NG} -3 by their degree sequences.

Theorem 17. Let G = (V, E) be a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ and $m = max\{i : d_i \ge i - 1\}$. Then $G \in (\mathcal{NG}-1)_n$ if and only if it satisfies

- (1) $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$, and
- (2) $d_m = m 1$ and m is the largest index for which $d_i = m 1$.

Similarly, $G \in (\mathcal{NG}-2)_n$ if and only if it satisfies (1) and

(2') $d_m = m - 1$ and m is the smallest index i for which $d_i = m - 1$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $deg(v_i) = d_i$ and let $A \cup B \cup C$ be the ABC-partition of G First, we show that in both directions of the proof, G must be a split graph. If $G \in (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)$, then G is a split graph by Proposition 5. Conversely, if G satisfies (1), it is a split graph by Theorem 14. Split graphs are perfect, so $\chi(G) = \omega(G)$.

Let $K = \{v_1, v_2, \dots, v_m\}$ and $S = \{v_{m+1}, v_{m+2}, \dots, v_n\}$. By Theorem 14 and Remark 15, we know K is a clique, S is a stable set and KS is a K-max partition of G. Thus $\omega(G) = |K| = m$ and $\chi(G) = m$.

For the forward direction, assume $G \in (\mathcal{NG}-1 \cup \mathcal{NG}-2)$. Then G is an unbalanced split graph by Theorem 8 and thus satisfies (1) of Theorem 14. It remains to show that $G \in \mathcal{NG}-1$ implies (2) and $G \in \mathcal{NG}-2$ implies (2').

First assume $G \in \mathcal{NG}$ -1. Then by Theorem 10, the partition $K = A \cup B$, S = C is the unique K-max partition of G. Since $A \neq \emptyset$ for NG-graphs and v_m has the smallest degree among vertices in K, we know that $v_m \in A$ and thus $d_m = \chi(G) - 1 = m - 1$. Moreover, $v_{m+1} \in C$, thus by Definition 1, $d_{m+1} < \chi(G) - 1 = m - 1$, proving (2).

Now assume $G \in \mathcal{NG}$ -2. Then by Theorem 10, any K-max partition of G has the form $K = B \cup \{a\}$, $S = (A \cup C) - \{a\}$ for some $a \in A$. By Definition 1, $deg(a) = \chi(G) - 1 = m - 1$ and deg(b) > m - 1 for all $b \in B$. Hence a is the unique vertex of smallest degree in K. Thus, $v_m = a$, and $deg(v_m) = m - 1$ and $deg(v_{m-1}) > m - 1$, proving (2').

Conversely, assume (1) holds. Since $\chi(G) = m$, A is the set of vertices of degree m-1 in G. By Theorem 16, m is either the largest index for which $d_i = m-1$ or the smallest (or both).

In the first instance, where (2) holds, each of the vertices in S has degree at most m-2 and $m-2 < \chi(G)-1$. Hence C=S and $A \cup B=K$. Each vertex in A is adjacent to the other m-1 vertices in K, thus is not adjacent to any vertex in C, which proves that $G \in \mathcal{NG}$ -1.

In the second instance, where (2') holds, each of the vertices $v_1, v_2, \ldots, v_{m-1}$ has degree at least m and $m > \chi(G) - 1$. Thus $B = \{v_1, v_2, \ldots, v_{m-1}\}$ and B is contained in the clique K. Since K is a clique and $deg(v_m) = m - 1$, vertex v_m is adjacent to each of the vertices in B and thus to no vertices in S. Each $v \in S$ with deg(v) = m - 1 is likewise adjacent to each vertex in B. Hence $A \cup C = S \cup \{v_m\}$ and is a stable set. Thus, $G \in \mathcal{NG}$ -2.

The next two examples illustrate Theorem 17.

Example 18. Let G_1 be the NG-1 graph whose ABC-partition is $A = \{v_4, v_5, v_6\}$, $B = \{v_1, v_2, v_3\}$ and $C = \{v_7, v_8, v_9, v_{10}\}$, and where the edge set between B and C is $E_2 = \{v_1v_7, v_1v_8, v_1v_9, v_2v_7, v_2v_8, v_3v_{10}\}$. The table below shows the vertex degrees $d_i = deg(v_i)$. Then m = 6 and as in Theorem 17, 6 is the largest index i for which $d_i = m - 1$.

i	1	2	3	4	5	<u>6</u>	7	8	9	10
d_i	8	7	6	5	5	<u>5</u>	2	2	1	1
i-1	0	1	2	3	4	<u>5</u>	6	7	8	9

Example 19. Let G_2 be the NG-2 graph formed by removing the 3 edges from G_1 between vertices in A. Again, the table below shows the vertex degrees. Now m = 4 and indeed, i = 4 is the smallest index i for which $d_i = m - 1$.

i	1	2	3	4	5	6	7	8	9	10
d_i	8	7	6	3_	3	3	2	2	1	1
i-1	0	1	2	<u>3</u>	4	5	6	7	8	9

The next corollary follows directly from Theorems 14, 16 and 17.

Corollary 20. $G \in (\mathcal{NG}-1 \cup \mathcal{NG}-2)$ iff G satisfies (1) of Theorem 17 and $d_m = m-1$.

The following characterization of the class of balanced split graphs follows immediately from Theorem 14 and Corollary 20. Observe that Corollary 21 does not make any reference to NG-graphs.

Corollary 21. Let G = (V, E) be a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ and $m = max\{i : d_i \ge i - 1\}$. Then G is a balanced split graph if and only if it satisfies

(i)
$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$$

(ii)
$$d_m > m - 1$$
.

In the next theorem, we characterize the class \mathcal{NG} -3 by degree sequences. In this instance, i is the middle index of five in which $d_i = m - 1$. An equivalent result appears in [13].

Theorem 22. Let G = (V, E) be a graph with degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ and let $m = max\{i : d_i \ge i - 1\}$. Then $G \in (\mathcal{NG}-3)_n$ if and only if the following conditions hold:

(i)
$$\sum_{i=1}^{m+2} d_i = (m+2)(m+1) - 10 + \sum_{i=m+3}^{n} d_i$$

(ii)
$$d_i = m-1$$
 if and only if $m-2 \le i \le m+2$

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where $deg(v_i) = d_i$. First we suppose that G satisfies the two conditions and show G is an NG-3 graph. Partition V(G) into sets B, A, C as follows and let $X = A \cup B$:

$$B = \{v_1, v_2, \dots, v_{m-3}\},\$$

$$A = \{v_{m-2}, v_{m-1}, v_m, v_{m+1}, v_{m+2}\},\$$

$$C = \{v_{m+3}, v_{m+4}, \dots, v_n\}.$$

We will show that this is the ABC-partition of G. Partition the edge set of G as $E_1 \cup E_2 \cup E_3$ where E_1 is the edge set of G[X], E_3 is the edge set of G[C] and E_2 is the set of edges in G with one endpoint in X and the other in G. Summing vertex degrees in G and in G we get

$$\sum_{i=1}^{m+2} d_i = 2|E_1| + |E_2| \text{ and } \sum_{i=m+3}^n d_i = 2|E_3| + |E_2|.$$

Using condition (i) in the hypothesis, we get

$$2|E_1| + |E_2| = (m+2)(m+1) - 10 + 2|E_3| + |E_2|,$$

or equivalently,

$$2|E_1| = (m+2)(m+1) - 10 + 2|E_3|. (2)$$

Since |X| = m + 2, each vertex in X has degree in G[X] of at most m + 1, and by condition (ii), the five vertices in A have degree m - 1 in G, thus their degree is at most m - 1 in G[X]. So summing degrees of vertices in the induced graph G[X] we get

$$2|E_1| = \sum_{i=1}^{m+2} deg_{G[X]}(v_i)$$

$$\leq 5(m-1) + (m-3)(m+1) = 5(m+1) - 10 + (m-3)(m+1)$$

$$= (m+2)(m+1) - 10.$$

Combining this with equation (2) we get $|E_3| = 0$ and conclude that C is a stable set. Now using $|E_3| = 0$ in (2) we get $|E_1| = {m+2 \choose 2} - 5$, so G[X] is a complete graph on m+2 vertices with five edges removed.

However, G[X] contains five vertices of degree m-1, namely those in A. Thus each vertex in A must be incident to exactly two of the removed edges, and the set of removed edges forms a 5-cycle. The edges remaining in G[A] also form a 5-cycle (since $\overline{C_5} = C_5$). Thus G[X] consists of a clique $G[B] \approx K_{m-3}$, a 5-cycle G[A] and all edges

between them. We have already verified conditions (i) – (iv) of Theorem 2. Since G[A] is a 5-cycle, any largest clique in G[X] consists of the vertices in B together with two adjacent vertices of A, thus $\omega(G) \geq \omega(G[X]) = m-1$. Each vertex in A has degree m-1 in G with m-3 neighbors in G[B] and 2 neighbors in G[A], thus vertices in A are not adjacent to any vertices of C, proving (v). This also means that vertices in C have degree at most |B| = m-3 and thus cannot participate in a clique of size m, hence $\omega(G) = m-1$.

However, $\chi(G) \geq m$ since the 5-cycle G[A] requires three colors and an additional m-3 colors are needed for the clique G[B]. This coloring can be extended to all of V(G) by assigning a color used in A to all vertices of C. Thus $\chi(G) = m$, the vertices in A have degree m-1, vertices in B have degree greater than m-1 and vertices in C have degree less than m-1, so the partition $V(G) = A \cup B \cup C$ is in fact an ABC-partition of G. By Theorem 2, G is an NG-graph and because G[A] is a 5-cycle, it is an NG-3 graph.

Conversely, suppose G is an NG-3 graph, so its ABC-partition satisfies the conditions of Theorem 2 with G[A] a 5-cycle. By the definition of an ABC-partition, vertices in B have degree greater than vertices in A which in turn have degree greater than those in C. Thus $B = \{v_1, v_2, \ldots, v_{|B|}\}$, $A = \{v_{|B|+1}, v_{|B|+2}, \ldots, v_{|B|+5}\}$, and $C = \{v_{|B|+6}, \ldots, v_n\}$. By the structure of NG-3 graphs, each vertex in A has degree 2 + |B|, thus $deg(v_{|B|+3}) \ge |B| + 2$ but $deg(v_{|B|+4}) < |B| + 3$ and hence m = |B| + 3. Therefore,

$$B = \{v_1, v_2, \dots, v_{m-3}\},\$$

$$A = \{v_{m-2}, v_{m-1}, v_m, v_{m+1}, v_{m+2}\},\$$

$$C = \{v_{m+3}, v_{m+4}, \dots, v_n\}.$$

Vertices in A have degree 2+|B|=m-1 and these are the only vertices of degree m-1 by definition of the ABC-partition. Thus the second condition in Theorem 22 holds. Vertices in C can only have neighbors in B, thus $deg(c) \leq |B| = m-3$ and we conclude that $d_i = m-1$ if and only if $m-2 \leq i \leq m+2$, which is the second condition of Proposition 22.

Finally, let E_1 be the edge set of $G[A \cup B]$ and E_2 be the set of edges in G with one endpoint in B and the other in C. Since $G[A \cup B]$ is a clique with a 5-cycle removed we have,

$$\sum_{i=1}^{m+2} d_i = 2|E_1| + |E_2| = (m+2)(m+1) - 10 + |E_2|.$$

Also, because C is a stable set and there are no edges between A and C, $\sum_{i=m+3}^{n} d_i = |E_2|$. Combining these gives the first condition in Proposition 22.

Example 23. Let G_3 be the NG-3 graph formed from G_1 in Example 18 by expanding A to be a 5-cycle. Thus, m = 6 which is the middle index of the 5 vertices of degree $d_m = 5$.

i	1	2	3	4	5	<u>6</u>	7	8	9	10	11	12
d_i	10	9	8	5	5	<u>5</u>	5	5	2	2	1	1
i-1	0	1	2	3	4	<u>5</u>	6	7	8	9	10	11

We know that split graphs are not NG-3 graphs by Proposition 5. Theorem 16 shows that the second condition of Theorem 22 is never satisfied for a split graph. The following example shows that it is possible for a split graph to satisfy the first condition of Theorem 22.

Example 24. Let G be the split graph with KS partition $K = \{v_1, v_2, v_3\}$, $S = \{v_4, v_5, v_6\}$ and $E(G) = E(K) \cup \{v_1v_3, v_2v_4\}$. The vertex degrees are: $d_1 = 3, d_2 = 3, d_3 = 2, d_4 = 1, d_5 = 1, d_6 = 0$, so m = 3. Notice that the first condition in Theorem 22 is satisfied, but the second is not.

Finally, since sorting the degrees of the vertices of a graph can be done in linear time using stable sort, Theorems 17 and 22 imply the next corollary.

Corollary 25. Given a graph G, determining if G is an NG-1, NG-2 or an NG-3 graph can be done in linear time.

4 Bijections

Throughout this section we are considering unlabeled graphs. Definition 3 divides the set of NG-graphs into three categories: \mathcal{NG} -1, \mathcal{NG} -2, and \mathcal{NG} -3, according to whether the set A of the ABC-partition induces a clique, a stable set or a 5-cycle. The first two categories overlap in the case where |A|=1. By removing this intersection as a separate class, we can partition the set of NG-graphs into four classes: $(\mathcal{NG}$ -1 - \mathcal{NG} -2), $(\mathcal{NG}$ -2 - \mathcal{NG} -1), $(\mathcal{NG}$ -1 - \mathcal{NG} -2) and \mathcal{NG} -3. Recall that we also divide split graphs (\mathcal{S}) into balanced (\mathcal{B}) and unbalanced (\mathcal{U}) . Let $(\mathcal{T})_{\leq n}$ be the set of split graphs on n or fewer vertices and recall that $(\mathcal{C})_n$ denotes the set of graphs with n vertices in a class \mathcal{C} .

In this section we provide bijections between classes of NG-graphs and classes of split graphs. In Section 5, we use these results to count the number of graphs in

each class. Table 1 summarizes the bijection results in this section and Theorem 26 records the locations of each of the proofs. The proofs are organized by the type of bijection used.

Theorem 26. There is a bijection between any two classes of graphs that appear in the same row of Table 1.

Proof. The bijections between the classes of graphs in row (a) appear in Theorem 28. The bijection between (iv) and (iv') in the proof of Theorem 28 gives the bijection between the classes in row (b). Complementation provides a bijection between $(\mathcal{NG}-1 - \mathcal{NG}-2)_n$ and $(\mathcal{NG}-2 - \mathcal{NG}-1)_n$ in row (c). A bijection between $(\mathcal{NG}-1 - \mathcal{NG}-2)_n$ and $(\mathcal{U})_{n-1}$ in row (c) is obtained by combining Remark 9 with the bijection given in the proof of Theorem 28 between the union of classes (i), (ii) and (iii) and the union of classes (i'), (ii') and (iii'). For row (d), Theorem 29 provides a bijection between $(\mathcal{NG}-3)_n$ and $(\mathcal{U})_{n-4}$. The remaining bijections appear in Theorem 30.

(/	$(\mathcal{NG}-1)_n$	$(\mathcal{NG}-2)_n$	$(\mathcal{S})_{n-1}$	
(b)	$(\mathcal{NG}-1\cap\mathcal{NG}-2)_n$	$(\mathcal{B})_{n-1}$		
(c)	$(\mathcal{NG}-1-\mathcal{NG}-2)_n$	$(\mathcal{NG}-2-\mathcal{NG}-1)_n$	$(\mathcal{U})_{n-1}$	$(\mathcal{T})_{\leq n-2}$
(d)	$(\mathcal{NG}-3)_n$	$(\mathcal{U})_{n-4}$	$(\mathcal{T})_{\leq n-5}$	

Table 1: There are bijections between all classes in the same row.

We begin with a lemma that identifies the type of KS-partition that results from removing A_G from an NG-graph G.

Lemma 27. Let G be an NG-graph and $V(G) = A \cup B \cup C$ be its ABC-partition. Then G - A is a split graph and has a KS-partition with K = B and S = C. Moreover,

- 1. if $G \in \mathcal{NG}$ -1 then KS is S-max, and
- 2. if $G \in \mathcal{NG}$ -2 then KS is K-max.

Proof. It is immediate that G-A is a split graph and that K=B, S=C constitutes a KS-partition of G-A. In the ABC-partition of an NG-1 graph G, every vertex in B has degree greater than every vertex in A so each vertex in B has a neighbor in C. Thus in our KS partition of G-A, no vertex in K can be moved to S. Hence, $\alpha(G-A)=|S|=|C|$. Similarly, in the ABC-partition of an NG-2 graph G, every vertex in G has degree less than any vertex in G, so every vertex of G has a non-neighbor in G. Hence in G has overtex in G can be moved to G. Thus, G has a non-neighbor in G

For an NG-graph G, let C'_G be the set of vertices in C_G whose neighbor set is all of B_G . In the proof of Theorem 28 we obtain our bijections by removing a single vertex from A_G .

Theorem 28. For $n \geq 1$, there are bijections between the following three classes: $(\mathcal{NG}-1)_n$, $(\mathcal{NG}-2)_n$ and $(\mathcal{S})_{n-1}$.

Proof. Taking graph complements provides a bijection between the classes $(\mathcal{NG}-1)_n$ and $(\mathcal{NG}-2)_n$, so it remains to show there exists a bijection between $(\mathcal{NG}-1)_n$ and $(\mathcal{S})_{n-1}$.

For $G \in (\mathcal{NG}-1)_n$, choose any $w \in A_G$ and let f(G) = G - w. We will show that f is our desired bijection. Indeed, we will do more. Partition the set of graphs in $(\mathcal{NG}-1)_n$ into four classes: (i) those G with $|A_G| \geq 3$, (ii) those G with $|A_G| = 2$ and $C'_G = \emptyset$, (iii) those G with $|A_G| = 2$ and $|A_G| = 2$ and (iv) those $|A_G| = 2$ the Likewise, partition the set of graphs in $|A_G| = 2$ and four classes: (i') $|A_G| = 2$ and $|A_$

Let $G \in \mathcal{NG}-1_n$ and let $A = A_G$, $B = B_G$, $C = C_G$ and $C' = C'_G$, so G[A] is a clique. Then $\chi(G) = |A| + |B|$ so each $a \in A$ has $deg_G(a) = |A| + |B| - 1$, each $b \in B$ has $deg_G(b) > |A| + |B| - 1$, and each $c \in C$ has $deg_G(c) < |A| + |B| - 1$. The graph G - w has $\chi(G - w) = |A| + |B| - 1$. Each $a \in A - w$ has $deg_{G-w}(a) = deg_G(a) - 1 = |A| + |B| - 2$, thus $a \in A_{G-w}$. Each $b \in B$ has $deg_{G-w}(b) = deg_B(b) - 1 > |A| + |B| - 2$ thus $b \in B_{G-w}$. However, each $c \in C$ has $deg_{G-w}(c) = deg_G(c)$, so some vertices in C may be part of A_{G-w} . We consider the cases mentioned above.

Case (i): $|A_G| \ge 3$. In this case, each $c \in C$ has $deg_{G-w}(c) = deg_G(c) \le |B| < |A| + |B| - 2 = \chi(G - w) - 1$, thus $c \in C_{G-w}$. The ABC-partition of G - w satisfies the five conditions of Theorem 2, A_{G-w} induces a clique, and $|A_{G-w}| \ge 2$, thus $G-w \in (\mathcal{NG}-1 - \mathcal{NG}-2)_{n-1}$.

Cases (ii) and (iii): $|A_G| = 2$. In this case, $A \cup C' - w$ forms a stable set and each vertex in $A \cup C' - w$ has neighbor set B. Then for any $c' \in C'$ we have $deg_{G-w}(c') = |B| = (|A| - 1) + |B| - 1 = \chi(G - w) - 1$ so $A_{G-w} = A \cup C' - w$, $B_{G-w} = B_G$, $C_{G-w} = C - C'$. The ABC-partition of G - w satisfies the five conditions of Theorem 2, A_{G-w} induces a stable set, thus $G - w \in (\mathcal{NG}-2)_{n-1}$. If $C'_G = \emptyset$ (case (ii)) then $|A_{G-w}| = 1$ and $G - w \in (\mathcal{NG}-1 \cap \mathcal{NG}-2)_{n-1}$. If $C'_G \neq \emptyset$ (case (iii)) then $|A_{G-w}| \geq 2$ and $G - w \in (\mathcal{NG}-2 - \mathcal{NG}-1)_{n-1}$.

Case (iv): |A| = 1. By Lemma 27, G - A is a split graph on n - 1 vertices with a KS-partition that is both K-max and S-max. By Definition 6, the graph G - A is a

balanced split graph, thus $G - A \in (\mathcal{B})_{n-1}$.

Next, we show the map f is onto by defining its inverse, g. Take any $H \in (\mathcal{S})_{n-1}$.

Case (i'): $H \in (\mathcal{NG}-1 - \mathcal{NG}-2)_{n-1}$. Thus $|A_H| \geq 2$ and A_H induces a clique. Add a vertex w to H that is adjacent to every vertex in $A_H \cup B_H$ to get H + w and define g(H) = H + w.

Then $\chi(H) = |A_H| + |B_H|$ so each $a \in A_H$ has $deg_H(a) = |A_H| + |B_H| - 1$, each $b \in B_H$ has $deg_H(b) > |A_H| + |B_H| - 1$, and each $c \in C_H$ has $deg_H(c) < |A_H| + |B_H| - 1$. The graph H + w has $\chi(H + w) = |A_H| + |B_H| + 1$. Each $a \in A_H \cup \{w\}$ has $deg_{H+w}(a) = |A_H| + |B_H|$ thus $a \in A_{H+w}$. Each $b \in B_H$ has $deg_{H+w}(b) = deg_H(b) + 1$ thus $b \in B_{H+w}$. Each $c \in C_H$ has $deg_{H+w}(c) = deg_H(c)$ thus $c \in C_{H+w}$. The ABC-partition of H + w satisfies the five conditions of Theorem 2, A_{H+w} induces a clique and $|A_{H+w}| \ge 3$, thus $H + w \in (\mathcal{NG}-1 - \mathcal{NG}-2)_{n-1}$ (case i).

Cases (ii') and (iii'): $H \in (\mathcal{NG}-2)_{n-1}$. In this case, A_H is a stable set. Let x be a vertex in A_H . Now $\chi(H) = |B_H| + 1$. Each $a \in A_H$ has $deg_H(a) = \chi(H) - 1 = |B_H|$, each $b \in B_H$ has $deg_H(b) > \chi(H) - 1 = |B_H|$, and each $c \in C_H$ has $deg_H(c) < \chi(H) - 1 = |B_H|$. Add a vertex w to H that is adjacent to x and all of B_H to get H + w and let g(H) = H + w. Then $\chi(H + w) = \chi(H) + 1 = |B_H| + 2$. The vertices x and w have $deg_{H+w}(x) = deg_{H+w}(w) = |B_H| + 1$, thus $x, w \in A_{H+w}$. Each $b \in B_H$ has $deg_{H+w}(b) = deg_H(b) + 1 > |B_H| + 1$, thus $b \in B_{H+w}$. Each $c \in A_H \cup C_H - \{x\}$ has $deg_{H+w}(c) \leq |B_H| < \chi(H+w) - 1$, thus $c \in C_{H+w}$. The ABC-partition of $c \in A_H$ satisfies the five conditions of Theorem 2, $c \in A_{H+w}$ induces a clique and $c \in A_{H+w} = 1$. Thus $c \in C_{H+w}$ induces a clique and $c \in A_H$ has $c \in A_H$ and $c \in A_H$ induces a clique and $c \in A_H$ and $c \in A_H$ has $c \in A_H$ induces a clique and $c \in A_H$ has $c \in A_H$ induces a clique and $c \in A_H$ has $c \in A_H$ induces a clique and $c \in A_H$ has $c \in A_H$ has $c \in A_H$ induces a clique and $c \in A_H$ has $c \in A_H$ h

Indeed, if $H \in (\mathcal{NG}\text{-}1 \cap \mathcal{NG}\text{-}2)$ (case ii') then $A_H = \{x\}$ and no vertices in H + w have degree $|B_H|$. But $|B_H| = |B_{H+w}|$, so no vertices in H + w have degree $|B_{H+w}|$ and $C'_{H+w} = \emptyset$ (case ii). Finally, if $H \in (\mathcal{NG}\text{-}2 - \mathcal{NG}\text{-}1)$ (case iii') then $|A_H| \geq 2$ and there exists at least one vertex $y \in A_H - \{x\}$. Then $deg_H(y) = deg_{H+w}(y) = |B_H| = |B_{H+w}|$ so $y \in C'_{H+w}$ and $C'_{H+w} \neq \emptyset$ (case iii).

Case (iv'): $H \in (\mathcal{B})_{n-1}$. In this case, H has a unique KS-partition by Proposition 12 such that $|K| = \omega(H) = \chi(H)$ and $|S| = \alpha(H)$. Add a vertex w to H that is adjacent to each vertex in K and let g(H) = H + w. Then $\chi(H + w) = |K| + 1$ and $deg(w) = |K| = \chi(H + w) - 1$. Each $x \in K$ has a neighbor in S since $|S| = \alpha(H)$, thus $deg_{H+w}(x) > |K| = \chi(H + w) - 1$. Similarly, each $y \in S$ has a non-neighbor in K since $|K| = \omega(H)$, thus $deg_{H+w}(y) < |K| = \chi(H + w) - 1$. The ABC-partition of H + w is $A = \{w\}$, B = K, C = S and this satisfies the five conditions of Theorem 2, with $|A_{H+w}| = 1$, thus $H + w \in (\mathcal{NG}-1)_n$ with |A| = 1 (case iv).

It is not hard to see that the function g is the inverse to f, thus we have described a bijection between $(\mathcal{NG}-1)_n$ and $(\mathcal{S})_{n-1}$.

Theorem 29. There is a bijection between $(\mathcal{NG}-3)_n$ and $(\mathcal{U})_{n-4}$.

Proof. We will demonstrate a bijection between $(\mathcal{NG}-3)_n$ and $(\mathcal{NG}-1-\mathcal{NG}-2)_{n-3}$. Then the desired bijection follows because we have already established bijections between the first three classes in row (c) of Table 1. Let $G \in (\mathcal{NG}-3)_n$ and let $V(G) = A \cup B \cup C$ be its ABC-partition. Then $G[A] = C_5$, by definition. Let $a_1, a_2 \in A$ such that $\{a_1, a_2\}$ is an edge in G[A], and define the map f by $f(G) = G[\{a_1, a_2\} \cup B \cup C]$. The graph f(G) has n-3 vertices and is a split graph with KS-partition $K = \{a_1, a_2\} \cup B$ and S = C. Since the neighborhood of a_1 does not include any vertices in C, then another KS-partition is K' = B and $S = C \cup \{a_1\}$. By Proposition 12, f(G) is an unbalanced split graph, hence $f(G) \in (\mathcal{NG}-1 \cup \mathcal{NG}-2)$. Note that $\omega(f(G)) = \chi(f(G)) = 2 + |B_G|$. The vertices in f(G) that have degree $1 + |B_G|$ include a_1 and a_2 , and also any vertices in B_G that have no neighbors in C_G . Hence $f(G) \in (\mathcal{NG}-1 - \mathcal{NG}-2)_{n-3}$

Now let $H \in (\mathcal{NG}\text{-}1 - \mathcal{NG}\text{-}2)_{n-3}$ with ABC-partition $A_H \cup B_H \cup C_H$ and let $a_1, a_2 \in A_H$. We define the map g as follows: g(H) is obtained by adding three vertices y_1, y_2, y_3 to H so that $H[\{a_1, a_2, y_1, y_2, y_3\}]$ is a 5-cycle and adding all edges between y_1, y_2, y_3 and $A_H - \{a_1, a_2\} \cup B_H$. Then $\chi(g(H)) = 3 + |A_H| - 2 + |B_H| = |A_H| + |B_H| + 1$. Each of a_1, a_2, y_1, y_2, y_3 has 2 neighbors in the 5-cycle and $|A_H| - 2 + |B_H|$ other neighbors, so these five vertices are in $A_{g[H]}$. The vertices in $A_H - \{a_1, a_2\} \cup B_H$ form a clique and each has at least $5 + |A_H| - 3 + |B_H| = 2 + |A_H| + |B_H|$ neighbors, so these vertices are in $B_{g[H]}$. Finally, the vertices in C_H form a stable set and each has at most $|B_H|$ neighbors, so these vertices are in $C_{g[H]}$. Hence the ABC-partition of g(H) is $A_{g(H)} = \{a_1, a_2, y_1, y_2, y_3\}, B_{g(H)} = A_H - \{a_1, a_2\} \cup B_H$, and $C_{g(H)} = C_H$, and $g(H) \in (\mathcal{NG}\text{-}3)_n$. It is straightforward to check that g is the inverse function of f.

For an NG-graph G, let B'_G be the set of vertices in B_G that have no neighbors in C_G . In the proof of Theorem 30, we obtain our bijections by removing A_G in part (1) and $A_G \cup B'_G$ in part (2). In both cases, the result is a split graph on n-2 or fewer vertices. This motivates our defining $(\mathcal{T})_{\leq n}$ to be the set of split graphs on n or fewer vertices and defining $|(\mathcal{S})_0| = |(\mathcal{T})_{\leq 0}| = 1$. Then by definition, $(\mathcal{S})_n = (\mathcal{U})_n \cup (\mathcal{B})_n$ and $(\mathcal{T})_{\leq n} = (\mathcal{S})_0 \cup (\mathcal{S})_1 \cup (\mathcal{S})_2 \cup \cdots \cup (\mathcal{S})_n$.

Theorem 30. Let G be an unlabeled NG-graph on n vertices. Then there are bijections between the following pairs of sets:

- 1. $(\mathcal{NG}-1 \mathcal{NG}-2)_n$ and $(\mathcal{T})_{\leq n-2}$.
- 2. $(\mathcal{NG}-3)_n$ and $(\mathcal{T})_{\leq n-5}$.

Proof. Let G be an NG-graph on n vertices and let $A \cup B \cup C$ be its ABC-partition.

Proof of (1): Let $G \in (\mathcal{NG}-1 - \mathcal{NG}-2)_n$. By definition of an NG-1 graph, deg(b) > deg(a) for any $a \in A, b \in B$, and hence every $b \in B$ has a neighbor in C. Define $f: (\mathcal{NG}-1 - \mathcal{NG}-2)_n \to (\mathcal{T})_{\leq n-2}$ by $f(G) = G[B \cup C]$. We will show f is our desired bijection. Since $|A| \geq 2$, the graph $G[B \cup C]$ has at most n-2 vertices, and by Lemma 27, it is a split graph, thus $G[B \cup C] \in (\mathcal{T})_{\leq n-2}$. Since every $b \in B$ has a neighbor in C, the KS-partition K = B, S = C is an S-max partition of $G[B \cup C]$.

Let $H \in (\mathcal{T})_{\leq n-2}$. By Theorem 7, we may choose a KS-partition of H that is S-max. Hence every $b \in K$ has a neighbor in S (otherwise b could be added to S). Define $g: (\mathcal{T})_{\leq n-2} \to (\mathcal{NG}\text{-}1 - \mathcal{NG}\text{-}2)_n$ by g(H) = G where G is the graph formed from H by adding a clique A of n - |V(H)| new vertices and joining every vertex in A to every vertex in K. Note that G has n vertices, and G is an split graph with KS-partition $K' = A \cup K$ and S. For $a \in A$, the sets $K' - \{a\}$ and $S \cup \{a\}$ provide another KS-partition of G, so G is an unbalanced split graph by Proposition 12.

Thus $G \in (\mathcal{NG}\text{-}1 \cup \mathcal{NG}\text{-}2)_n$ by Theorem 8. Since $\chi(G) = \omega(G) = |A| + |K|$, the ABC-partition of G is $A \cup K \cup S$ and $G \in (\mathcal{NG}\text{-}1 - \mathcal{NG}\text{-}2)_n$ because A is a clique and $|A| \geq 2$. It is straightforward to check that g is the inverse function of f.

Proof of (2): Let $G \in (\mathcal{NG}\text{-}3)_n$. Define $f:(\mathcal{NG}\text{-}3)_n \to (\mathcal{T})_{\leq n-5}$ by $f(G) = G[(B-B') \cup C]$. We will show f is our desired bijection. Since |A| = 5, the graph f(G) has at most n-5 vertices, and it is a split graph with KS-partition K = B - B', S = C. Thus $f(G) \in (\mathcal{T})_{\leq n-5}$. Furthermore, since every $b \in B - B'$ has a neighbor in C, the KS-partition K = B - B', S = C is an S-max partition of f(G).

Let $H \in (\mathcal{T})_{\leq n-5}$. By Theorem 7, we may choose a KS-partition of H that is S-max. Hence every $b \in K$ has a neighbor in S. Define $g: (\mathcal{T})_{\leq n-5} \to (\mathcal{NG}-3)_n$ by g(H) = G where G is the graph formed from H by adding a set D of n - |V(H)| new vertices such that G[D] is a clique minus the edges of a 5-cycle, and every vertex in D is adjacent to every vertex in K but to no vertices in S. Note that G has n vertices, and $\chi(G) = |D| - 2 + |K|$ since the vertices in D in the 5-cycle may be colored with 3 instead of 5 colors. Five vertices in D have degree |D| - 3 + |K| and the rest have degree |D| - 1 + |K|. For all $b \in K$, $deg_G(b) > |D| + |K| - 1$ and for all $c \in S$, $deg(c) \leq |K|$. Thus in the ABC-partition of G, A is the set of five vertices in D that induce a 5-cycle, $B = (D - A) \cup K$ and C = S. It follows from Definition 3 that $G \in (\mathcal{NG}-3)_n$ and it is straightforward to check that g is the inverse function of f. \square

In the proof of Theorem 30 we used a more complicated bijection in part (2) because the function $f(G) = G[B \cup C]$ is not injective when applied to graphs in $(\mathcal{NG}-3)_n$. This is illustrated in Example 31.

Example 31. Let G be the NG-3 graph where $A_G = \{a_1, a_2, a_3, a_4, a_5\}$, $B_G = \{b_1, b_2, b_3\}$, $C_G = \{c_1, c_2\}$ and b_1c_1 is the only edge between B_G and C_G . Note that $\chi(G) = 6$, the vertices in A_G have degree 5, those in B_G have degree greater than 5, and those in C_G have degree less than 5, as needed. The graph $G - A_G$ consists of a triangle (with vertices b_1, b_2, b_3), an edge between b_1 and c_1 and an isolated vertex (c_2) .

Now let H be the NG-3 graph where again $A_H = \{a_1, a_2, a_3, a_4, a_5\}$, $B_H = \{b_1, b_2\}$, $C_H = \{c_1, c_2, c_3\}$ and the edge set between B_H and C_H is $\{b_1c_1, b_1c_3, b_2c_3\}$. Now $\chi(G) = 5$, the vertices in A_H have degree 4, those in B_H have degree greater than 4, and those in C_H have degree less than 4, as needed. The graph $H - A_H$ consists of a triangle (with vertices b_1, b_2, c_3), an edge between b_1 and c_1 and an isolated vertex (c_2) . Thus G and H are not isomorphic, yet $G - A_G$ and $H - A_H$ are isomorphic.

Theorem 26 includes a bijection between the classes $(\mathcal{U})_{n-4}$ and $(\mathcal{T})_{\leq n-5}$, which implies that $|(\mathcal{U})_n| = |(\mathcal{T})_{\leq n-1}|$. We provide a second proof of this equality using the classic presentation of split graphs found in [11] and this second proof does not rely on NG-graphs. Let $(\mathcal{U})_n^K$ be the set of triples (G, K_G, S_G) where G is an unlabeled, unbalanced split graph on n vertices, and $K_G S_G$ is a KS-partition that is K-max. Similarly, define $(\mathcal{U})_n^S$ where the KS-partition $K_G S_G$ is S-max. Recall from Proposition 12 that balanced split graphs have a unique KS-partition, so we may define $(\mathcal{B})_n^{KS}$ to be the set of triples (G, K_G, S_G) of balanced split graphs on n vertices together with their unique KS-partition.

Theorem 32. There is a bijection between $(\mathcal{U})_n^K$ and $(\mathcal{U})_{n-1}^K \cup (\mathcal{U})_{n-1}^S \cup \mathcal{B}_{n-1}^{KS}$, and $|(\mathcal{U})_n| = |(\mathcal{T})_{\leq n-1}|$.

Proof. Let $(G, K_G, S_G) \in (\mathcal{U})_{n-1}^K \cup (\mathcal{U})_{n-1}^S \cup B_{n-1}^{KS}$. Create a new graph H consisting of G together with a new vertex w that is adjacent to every vertex in K_G . Then H is a split graph and the sets $K = K_G \cup \{w\}$ and $S = S_G$ form a KS-partition of H. Further, H is an unbalanced split graph because w is not adjacent to any vertex in S, hence $K' = K_G$, $S' = S_G \cup \{w\}$ is another KS-partition of H. Thus, KS is a K-max partition of H. Define $\phi: (\mathcal{U})_{n-1}^K \cup (\mathcal{U})_{n-1}^S \cup B_{n-1}^{KS} \to (\mathcal{U})_n^K$ by $\phi(G) = H$. We next show that ϕ is a reversible map. Take any $(H, K_H, S_H) \in (\mathcal{U})_n^K$. By

We next show that ϕ is a reversible map. Take any $(H, K_H, S_H) \in (\mathcal{U})_n^K$. By Theorem 7, there exists $w \in K_H$ so that $K_H - \{w\}$ is complete and $S_H \cup \{w\}$ is a stable set. Let $\psi(H) = H - w$. Note that $\psi(H)$ is the same unlabeled graph for any

possible choice of w. Then H-w is a split graph on n-1 vertices and $K=K_H-\{w\}$, $S=S_H$ is a KS-partition of H-w. Therefore, ψ reverses the operation of ϕ .

Finally, we prove $|(\mathcal{U})_n| = |(\mathcal{T})_{\leq n-1}|$. The sets $(\mathcal{U})_{n-1}^k$, $(\mathcal{U})_{n-1}^S$, and B_{n-1}^{KS} are disjoint by definition, so $|(\mathcal{U})_n^K| = |(\mathcal{U})_{n-1}^k| + |(\mathcal{U})_{n-1}^S| + |B_{n-1}^{KS}|$. We know from Corollary 13 that every unbalanced split graph has exactly two non-isomorphic KS-partitions, one K-max and the other S-max, thus $|(\mathcal{U})_n| = |(\mathcal{U})_n^K| = |(\mathcal{U})_n^S|$. Balanced split graphs have a unique KS-partition (Proposition 12) thus $|(\mathcal{B})_n| = |(\mathcal{B})_n^{KS}|$. Thus

$$|(\mathcal{U})_n| = |(\mathcal{U})_n^K| = |\mathcal{U}_{n-1}^K| + |\mathcal{U}_{n-1}^S| + |\mathcal{B}_{n-1}^{KS}| = 2|(\mathcal{U})_{n-1}| + |(\mathcal{B})_{n-1}|.$$

$$|(\mathcal{U})_n| = |(\mathcal{U})_{n-1}| + |(\mathcal{S})_{n-1}|.$$

Since $|(\mathcal{T})_{\leq n-1}| = |(\mathcal{T})_{\leq n-2}| + |(\mathcal{S})_{n-1}|$, and $|\mathcal{U}_1| = 1 = |\mathcal{T}_0|$, we see that $|(\mathcal{U})_n|$ and $|(\mathcal{T})_{\leq n-1}|$ satisfy the same recurrence and initial condition and therefore are equal. \square

5 Counting

In this section we use the bijections from Section 4 to calculate the size of classes of split graphs, NG-graphs and pseudo-split graphs. In [6], Clarke gives an expression for the number of minimal k-covers of a set of n indistinguishable objects. Royle [15] then describes a bijection between such k-covers and the set of split graphs on n vertices with exactly k maximal cliques that each contain a vertex in none of the other maximal cliques. Summing over k from 1 to n, Royle obtains a formula for $|(\mathcal{S})_n|$, the number of split graphs on n vertices. Unfortunately, the formula is quite complicated.

Our bijections from Section 4 allow us to calculate the number of balanced and unbalanced split graphs, all categories of NG-graphs and pseudo-split graphs solely in terms of the number of split graphs. Let $(\mathcal{PS})_n$ denote the set of pseudo-split graphs on n vertices. The corollary below describes the exact formulas.

Corollary 33. The following equalities hold for each $n \geq 1$.

(1)
$$|(\mathcal{U})_n| = \sum_{i=0}^{n-1} |(\mathcal{S}_i)|$$

(2)
$$|(\mathcal{B})_n| = |(\mathcal{S})_n| - \sum_{i=0}^{n-1} |(\mathcal{S}_i)|$$

(3)
$$|(\mathcal{NG}-1)|_n = |(\mathcal{NG}-2)|_n = |(\mathcal{S})_{n-1}|$$

$$(4) |(\mathcal{NG}-3)|_n = \sum_{i=0}^{n-5} |(\mathcal{S}_i)|$$

(5)
$$|(\mathcal{NG})_n| = \sum_{i=0}^{n-1} |(\mathcal{S}_i)| + \sum_{i=0}^{n-5} |(\mathcal{S}_i)|$$

(6)
$$|(\mathcal{PS})_n| = |(\mathcal{S})_n| + \sum_{i=0}^{n-5} |(\mathcal{S}_i)|$$

Proof. The formulas in (1), (3) and (4) follow directly from the bijections in Section 4. To prove (2), we use (1) and the fact that a split graph is either balanced or unbalanced, and to prove (5) we apply (1) and (4) and Remark 9. Finally, to prove (6), we use Remark 4 and apply the formula from (4).

Recall that $|(\mathcal{T})_{\leq n}| = \sum_{i=0}^{n} |(\mathcal{S}_i)|$. In Table 2, we make use of the values computed by Royle [15] for $|(\mathcal{S})_n|$ and the formulas in Corollary 33 to determine the values for $|(\mathcal{T})_{\leq n}|$, $|(\mathcal{U})_n|$, $|(\mathcal{B})_n|$, $|(\mathcal{NG})_n|$ and $|(\mathcal{PS})_n|$ for $n=0,\ldots,11$. The table in [15] shows rapid growth in the number of split graphs. Royle does not divide split graphs into balanced and unbalanced as we do. Table 3 shows the ratio of the number of balanced split graphs to the number of split graphs on n vertices, indicating that the rapid growth in the number of split graphs comes from the balanced category. We conjecture that this ratio approaches 1 as n goes to infinity. We now show why this may be the case.

Theorem 34. If
$$\lim_{n\to\infty} \frac{|(\mathcal{S})_{n-1}|}{|(\mathcal{S})_n|} \to 0$$
 then $\lim_{n\to\infty} \frac{|(\mathcal{B})_n|}{|(\mathcal{S})_n|} \to 1$.

Proof. From Table 1, $|(\mathcal{U})_n| = |(\mathcal{T})_{\leq n-1}|$, hence $|(\mathcal{U})_n| - |(\mathcal{U})_{n-1}| = |(\mathcal{S})_{n-1}|$. Thus,

$$|(\mathcal{B})_n| - |(\mathcal{B})_{n-1}| = (|(\mathcal{S})_n| - |(\mathcal{U})_n|) - (|(\mathcal{S})_{n-1}| - |(\mathcal{U})_{n-1}|)$$

$$= (|(\mathcal{S})_n| - |(\mathcal{S})_{n-1}|) - (|(\mathcal{U})_n| - |(\mathcal{U})_{n-1}|) = |(\mathcal{S})_n| - 2|(\mathcal{S})_{n-1}|.$$

Thus, by the hypothesis and the Sandwich Theorem.

$$1 \ge \lim_{n \to \infty} \frac{|(\mathcal{B})_n|}{|(\mathcal{S})_n|} \ge \lim_{n \to \infty} \frac{|(\mathcal{B})_n| - |(\mathcal{B})_{n-1}|}{|(\mathcal{S})_n|} = \lim_{n \to \infty} \frac{|(\mathcal{S})_n| - 2|(\mathcal{S})_{n-1}|}{|(\mathcal{S})_n|}$$
$$= \lim_{n \to \infty} 1 - 2\left(\frac{|(\mathcal{S})_{n-1}|}{|(\mathcal{S})_n|}\right) = 1.$$

6 Acknowledgments

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n	0	1	2	3	4	5	6	7	8	9	10	11
$ (\mathcal{S})_n $	1	1	2	4	9	21	56	164	557	2,223	10,766	64,956
$ (\mathcal{T})_{\leq n} $	1	2	4	8	17	38	94	258	815	3,038	13,804	78,760
$ (\mathcal{U})_n $	0	1	2	4	8	17	38	94	258	815	3,038	13,804
$ (\mathcal{B})_n $	1	0	0	0	1	4	18	70	299	1,408	7,728	51,152
$ \left(\mathcal{NG} ight)_n $	0	1	2	4	8	18	40	98	266	832	3,076	13,898
$ (\mathcal{PS})_n $	1	1	2	4	9	22	58	168	565	2,240	10,804	65,050

Table 2: The number of split graphs (total, balanced, unbalanced) and NG-graphs on n vertices.

n	4	5	6	7	8	9	10	11	12	13	14	15	16
ratio	.11	.19	.32	.42	.54	.63	.72	.79	.84	.89	.92	.94	.96

Table 3: The ratio of the number of balanced split graphs to split graphs on n vertices.

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