

# New self-dual additive $\mathbb{F}_4$ -codes constructed from circulant graphs

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August 14, 2018

## Abstract

In order to construct quantum  $[[n, 0, d]]$  codes for  $(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20)$ , we construct self-dual additive  $\mathbb{F}_4$ -codes of length  $n$  and minimum weight  $d$  from circulant graphs. The quantum codes with these parameters are constructed for the first time.

## 1 Introduction

Let  $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$  be the finite field with four elements, where  $\bar{\omega} = \omega^2 = \omega + 1$ . An *additive*  $\mathbb{F}_4$ -code of length  $n$  is an additive subgroup of  $\mathbb{F}_4^n$ . An element of  $C$  is called a *codeword* of  $C$ . An additive  $(n, 2^k)$   $\mathbb{F}_4$ -code is an additive  $\mathbb{F}_4$ -code of length  $n$  with  $2^k$  codewords. The (Hamming) weight of a codeword  $x$  of  $C$  is the number of non-zero components of  $x$ . The minimum non-zero weight of all codewords in  $C$  is called the *minimum weight* of  $C$ .

Let  $C$  be an additive  $\mathbb{F}_4$ -code of length  $n$ . The symplectic dual code  $C^*$  of  $C$  is defined as  $\{x \in \mathbb{F}_4^n \mid x * y = 0 \text{ for all } y \in C\}$  under the trace inner product:

$$x * y = \sum_{i=1}^n (x_i y_i^2 + x_i^2 y_i)$$

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for  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n) \in \mathbb{F}_4^n$ . An additive  $\mathbb{F}_4$ -code  $C$  is called (symplectic) *self-orthogonal* (resp. *self-dual*) if  $C \subset C^*$  (resp.  $C = C^*$ ).

Calderbank, Rains, Shor and Sloane [3] gave the following useful method for constructing quantum codes from self-orthogonal additive  $\mathbb{F}_4$ -codes (see [3] for more details on quantum codes). A self-orthogonal additive  $(n, 2^{n-k})$   $\mathbb{F}_4$ -code  $C$  such that there is no element of weight less than  $d$  in  $C^* \setminus C$ , gives a quantum  $[[n, k, d]]$  code, where  $k \neq 0$ . In addition, a self-dual additive  $\mathbb{F}_4$ -code of length  $n$  and minimum weight  $d$  gives a quantum  $[[n, 0, d]]$  code. Let  $d_{\max}(n, k)$  denote the maximum integer  $d$  such that a quantum  $[[n, k, d]]$  code exists. It is a fundamental problem to determine the value  $d_{\max}(n, k)$  for a given  $(n, k)$ . A table on  $d_{\max}(n, k)$  is given in [3, Table III] for  $n \leq 30$ , and an extended table is available online [5].

In this note, we construct self-dual additive  $\mathbb{F}_4$ -codes of length  $n$  and minimum weight  $d$  for

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), \\ (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20). \quad (1)$$

These codes are obtained from adjacency matrices of some circulant graphs. The above self-dual additive  $\mathbb{F}_4$ -codes allow us to construct quantum  $[[n, 0, d]]$  codes for the  $(n, d)$  given in (1). These quantum codes improve the previously known lower bounds on  $d_{\max}(n, 0)$  for the above  $n$ .

The data of these new quantum codes has already been included in [5]. All computer calculations in this note were performed using MAGMA [1].

## 2 Self-dual additive $\mathbb{F}_4$ -codes from circulant graphs

A *graph*  $\Gamma$  consists of a finite set  $V$  of vertices together with a set of edges, where an edge is a subset of  $V$  of cardinality 2. All graphs in this note are simple, that is, graphs are undirected without loops and multiple edges. The *adjacency matrix* of a graph  $\Gamma$  with  $V = \{x_1, x_2, \dots, x_v\}$  is a  $v \times v$  matrix  $A_\Gamma = (a_{ij})$ , where  $a_{ij} = a_{ji} = 1$  if  $\{x_i, x_j\}$  is an edge and  $a_{ij} = 0$  otherwise. Let  $\Gamma$  be a graph and let  $A_\Gamma$  be the adjacency matrix of  $\Gamma$ . Let  $C(\Gamma)$  denote the additive  $\mathbb{F}_4$ -code generated by the rows of  $A_\Gamma + \omega I$ , where  $I$  denotes the identity matrix. Then  $C(\Gamma)$  is a self-dual additive  $\mathbb{F}_4$ -code [4].

Two additive  $\mathbb{F}_4$ -codes  $C_1$  and  $C_2$  of length  $n$  are *equivalent* if there is a map from  $S_3^n \times S_n$  sending  $C_1$  onto  $C_2$ , where the symmetric group  $S_n$  acts on the set of the  $n$  coordinates and each copy of the symmetric group  $S_3$  permutes the non-zero elements  $1, \omega, \bar{\omega}$  of the field in the respective coordinate. For any self-dual additive  $\mathbb{F}_4$ -code  $C$ , it was shown in [4, Theorem 6] that there is a graph  $\Gamma$  such that  $C(\Gamma)$  is equivalent to  $C$ . Using this characterization, all self-dual additive  $\mathbb{F}_4$ -codes were classified for lengths up to 12 [4, Section 5].

An  $n \times n$  matrix is *circulant* if it has the following form:

$$M = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_n & r_1 & \cdots & r_{n-2} & r_{n-1} \\ r_{n-1} & r_n & \ddots & r_{n-3} & r_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ r_2 & r_3 & \cdots & r_n & r_1 \end{pmatrix}. \quad (2)$$

Trivially, the matrix  $M$  is fully determined by its first row  $(r_1, r_2, \dots, r_n)$ . A graph is called *circulant* if it has a circulant adjacency matrix. For a circulant adjacency matrix of the form (2), we have

$$r_1 = 0 \quad \text{and} \quad r_i = r_{n+2-i} \quad \text{for } i = 2, \dots, \lfloor n/2 \rfloor. \quad (3)$$

Circulant graphs and their applications have been widely studied (see [7] for a recent survey on this subject). For example, it is known that the number of non-isomorphic circulant graphs is known for orders up to 47 (see the sequence A049287 in [8]). In this note, we concentrate on self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma)$  generated by the rows of  $A_\Gamma + \omega I$ , where  $A_\Gamma$  are the adjacency matrices of circulant graphs  $\Gamma$ . These codes were studied, for example, in [6] and [9].

### 3 New self-dual additive $\mathbb{F}_4$ -codes and quantum codes from circulant graphs

#### 3.1 Lengths up to 50

Throughout this section, let  $\Gamma$  denote a circulant graph with adjacency matrix  $A_\Gamma$ . Let  $C(\Gamma)$  denote the self-dual additive  $\mathbb{F}_4$ -code generated by the

rows of  $A_\Gamma + \omega I$ . Let  $d_{\max}^\Gamma(n)$  denote the maximum integer  $d$  such that a self-dual additive  $\mathbb{F}_4$ -code  $C(\Gamma)$  of length  $n$  and minimum weight  $d$  exists. Varbanov [9] gave a classification of self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma)$  for lengths  $n = 13, 14, \dots, 29, 31, 32, 33$  and determined the values  $d_{\max}^\Gamma(n)$  for lengths up to 33.

Table 1: Self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma_n)$  of lengths  $n = 34, 35, \dots, 50$

$n$	$d_{\max}^\Gamma(n)$	Support of the first row of $A_{\Gamma_n}$	$d_{\max}(n, 0)$
34	10	2, 3, 6, 8, 9, 27, 28, 30, 33, 34	10–12
35	<i>10</i>	2, 4, 6, 7, 10, 27, 30, 31, 33, 35	11–13
36	<i>11</i>	2, 3, 4, 5, 7, 9, 13, 14, 24, 25, 29, 31, 33, 34, 35, 36	12–14
37	11	5, 6, 7, 9, 11, 12, 27, 28, 30, 32, 33, 34	11–14
38	12	2, 3, 5, 7, 10, 11, 20, 29, 30, 33, 35, 37, 38	12–14
39	11	2, 4, 5, 6, 7, 10, 11, 30, 31, 34, 35, 36, 37, 39	11–14
40	12	2, 3, 5, 8, 10, 21, 32, 34, 37, 39, 40	12–14
41	12	2, 3, 4, 5, 6, 10, 11, 13, 30, 32, 33, 37, 38, 39, 40, 41	12–15
42	12	2, 3, 13, 15, 16, 18, 21, 22, 23, 26, 28, 29, 31, 41, 42	12–16
43	<i>12</i>	3, 4, 7, 9, 10, 12, 33, 35, 36, 38, 41, 42	13–16
44	14	4, 5, 8, 10, 13, 17, 18, 21, 23, 25, 28, 29, 33, 36, 38, 41, 42	14–16
45	13	2, 4, 5, 9, 10, 12, 14, 15, 17, 18, 20, 27, 29, 30, 32, 33, 35, 37, 38, 42, 43, 45	13–16
46	14	4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 24, 29, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44	14–16
47	13	4, 8, 11, 13, 14, 15, 34, 35, 36, 38, 41, 45	13–17
48	14	3, 4, 5, 10, 12, 14, 15, 16, 25, 34, 35, 36, 38, 40, 45, 46, 47	14–18
49	13	4, 5, 7, 8, 9, 10, 13, 14, 37, 38, 41, 42, 43, 44, 46, 47	13–18
50	14	3, 7, 8, 9, 11, 12, 13, 17, 20, 22, 24, 25, 26, 27, 28, 30, 32, 35, 39, 40, 41, 43, 44, 45, 49	14–18

For lengths  $n = 13, 14, \dots, 50$ , by exhaustive search, we determined the largest minimum weights  $d_{\max}^\Gamma(n)$ . In Table 1, for lengths  $n = 34, 35, \dots, 50$ , we list  $d_{\max}^\Gamma(n)$  and an example of a self-dual additive  $\mathbb{F}_4$ -code  $C(\Gamma_n)$  having minimum weight  $d_{\max}^\Gamma(n)$ , where the support of the first row of the circulant adjacency matrix  $A_{\Gamma_n}$  is given. Our present state of knowledge about the upper bound  $d_{\max}(n, 0)$  on the minimum distance is also listed in the table. For most lengths, the self-dual additive  $\mathbb{F}_4$ -codes give quantum  $[[n, 0, d]]$  codes such that  $d = d_{\max}(n, 0)$  or  $d$  attains the currently known lower bound on  $d_{\max}(n, 0)$ ; three exceptions (lengths 35, 36 and 43) are typeset in *italics*.

Note that  $d_{\max}^\Gamma(36) = 11$ . For lengths 34, 35 and 36, self-dual additive

Table 2: Weight distribution of  $C(\Gamma_{36})$

$i$	$A_i$	$i$	$A_i$	$i$	$A_i$	$i$	$A_i$
0	1	17	16 145 280	24	5 144 050 296	31	3 388 554 144
11	1 584	18	51 147 440	25	7 408 053 504	32	1 588 252 581
12	9 936	19	145 391 760	26	9 402 473 952	33	577 571 712
13	52 992	20	370 815 624	27	10 446 604 880	34	152 925 552
14	265 392	21	847 669 248	28	10 073 332 800	35	26 213 616
15	1 168 032	22	1 733 647 968	29	8 336 897 280	36	2 179 688
16	4 578 786	23	3 165 414 336	30	5 836 058 352		

$\mathbb{F}_4$ -codes  $C(\Gamma)$  with minimum weight 10 were constructed in [9]. For length 36, we found a self-dual additive  $\mathbb{F}_4$ -code  $C(\Gamma_{36})$  of length 36 and minimum weight 11 (see Table 1). The weight distribution of the code  $C(\Gamma_{36})$  is listed in Table 2, where  $A_i$  denotes the number of codewords of weight  $i$ .

**Proposition 1.** *The largest minimum weight  $d_{\max}^\Gamma(36)$  among all self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma)$  of length 36 from circulant graphs is 11.*

A self-dual additive  $\mathbb{F}_4$ -code is called Type II if it is even. It is known that a Type II additive  $\mathbb{F}_4$ -code must have even length. A self-dual additive  $\mathbb{F}_4$ -code, which is not Type II, is called Type I. Although the following proposition is somewhat trivial, we give a proof for completeness.

**Proposition 2.** *Let  $C(\Gamma)$  be the self-dual additive  $\mathbb{F}_4$ -code of even length  $n$  generated by the rows of  $A_\Gamma + \omega I$ , where  $A_\Gamma$  is circulant. Let  $S$  be the support of the first row of  $A_\Gamma$ . Then  $C(\Gamma)$  is Type II if and only if  $n/2 + 1 \in S$ .*

*Proof.* It was shown in [4, Theorem 15] that the codes  $C(\Gamma)$  are Type II if and only if all the vertices of  $\Gamma$  have odd degree. For a circulant graph  $\Gamma$ , the degree of the vertices is constant and equals the size of the support  $S$  of the first row of  $A_\Gamma$ . From (3) it follows that the size of the support  $S$  is odd if and only if  $r_{n/2+1} = 1$ , i.e.,  $n/2 + 1 \in S$ .  $\square$

Note that (3) also implies that the size of the support  $S$  of the first row of  $A_\gamma$  is always even when  $n$  is odd, i.e., self-dual codes of odd length from circulant graphs cannot be Type II.

By Proposition 2, the codes  $C(\Gamma_n)$  ( $n = 38, 40, 42, 44, 46, 48, 50$ ) are Type II. In addition, the other codes in Table 1 are Type I. Let  $d_{\max, I}^\Gamma(n)$

denote the maximum integer  $d$  such that a Type I additive  $\mathbb{F}_4$ -code  $C(\Gamma)$  of length  $n$  and minimum weight  $d$  exists. By exhaustive search, we verified that  $d_{\max, I}^\Gamma(44) = d_{\max}^\Gamma(44) - 2$ ,  $d_{\max, I}^\Gamma(n) = d_{\max}^\Gamma(n) - 1$  ( $n = 38, 40, 46, 48$ ) and  $d_{\max, I}^\Gamma(n) = d_{\max}^\Gamma(n)$  ( $n = 42, 50$ ). For  $(n, d) = (42, 12)$  and  $(50, 14)$ , we list an example of Type I additive  $\mathbb{F}_4$ -code  $C(\Gamma'_n)$  of length  $n$  and minimum weight  $d$ , where the support of the first row of the circulant adjacency matrix  $A_{\Gamma'_n}$  is given in Table 3.

Table 3: Type I additive  $\mathbb{F}_4$ -codes  $C(\Gamma'_n)$  of lengths 42, 50

$n$	$d$	Support of the first row of $A_{\Gamma'_n}$
42	12	2, 3, 5, 6, 8, 11, 12, 13, 31, 32, 33, 36, 38, 39, 41, 42
50	14	5, 6, 7, 9, 10, 11, 12, 20, 32, 40, 41, 42, 43, 45, 46, 47

### 3.2 Sporadic lengths $n \geq 51$

For lengths  $n \geq 51$ , by non-exhaustive search, we tried to find self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma)$  with large minimum weight, where  $\Gamma$  is a circulant graph. By this method, we found new self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma_n)$  of length  $n$  and minimum weight  $d$  for

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), \\ (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).$$

For each self-dual additive  $\mathbb{F}_4$ -code  $C(\Gamma_n)$ , the support of the first row of the circulant adjacency matrix  $A_{\Gamma_n}$  is listed in Table 4. Additionally, for  $n = 51, \dots, 55, 59, 60, 64, 65, 66, 69, 72, \dots, 78, 81, 82, 84, 88, 94, 100$ , we found self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma_n)$  from circulant graphs matching the known lower bound on the minimum distance of quantum codes  $[[n, 0, d]]$ . For the remaining lengths, our non-exhaustive computer search failed to discover a self-dual additive  $\mathbb{F}_4$ -code from a circulant graph matching the known lower bound.

For the codes  $C(\Gamma_n)$  ( $n = 56, 57, 58, 63, 67, 70, 71, 79$ ), we give in Table 5 part of the weight distribution. Due to the computational complexity, we calculated the number  $A_i$  of codewords of weight  $i$  for only  $i = 15, 16, \dots, 19$ . As some basic properties of the graphs  $\Gamma_n$ , we give in Table 6 the valency

Table 4: New self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma_n)$

Code	Support of the first row of $A_{\Gamma_n}$
$C(\Gamma_{56})$	2, 3, 7, 8, 12, 14, 15, 16, 17, 20, 22, 26, 28, 30, 32, 36, 38, 41, 42, 43, 44, 46, 50, 51, 55, 56
$C(\Gamma_{57})$	7, 8, 10, 12, 17, 18, 22, 23, 24, 35, 36, 37, 41, 42, 47, 49, 51, 52
$C(\Gamma_{58})$	2, 3, 7, 10, 13, 14, 15, 17, 21, 25, 27, 29, 30, 31, 33, 35, 39, 43, 45, 46, 47, 50, 53, 57, 58
$C(\Gamma_{63})$	2, 5, 6, 9, 13, 14, 15, 16, 17, 19, 46, 48, 49, 50, 51, 52, 56, 59, 60, 63
$C(\Gamma_{67})$	4, 5, 6, 11, 12, 14, 15, 16, 17, 18, 21, 25, 26, 27, 28, 30, 39, 41, 42, 43, 44, 48, 51, 52, 53, 54, 55, 57, 58, 63, 64, 65
$C(\Gamma_{70})$	2, 6, 7, 8, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 28, 29, 30, 32, 33, 35, 36, 37, 39, 40, 42, 43, 44, 48, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 64, 65, 66, 70
$C(\Gamma_{71})$	2, 3, 5, 11, 12, 15, 17, 20, 23, 26, 27, 28, 31, 34, 35, 38, 39, 42, 45, 46, 47, 50, 53, 56, 58, 61, 62, 68, 70, 71
$C(\Gamma_{79})$	2, 4, 7, 10, 13, 15, 18, 19, 20, 21, 23, 24, 25, 29, 30, 31, 32, 35, 36, 37, 39, 42, 44, 45, 46, 49, 50, 51, 52, 56, 57, 58, 60, 61, 62, 63, 66, 68, 71, 74, 77, 79
$C(\Gamma_{83})$	3, 4, 5, 7, 9, 11, 14, 19, 20, 21, 22, 23, 24, 27, 28, 30, 31, 32, 33, 34, 36, 38, 41, 44, 47, 49, 51, 52, 53, 54, 55, 57, 58, 61, 62, 63, 64, 65, 66, 71, 74, 76, 78, 80, 81, 82
$C(\Gamma_{87})$	7, 11, 12, 13, 14, 15, 20, 23, 24, 25, 27, 28, 29, 30, 31, 34, 35, 37, 40, 41, 42, 47, 48, 49, 52, 54, 55, 58, 59, 60, 61, 62, 64, 65, 66, 69, 74, 75, 76, 77, 78, 82
$C(\Gamma_{89})$	3, 4, 7, 10, 14, 15, 18, 19, 21, 23, 25, 26, 30, 32, 34, 35, 37, 39, 40, 45, 46, 51, 52, 54, 56, 57, 59, 61, 65, 66, 68, 70, 72, 73, 76, 77, 81, 84, 87, 88
$C(\Gamma_{95})$	4, 5, 6, 11, 12, 14, 15, 18, 19, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 38, 40, 42, 43, 45, 47, 50, 52, 54, 55, 57, 59, 61, 62, 63, 64, 65, 66, 67, 69, 70, 71, 78, 79, 82, 83, 85, 86, 91, 92, 93

Table 5: Number  $A_i$  of codewords of weight  $i$  ( $i = 15, 16, \dots, 19$ )

Code	$d$	$A_{15}$	$A_{16}$	$A_{17}$	$A_{18}$	$A_{19}$
$C(\Gamma_{56})$	15	4 032	25 508	173 264	1 124 648	6 839 224
$C(\Gamma_{57})$	15	1 938	18 126	120 783	838 451	5 093 409
$C(\Gamma_{58})$	16		24 882	0	1 205 240	0
$C(\Gamma_{63})$	16		2 142	12 726	113 568	757 575
$C(\Gamma_{67})$	17			2 278	23 785	193 429
$C(\Gamma_{70})$	18				15 260	0
$C(\Gamma_{71})$	18				6 745	43 949
$C(\Gamma_{79})$	19					1 343

$k(\Gamma_n)$ , the diameter  $d(\Gamma_n)$ , the girth  $g(\Gamma_n)$ , the size  $\omega(\Gamma_n)$  of the maximum clique and the order  $|\text{Aut}(\Gamma_n)|$  of the automorphism group. With the exception of  $n = 53$ , the automorphism group is the dihedral group on  $n$  points of order  $2n$ . Note, however, that the notion of equivalence for graphs and codes are different, i. e., the graph invariants are not preserved with respect to code equivalence [2]. By Proposition 2, the codes  $C(\Gamma_{58})$  and  $C(\Gamma_{70})$  are Type II.

Finally, by the method in [3], the existence of our self-dual additive  $\mathbb{F}_4$ -codes  $C(\Gamma_n)$  yields the following:

**Theorem 3.** *There are a quantum  $[[n, 0, d]]$  codes for*

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), \\ (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).$$

The above quantum  $[[n, 0, d]]$  codes improve the previously known lower bounds on  $d_{\max}(n, 0)$  ( $n = 56, 57, 58, 63, 67, 70, 71, 79, 87, 89$ ). More precisely,

Table 6: Properties of the graphs  $\Gamma_n$ 

Graph	$d_{\min}(C(\Gamma_n))$	$k(\Gamma_n)$	$d(\Gamma_n)$	$g(\Gamma_n)$	$\omega(\Gamma_n)$	$ \text{Aut}(\Gamma_n) $
$\Gamma_{51}$	14	24	2	3	6	102
$\Gamma_{52}$	14	16	3	3	4	104
$\Gamma_{53}$	15	26	2	3	5	1378
$\Gamma_{54}$	16	29	2	3	8	108
$\Gamma_{55}$	14	14	3	3	4	110
$\Gamma_{56}$	15	26	2	3	19	112
$\Gamma_{57}$	15	18	2	3	5	114
$\Gamma_{58}$	16	25	2	3	7	116
$\Gamma_{59}$	15	30	2	3	8	118
$\Gamma_{60}$	16	31	2	3	6	120
$\Gamma_{63}$	16	20	2	3	5	126
$\Gamma_{64}$	16	43	2	3	12	128
$\Gamma_{65}$	16	28	2	3	6	130
$\Gamma_{66}$	16	33	2	3	6	132
$\Gamma_{67}$	17	32	2	3	6	134
$\Gamma_{69}$	17	38	2	3	7	138
$\Gamma_{70}$	18	45	2	3	10	140
$\Gamma_{71}$	18	30	2	3	6	142
$\Gamma_{72}$	18	27	2	3	6	144
$\Gamma_{73}$	18	40	2	3	8	146
$\Gamma_{74}$	18	32	2	3	6	148
$\Gamma_{75}$	18	34	2	3	6	150
$\Gamma_{76}$	18	37	2	3	8	152
$\Gamma_{77}$	18	48	2	3	10	154
$\Gamma_{78}$	18	35	2	3	7	156
$\Gamma_{79}$	19	42	2	3	8	158
$\Gamma_{81}$	19	40	2	3	7	162
$\Gamma_{82}$	20	43	2	3	7	164
$\Gamma_{83}$	20	46	2	3	9	166
$\Gamma_{84}$	20	25	2	3	6	168
$\Gamma_{87}$	20	42	2	3	7	174
$\Gamma_{88}$	20	37	2	3	6	176
$\Gamma_{89}$	21	40	2	3	6	178
$\Gamma_{94}$	20	44	2	3	10	188
$\Gamma_{95}$	20	50	2	3	7	190
$\Gamma_{100}$	20	48	2	3	7	200

we give our present state of knowledge about  $d_{\max}(n, 0)$  [5]:

$$\begin{aligned} 15 &\leq d_{\max}(56, 0) \leq 20, & 15 &\leq d_{\max}(57, 0) \leq 20, \\ 16 &\leq d_{\max}(58, 0) \leq 20, & 16 &\leq d_{\max}(63, 0) \leq 22, \\ 17 &\leq d_{\max}(67, 0) \leq 24, & 18 &\leq d_{\max}(70, 0) \leq 24, \\ 18 &\leq d_{\max}(71, 0) \leq 25, & 19 &\leq d_{\max}(79, 0) \leq 28, \\ 20 &\leq d_{\max}(83, 0) \leq 29, & 20 &\leq d_{\max}(87, 0) \leq 30, \\ 21 &\leq d_{\max}(89, 0) \leq 31, & 20 &\leq d_{\max}(95, 0) \leq 33. \end{aligned}$$

**Acknowledgment.** The authors would like to thank the anonymous referees for helpful comments. This work is supported by JSPS KAKENHI Grant Number 15H03633.

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