A NEW CYCLIC SIEVING PHENOMENON FOR CATALAN OBJECTS

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Abstract. Based on computational experiments, Jim Propp and Vic Reiner suspected that there might exist a sequence of combinatorial objects X_n, each carrying a natural action of the cyclic group C_{n-1} of order $n-1$ such that the triple $\left(X_n, C_{n-1}, \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q\right)$ exhibits the cyclic sieving phenomenon. We prove their suspicion right.

1. Introduction

1.1. **The Cyclic Sieving Phenomenon.** Reiner, Stanton and White have observed that the following situation often occurs: one has a combinatorial object X, a cyclic group C that acts on X and a "nice" polynomial $X(q)$ whose evaluations at $|C|$-th roots of unity encode the cardinalities of the fixed point sets of the elements of C acting on X. They termed this the cyclic sieving phenomenon.

Definition 1.1 ([RSW04])**.** *Let X be a finite set carrying an action of a cyclic group C and let $X(q)$ be a polynomial in q with nonnegative integer coefficients. Fix an isomorphism ω from C to the set of $|C|$-th roots of unity, that is an embedding $\omega : C \hookrightarrow \mathbb{C}^*$. We say that the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon (CSP) if*

$$|\{x \in X : c(x) = x\}| = X(q)_{q=\omega(c)} \text{ for every } c \in C.$$

In particular, if $(X, C, X(q))$ exhibits the CSP then $|X| = X(1)$. So $X(q)$ is a *q-analogue* of $|X|$.

1.2. **Catalan numbers.** One of the most famous number sequences in combinatorics is the sequence $1, 1, 2, 5, 14, 42, 132, \ldots$ of *Catalan numbers* given by the formula

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

A vast variety of combinatorial objects are counted by the Catalan number C_n, for example the set of triangulations of a convex $(n+2)$-gon and the set of noncrossing matchings of $\{1, 2, \ldots, 2n\}$. The (MacMahon) *q-Catalan number* $C_n(q)$ is the natural q-analogue of C_n, defined as

$$C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q,$$

where $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$, $[n]_q! := [1]_q[2]_q \cdots [n]_q$ and $\begin{bmatrix}n\\k\end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$. It is a polynomial in q with nonnegative integer coefficients.

The q-Catalan number has the distinction of occurring in two entirely different CSPs for Catalan objects:

Theorem 1.2 ([RSW04, Theorem 7.1])**.** *Let Δ_n be the set of triangulations of a convex $(n+2)$-gon and let C_{Δ_n} be the cyclic group of order $n+2$ acting on Δ_n by rotation. Then $(\Delta_n, C_{\Delta_n}, C_n(q))$ exhibits the cyclic sieving phenomenon.*

Theorem 1.3 ([PPR09, Theorems 1.4 and 1.5])**.** *Let M_n be the set of noncrossing matchings of $[2n] := \{1, 2, \dots, 2n\}$ and let C_{M_n} be the cyclic group of order $2n$ acting on M_n by rotation. Then $(M_n, C_{M_n}, C_n(q))$ exhibits the cyclic sieving phenomenon.*

Computational experiments by Jim Propp and Vic Reiner suggested that substituting an $(n-1)$-th root of unity into $C_n(q)$ always yields a positive integer. So they suspected that there might also be cyclic sieving phenomenon involving $C_n(q)$ and a cyclic group of order $n-1$. The main result of this note proves that their suspicion is correct.

Theorem 1.4. *For any $n > 0$, there exists an explicit set X_n that carries an action of the cyclic group C_{X_n} of order $n-1$ such that the triple $(X_n, C_{X_n}, C_n(q))$ exhibits the cyclic sieving phenomenon.*

2. Proof of Theorem 1.4

The first order of business is to define the set X_n. Call a subset of $[m]$ a *ball* if it has cardinality 1 and an *arc* if it has cardinality 2. Define a *$(1,2)$-configuration* on $[m]$ as a set of pairwise disjoint balls and arcs. Say that a $(1,2)$-configuration F has a *crossing* if it contains arcs $\{i_1, i_2\}$ and $\{j_1, j_2\}$ with $i_1 < j_1 < i_2 < j_2$. If F has no crossing it is called *noncrossing*.

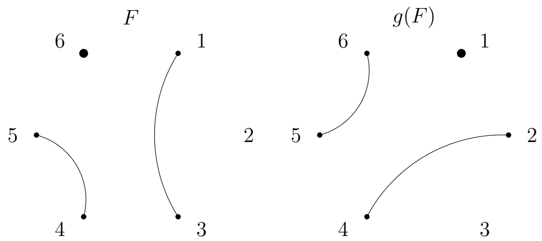

Figure 1. The noncrossing $(1,2)$-configuration $F = \{\{1,3\},\{4,5\},\{6\}\}$ of $[6]$ and its rotation $g(F)$.

For $n > 0$, define X_n to be the set of noncrossing $(1,2)$-configurations of $[n-1]$. This is a corrected variant of (e^8) in Stanley's Catalan addendum [Sta].

Theorem 2.1. *$|X_n| = C_n$ for all $n > 0$.*

Proof. To choose a noncrossing $(1,2)$-configuration F of $[n-1]$, first pick the number a of arcs in it. Then pick the subset A of $[n-1]$ to be covered by arcs in one of $\binom{n-1}{2a}$ ways. Then choose a noncrossing matching of A in one of $C_a = \frac{1}{a+1}\binom{2a}{a}$ ways. Finally choose the set of balls in F from $[n-1] \backslash A$ in one of 2^{n-1-2a} ways. Thus

$$|X_n| = \sum_{a \geq 0} \binom{n-1}{2a} \frac{1}{a+1} \binom{2a}{a} 2^{n-1-2a} = \frac{1}{n+1}\binom{2n}{n}.$$

The last equality can be proven in many ways, for example using "snake oil" [Wil06]. □

Define C_{X_n} as the cyclic group of order $n-1$ acting on $[n-1]$ by cyclically permuting its elements. The corresponding action of C_{X_n} on the set of $(1,2)$-configurations on $[n-1]$ preserves crossings, so it restricts to an action on X_n.

Proof of Theorem 1.4. We proceed by direct computation. Let

$$\begin{aligned} g : [n-1] &\to [n-1] \\ i &\mapsto i+1 \text{ if } i \neq n-1 \\ n-1 &\mapsto 1 \end{aligned}$$

be a generator of C_{X_n} and let $\omega : g^k \mapsto e^{\frac{2\pi i k}{n-1}}$ be an embedding $C_{X_n} \hookrightarrow \mathbb{C}^*$. In order to show that

$$|\{x \in X_n : g^k(x) = x\}| = C_n(q)_{q=e^{\frac{2\pi i k}{n-1}}} \text{ for every } k \tag{1}$$

we simply compute both sides. Without loss of generality, we may assume that k divides $n-1$, say $dk = n-1$.

First we compute the right-hand side of (1). If $d = 1$, it equals $C_n(1) = C_n$. If $d = 2$, it equals $\binom{n}{\frac{n-1}{2}}$ using $C_n(q) = \frac{1}{[n]_q}\left[{2n \atop n+1}\right]_q$ and [RSW04, Proposition 4.2 (iii)]. If $d \neq 1, 2$ it equals $\binom{2k}{k}$ using $C_n(q) = \frac{[2n]_q}{[n]_q[n+1]_q}\left[{2n-1 \atop n}\right]_q$ and [RSW04, Proposition 4.2 (iii)].

Next we compute the left-hand side of (1). To choose a noncrossing $(1,2)$-configuration F of $[n-1]$ that fixed by g^k, first pick the number a of points in $[k]$ that are covered by arcs of F. Then pick the subset of $[k]$ covered by arcs of F in one of $\binom{k}{a}$ ways. The g^k-invariance of F then determines the entire subset A of $[n-1]$ covered by arcs of F. In particular $|A| = da$. Next choose a g^k-invariant noncrossing matching of A. These are in natural bijection with the c^a-invariant noncrossing matchings of $[da]$ (where c is the generator of the natural cyclic action on $[da]$). So using Theorem 1.3 their number is $C_{\frac{da}{2}}(q)_{q=e^{\frac{2\pi i a}{da}}}$ (taken to be 0 if da is odd). Finally, choose the balls of F in $[k]$ in one of 2^{k-a} ways. By g^k-invariance these determine all the balls of F. Putting it all together we have

$$|\{x \in X_n : g^k(x) = x\}| = \sum_{a \geq 0} \binom{k}{a} C_{\frac{da}{2}}(q)_{q=e^{\frac{2\pi i a}{da}}} 2^{k-a} \tag{2}$$

If $d = 1$, then

$$|\{x \in X_n : g^k(x) = x\}| = \sum_{a \geq 0} \binom{n-1}{2a} \frac{1}{a+1}\binom{2a}{a} 2^{n-1-2a} = \frac{1}{n+1}\binom{2n}{n}$$

as in Theorem 2.1.

Now consider the case $d > 1$. If $2 \mid a$, then

$$C_{\frac{da}{2}}(q)_{q=e^{\frac{2\pi i a}{da}}} = \binom{a}{\frac{a}{2}}$$

using [RSW04, Proposition 4.2 (ii)]. If $2 \nmid a$, then using $\left[{2n \atop n}\right]_q - \left[{2n \atop n+1}\right]_q = q^n C_n(q)$ and [RSW04, Proposition 4.2 (ii)] gives

$$C_{\frac{da}{2}}(q)_{q=e^{\frac{2\pi i a}{da}}} = \binom{a}{\frac{a-1}{2}} \text{ if } d = 2,$$

and

$$C_{\frac{da}{2}}(q)_{q=e^{\frac{2\pi i a}{da}}} = 0 \text{ if } d > 2.$$

So we calculate that for $d=2$ we have

$$\begin{aligned}&|\{x\in X_n : g^k(x)=x\}|\\&=\sum_{a\geq 0}\binom{\frac{n-1}{2}}{2a}\binom{2a}{a}2^{\frac{n-1}{2}-2a}+\sum_{a\geq 0}\binom{\frac{n-1}{2}}{2a+1}\binom{2a+1}{a}2^{\frac{n-1}{2}-2a-1}\\&=\binom{n}{\frac{n-1}{2}}.\end{aligned}$$

For $d>2$ we have

$$|\{x\in X_n : g^k(x)=x\}|=\sum_{a\geq 0}\binom{k}{2a}\binom{2a}{a}2^{k-2a}=\binom{2k}{k}$$

as required. □

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