

RESOLUTION OF INDECOMPOSABLE INTEGRAL FLOWS ON SIGNED GRAPHS

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ABSTRACT. It is well known that each nonnegative integral flow on a graph can be decomposed into a sum of nonnegative graphic circuit flows, which cannot be further decomposed into nonnegative integral sub-flows. This is equivalent to saying that the indecomposable flows on graphs are those graphic circuit flows. Turning from graphs to signed graphs, the indecomposable flows are much richer than those of unsigned graphs. This paper gives a complete description of indecomposable flows on signed graphs from the viewpoint of resolution of singularities by means of double covering graphs.

1. INTRODUCTION

A *signed graph* is a graph in which each edge is given either a positive or a negative sign. A real or integral flow (or “circulation”) on an ordinary unsigned graph is a real- or integer-valued function on (oriented) edges such that the net inflow to each vertex is zero. Analogously, a *real flow* on a signed graph is a real-valued function on (oriented) signed edges such that the net inflow to each vertex is zero, and an *integral flow*, a concept introduced by Bouchet [4], is a flow whose values are integers. The theory of flows on ordinary graphs is the specialization of the signed-graph theory to the case in which all edges are positive.

There are many reasons to be interested in integral flows on graphs; important ones are their connection to integer programming through network optimization and their relationship to graph structure through the analysis of *conformally indecomposable flows*, that is, integral flows that cannot be decomposed as the sum of two integral flows whose flow values have the same sign on each edge (both ≥ 0 or both ≤ 0). It is well known, and an important observation in the theory of integral network flows, that the indecomposable flows are identical to the *circuit flows*, which are flows on circuits of the graphic matroid and which (in a suitable orientation of the graph) have value 1 on the edges of a graph circuit (i.e., a connected 2-regular subgraph) and 0 elsewhere. The extension of the theory of indecomposable integral flows to signed graphs by Chen and Wang [7], carried out by an algorithmic method, led to the remarkable discovery that, besides the anticipated circuit flows, which are already more complicated in signed graphs than in ordinary graphs, there are many “strange” indecomposable flows with elaborate self-intersection structure not describable by circuits of the signed graph. Chen and Wang obtained a classification of indecomposable flows by means of their algorithm.

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The present paper, by contrast, has a structural approach. We characterize indecomposable flows by the method of covering graphs, lifting each vertex and each edge of a signed graph to two vertices and two edges of a double covering graph. The strange indecomposable flows are regarded as singular phenomena, which we resolve by lifting them (blowing up repeated vertices and overlapping edges) to ordinary circuit flows in the covering graph. In comparison to the algorithmic approach in [7], the present paper hints at a connection (at least conceptually) between graph theory and resolution of singularities through covering spaces, as we think of lifting as a combinatorial analogue of resolution of singularities in algebraic geometry. We believe this connection may be useful for studying gain graphs [12], which are more complicated than signed graphs.

The resolution process leads, via “sesqui-Eulerian circle-trees” (Definition 4.1), to the following half-integral conformal decomposition (see Theorem 4.9(d)): *Every nonzero integral flow on a signed graph can be conformally decomposed into a half-integral positive linear combination of signed-graphic circuit flows.* (For this purpose the half-integers include the integers.)

2. GRAPHS AND SIGNED GRAPHS

2.1. Graphs.

A *graph* G is a pair consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$, such that each edge $x \in E(G)$ is associated with a multiset $\text{End}(x)$ of two vertices, called the endpoints of x . (We use letters e, x, y, z for edges.) The edge x is called a *link* if the two vertices of $\text{End}(x)$ are distinct and is called a *loop* if the two vertices are identical. Let x be an edge with $\text{End}(x) = \{u, v\}$; we say that x is *incident* with u and v , and that u and v are *adjacent* by x .

An incidence can be treated as an object, denoted by (u, x) (which we think of as the *end* of x at u); thus the edge x has the two incidences, or *ends*, (u, x) and (v, x) . Although a loop x (where $u = v$) has the same endpoint twice, we treat its two ends as two distinct objects; this is necessary in order to define orientation of signed edges. When occasionally the notation must distinguish the two ends of a loop we write them as $(u, x), (v, x)$ as if x were a link. For a vertex v we define

$$\text{End}(v) = \{\text{edge ends } (v, e) \text{ incident with } v\};$$

in particular a loop at v has two distinct ends in $\text{End}(v)$.

A *cut-edge* is an edge whose deletion increases the number of connected components. A *cut-vertex* is a vertex whose deletion, together with all incident edges, increases the number of components, or that is incident with a loop and at least one other edge. A *circle* (also known as a circuit, cycle, or polygon) is a subgraph that is connected and regular of degree 2, or the edge set of such a subgraph. The collection of circles of a graph G forms the circuit system of a matroid, known as the *graphic matroid*, on the edge set of G .

A *block* is a maximal connected subgraph without cut-vertices. Thus, loops, cut-edges, and isolated vertices are blocks. We call blocks *adjacent* if they have a common vertex (which is necessarily a cut-vertex). An *end block* is a block adjacent to exactly one other block. A *circle block* is a block that is a circle.

A *walk* of length n in a graph is a sequence of vertices and edges,

$$W = v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

such that $\text{End}(x_i) = \{u_{i-1}, u_i\}$. The *initial vertex* is v_0 , v_n is the *terminal vertex*, and v_1, \dots, v_{n-1} are *internal vertices* of W . We write

$$W^{-1} = v_n e_n v_{n-1} \cdots e_2 v_1 e_1 v_0$$

for the same walk in the reverse direction; its initial vertex is v_n and its terminal vertex is v_0 . A *subwalk* of W is a subsequence of the form $v_i e_{i+1} v_{i+1} e_{i+2} \cdots v_{k-1} e_k v_k$. A walk is *closed* if $n \geq 1$ and $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if it has no repeating edges, and an *open path* if it has no repeating vertices (and consequently no repeating edges), and a *closed path* if it has no repeating vertices except that $v_0 = v_n$. The graph of a closed path is a circle. (The difference is that a closed path has an initial and terminal vertex and a direction.) Note that every walk, including a closed walk, has an initial vertex and a terminal vertex, which are identical if the walk is closed.

2.2. Signed Graphs.

A *signed graph* $\Sigma = (G, \sigma)$ consists of an unsigned graph G together with a *sign function* $\sigma : E(G) \rightarrow \{+1, -1\}$. We usually write $V(G)$ and $E(G)$ as $V(\Sigma)$ and $E(\Sigma)$, respectively. Subgraphs of Σ inherit the edge signs from Σ .

The sign of a walk $W = v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$ is the product

$$\sigma(W) = \prod_{i=1}^n \sigma(e_i).$$

In particular, the sign of a circle is the product of the signs of its edges. A subgraph (or its edge set) is *balanced* if every circle in it has positive sign.

A *signed-graph circuit* is a subgraph (or its edge set) of the following three types:

- (1) A positive circle, said to be of *Type I*.
- (2) A pair of negative circles whose intersection is a single vertex, said to be of *Type II*.
- (3) A pair of vertex-disjoint negative circles together with a path of positive length that connects the two circles and is internally disjoint from the two circles, said to be of *Type III*.

The connecting path in Type III and the common vertex in Type II are called the *circuit path* of the circuit.

The circuits of a signed graph Σ form the circuit system of a matroid on the edge set of Σ [11], the *frame matroid* of the signed graph Σ ; such a matroid is called a *signed-graphic matroid*. (Type II and Type III circuits are named contrabalanced tight handcuffs and contrabalanced loose handcuffs respectively by Zaslavsky [11]. We do not use these names here.) An ordinary unsigned graph is viewed as a signed graph whose edges are all positive; so all of its circles have positive sign and the frame matroid of an unsigned graph coincides with the graphic matroid.

2.3. Orientation.

A *bidirection* of a graph (a concept introduced by Edmonds [8]) is a function ω from the set of all edge ends to the sign group, $\{-1, +1\}$. We view a positive value $\omega(u, e)$ as denoting an arrow at the end (u, e) directed along the edge e toward the endpoint u , and a negative value as an arrow directed away from the endpoint. Recall that we treat a loop $e = uv$ (with $u = v$) as having distinguishable ends (u, e) and (v, e) .

An *orientation* of a signed graph Σ [13] is a bidirection ω on its underlying graph such that for each edge e , with endpoints u and v ,

$$\sigma(e) = -\omega(u, e)\omega(v, e).$$

So a positive edge e must have two arrows in the same direction along e , indicating a direction of e as in an ordinary directed graph. A negative edge e has two opposite arrows, which both point toward or both point away from its endpoints. We let (Σ, ω) denote an oriented signed graph throughout.

A *sink* in (Σ, ω) is a vertex v at which all edges point toward v , that is, $\omega(v, e) = +1$ for all edges e at v . Conversely, a *source* is a vertex v at which all edges at v point away from v , that is, $\omega(v, e) = -1$ for all edges e at v .

Two oriented edges $e_1, e_2 \in E$ having a common endpoint v , with the orientations $\omega(v, e_1)$ and $\omega(v, e_2)$, are *coherent at v* if

$$\omega(v, e_1)\omega(v, e_2) = -1, \quad \text{i.e.,} \quad \omega(v, e_1) + \omega(v, e_2) = 0.$$

This means that e_1 and e_2 have a common direction (locally) at their common endpoint v . A walk $W = v_0e_1v_1e_2 \cdots v_{n-1}e_nv_n$ in (Σ, ω) is *coherent at v_i* when e_i and e_{i+1} are coherent at v_i ; that is, when

$$\omega(v_i, e_i)\omega(v_i, e_{i+1}) = -1, \quad \text{i.e.,} \quad \omega(v_i, e_i) + \omega(v_i, e_{i+1}) = 0;$$

and similarly a closed walk is *coherent at v_0* ($= v_n$) when

$$\omega(v_0, e_1)\omega(v_n, e_n) = -1, \quad \text{i.e.,} \quad \omega(v_0, e_1) + \omega(v_n, e_n) = 0.$$

The walk W is a *coherent walk* when it is coherent at its internal vertices and, if it is closed, also coherent at the initial and terminal vertex. (Coherence is meaningless at the initial and terminal vertices of an open walk.) A simple fact is:

Lemma 2.1. *In an oriented signed graph, the sign of a walk of positive length equals $(-1)^{l+1}\omega(v_0, e_1)\omega(v_n, e_n)$, where l is the number of incoherent internal vertices. If the walk is closed, its sign equals $(-1)^k$, where k is the total number of times that the walk is incoherent at vertices, including the initial and terminal vertex (which is counted as a single vertex).*

Proof. We perform a short calculation. Let $W = v_0e_1v_1 \cdots e_nv_n$ be the walk. Its sign is

$$\begin{aligned} \sigma(W) &= \prod_{i=1}^n \sigma(e_i) = \prod_{i=1}^n (-\omega(v_{i-1}, e_i)\omega(v_i, e_i)) \\ &= -\omega(v_0, e_1)\omega(v_n, e_n) \prod_{i=1}^{n-1} (-\omega(v_i, e_i)\omega(v_i, e_{i+1})) \\ &= -(-1)^l \omega(v_0, e_1)\omega(v_n, e_n) \\ &= (-1)^k \quad \text{if } W \text{ is closed.} \quad \square \end{aligned}$$

A *direction* of W is an assignment ω_W to each edge e_i in W of an orientation that is coherent at all internal vertices. (The walk orientation ω_W is separate from the orientation ω of Σ .) Every walk of positive length has exactly two directions, opposite to each other. A *directed walk* (W, ω_W) is a walk W with a direction ω_W .

Let $S \subseteq E$ be an edge set. The *reorientation* of Σ by S is the orientation ω_S obtained from ω by reversing the orientations of the edges in S and keeping the orientations of edges outside S unchanged. Thus ω_S is given by

$$\omega_S(v, e) = \begin{cases} -\omega(v, e) & \text{if } e \in S, \\ \omega(v, e) & \text{if } e \notin S. \end{cases}$$

Let ω_i be orientations on subgraphs Σ_i of Σ , $i = 1, 2$. The *coupling* of ω_1 and ω_2 is a function $[\omega_1, \omega_2] : E(\Sigma) \rightarrow \{-1, 0, +1\}$, defined for each edge e (having endpoint v) by

$$[\omega_1, \omega_2](e) = \begin{cases} 1 & \text{if } e \in E(\Sigma_1) \cap E(\Sigma_2), \omega_1(v, e) = \omega_2(v, e), \\ -1 & \text{if } e \in E(\Sigma_1) \cap E(\Sigma_2), \omega_1(v, e) \neq \omega_2(v, e), \\ 0 & \text{otherwise.} \end{cases}$$

(The definition is independent of which endpoint v is.) One may extend ω_i to Σ by requiring $\omega_i(v, e) = 0$ whenever the edge e is not incident with the vertex v in Σ_i . We always assume this extension automatically. Then alternatively,

$$[\omega_1, \omega_2](e) = \omega_1(v, e) \omega_2(v, e).$$

2.4. The Double Covering Graph.

2.4.1. *Defining the covering graph.* The *double covering graph* of Σ is an unsigned graph $\tilde{\Sigma}$ whose vertex and edge sets are

$$V(\tilde{\Sigma}) = V(\Sigma) \times \{+, -\} \quad \text{and} \quad E(\tilde{\Sigma}) = E(\Sigma) \times \{+, -\},$$

with adjacency defined as follows: If vertices $u, v \in V(\Sigma)$ are adjacent by an edge $e \in E(\Sigma)$, then the vertices (u, α) and $(v, \alpha \sigma(e))$ in $V(\tilde{\Sigma})$ are adjacent by an edge in $E(\tilde{\Sigma})$, and the vertices $(u, -\alpha)$ and $(v, -\alpha \sigma(x))$ in $V(\tilde{\Sigma})$ are adjacent by another edge in $E(\tilde{\Sigma})$. We denote these two edges by \tilde{e} and \tilde{e}^* (the signs on edges in $E(\tilde{\Sigma})$ are not edge signs; they are a notational convenience to ensure that $E(\tilde{\Sigma})$ contains two copies of each edge of Σ). For simplicity, we write $u^\alpha = (u, \alpha)$. When e is a negative loop at its unique endpoint v , the edges \tilde{e} and \tilde{e}^* are two parallel edges in $\tilde{\Sigma}$ with the endpoints v^+ and v^- .

We may think of $V(\tilde{\Sigma})$ as having two levels:

$$\begin{aligned} V^+ &= \{v^+ : v \in V(\Sigma)\} \quad (\text{the positive level}), \\ V^- &= \{v^- : v \in V(\Sigma)\} \quad (\text{the negative level}). \end{aligned}$$

A positive edge is lifted to two edges, one inside the positive level and the other inside the negative level; a negative edge is lifted to two edges crossing between the two levels. It is therefore impossible to lift all edges of Σ to the same level when Σ is unbalanced.

The asterisk marks a canonical involutory, fixed-point-free graph automorphism $*$ of $\tilde{\Sigma}$, defined by

$$(v^\alpha)^* = v^{-\alpha}, \quad (\tilde{e})^* = \tilde{e}^*, \quad (\tilde{e}^*)^* = \tilde{e}.$$

There is also a canonical graph homomorphism $\pi : \tilde{\Sigma} \rightarrow \Sigma$, called the *projection* of $\tilde{\Sigma}$ to Σ , which is a pair of functions $\pi_V : V(\tilde{\Sigma}) \rightarrow V(\Sigma)$ and $\pi_E : E(\tilde{\Sigma}) \rightarrow E(\Sigma)$, defined respectively by

$$\pi_V(v^\alpha) = v \quad \text{and} \quad \pi_E(\tilde{e}) = e.$$

Usually, we write π_V and π_E simply as π .

Often it is convenient to denote \tilde{e} and \tilde{e}^* by e^β and $e^{-\beta}$, respectively, for some (arbitrary) choice of $\beta \in \{+, -\}$. If u and v are the endpoints of e , then

$$e^\beta = u^\alpha v^{\alpha\sigma(e)}, \quad e^{-\beta} = u^{-\alpha} v^{-\alpha\sigma(e)}, \quad \text{for some } \alpha \in \{+, -\}.$$

In the symbol e^β , β is not related to the sign of the edge e in Σ . (The choice of whether \tilde{e} is called e^+ or e^- does not change the double covering graph; it is only a choice of names for edges.)

2.4.2. *Orienting the double covering graph.* An orientation ω on Σ lifts to an orientation $\tilde{\omega}$ on $\tilde{\Sigma}$, called the *lift* of ω . Let $e \in E(\Sigma)$ be an edge incident with a vertex $v \in V(\Sigma)$, and let e be lifted to an edge $e^\beta \in E(\tilde{\Sigma})$ incident with a vertex $v^\alpha \in V(\tilde{\Sigma})$. Define

$$\tilde{\omega}(v^\alpha, e^\beta) = \alpha \omega(v, e). \quad (2.1)$$

Since $v^{-\alpha}$ is an endpoint of the lifted edge $e^{-\beta}$, then by definition

$$\tilde{\omega}(v^{-\alpha}, e^{-\beta}) = -\alpha \omega(v, e).$$

The two arrows on each lifted edge e^β are directed the same way along the edge, regardless of the sign $\sigma(e)$. In fact, for each edge e with endpoints u and v (possibly $u = v$), we have

$$\tilde{\omega}(u^\alpha, e^\beta) \tilde{\omega}(v^{\alpha\sigma(e)}, e^\beta) = \alpha \omega(u, e) \alpha \sigma(e) \omega(v, e) = -1.$$

This means that the two arrows on e^β have the same direction, as in Figure 1 for a link and Figure 2 for a loop. So $(\tilde{\Sigma}, \tilde{\omega})$ is an ordinary oriented unsigned graph.

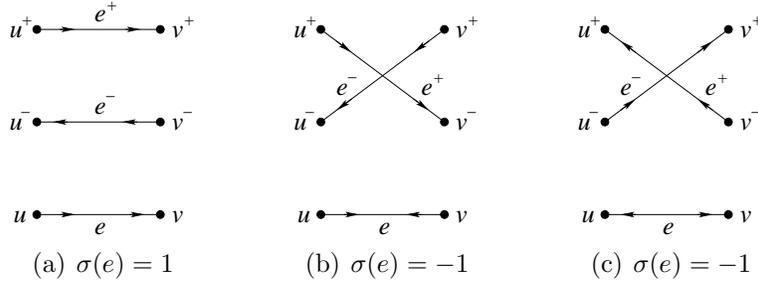


FIGURE 1. Lifting of a link and its orientation.

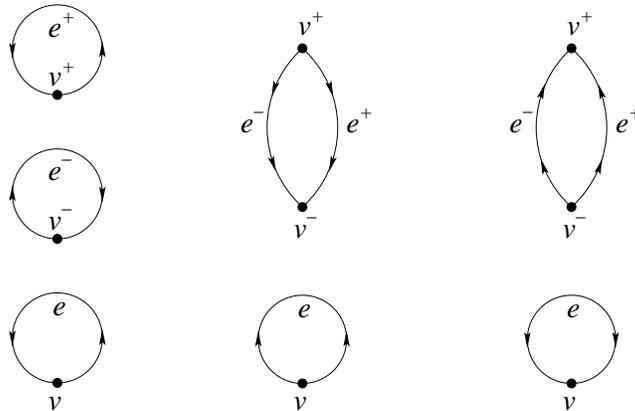


FIGURE 2. Lifting of a loop and its orientation.

The projection π maps an edge e^β with orientation $\tilde{\omega}(v^\alpha, e^\beta)$ in $\tilde{\Sigma}$ to the edge e with orientation $\omega(v, e) := \alpha\tilde{\omega}(v^\alpha, e^\beta)$.

Lifting of orientations preserves coherence. If e_1 and e_2 are edges with a common endpoint v , with the orientations $\omega(v, e_1)$ and $\omega(v, e_2)$, then the concatenated lift $\tilde{e}_1 v^\alpha \tilde{e}_2$ of $e_1 v e_2$, with the orientations $\tilde{\omega}(v^\alpha, \tilde{e}_1)$ and $\tilde{\omega}(v^\alpha, \tilde{e}_2)$, is coherent at v^α if and only if $e_1 v e_2$ is coherent at v . This fact follows from

$$\tilde{\omega}(v^\alpha, \tilde{e}_1)\tilde{\omega}(v^\alpha, \tilde{e}_2) = \alpha\omega(v, e_1) \cdot \alpha\omega(v, e_2) = \omega(v, e_1)\omega(v, e_2).$$

Let ω_i be orientations on subgraphs Σ_i of Σ , $i = 1, 2$. The lifted graphs $\tilde{\Sigma}_i$ are subgraphs of $\tilde{\Sigma}$, and the $(\tilde{\Sigma}_i, \tilde{\omega}_i)$ are subgraphs of the oriented graph $(\tilde{\Sigma}, \tilde{\omega})$. Moreover, lifting of orientations preserves the coupling, that is,

$$[\tilde{\omega}_1, \tilde{\omega}_2](e^\beta) = [\omega_1, \omega_2](e) \quad (2.2)$$

for each lift e^β of an edge e in $E(\Sigma)$. Indeed,

$$[\tilde{\omega}_1, \tilde{\omega}_2](e^\beta) = \tilde{\omega}_1(v^\alpha, e^\beta)\tilde{\omega}_2(v^\alpha, e^\beta) = \alpha\omega_1(v, e) \cdot \alpha\omega_2(v, e) = [\omega_1, \omega_2](e).$$

Let W be a walk in Σ of length n with the vertex-edge sequence $v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$ and a direction ω_W . We may lift W to a walk \tilde{W} in $\tilde{\Sigma}$ as follows: Select an initial vertex v^{α_0} ; define

$$\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \tilde{e}_2 \cdots v_{n-1}^{\alpha_{n-1}} \tilde{e}_n v_n^{\alpha_n}, \quad \text{where } \alpha_i = \alpha_{i-1} \sigma(e_i), \quad \tilde{e}_i = v_{i-1}^{\alpha_{i-1}} v_i^{\alpha_i}. \quad (2.3)$$

We call \tilde{W} a *lift* of W . A lift is a *resolution* of W if \tilde{W} is an open or closed path in $\tilde{\Sigma}$. There are exactly two lifts of W since there are exactly two choices for α_0 . Moreover, the orientation $\tilde{\omega}_W$ on the edges in \tilde{W} lifted from ω_W by (2.1) forms a direction of \tilde{W} . Thus (W, ω_W) lifts to exactly two directed walks, $(\tilde{W}, \tilde{\omega}_W)$ and $(\tilde{W}^*, \tilde{\omega}_W^*)$, where

$$\begin{aligned} \tilde{W}^* &= v_0^{-\alpha_0} \tilde{e}_1^* v_1^{-\alpha_1} \tilde{e}_2^* \cdots v_{n-1}^{-\alpha_{n-1}} \tilde{e}_n^* v_n^{-\alpha_n}, \\ \tilde{\omega}_W^*(v_i^{-\alpha_i}, e_i) &= -\alpha_i \omega_W(v_i, e_i), \quad 1 \leq i \leq n. \end{aligned}$$

We call W the *projection* of \tilde{W} and \tilde{W}^* and we write $W = \pi(\tilde{W}) = \pi(\tilde{W}^*)$.

Lemma 2.2. (a) *The projection $\pi : \tilde{\Sigma} \rightarrow \Sigma$ induces an incidence-preserving bijection from $\text{End}(v^\alpha)$ to $\text{End}(v)$ for each $v \in V(\Sigma)$ and $\alpha \in \{+, -\}$.*

(b) *The projections of directed walks in $\tilde{\Sigma}$ are directed walks in Σ . The projections of closed walks are closed.*

(c) *Let W be a closed walk in Σ with initial and terminal vertices v_0 and v_n , and let ω_W be a direction of W . Let $(\tilde{W}, \tilde{\omega}_W)$ be a lift of the directed walk (W, ω_W) . If W is positive, then $(\tilde{W}, \tilde{\omega}_W)$ is a directed closed walk. If W is negative, then $(\tilde{W}, \tilde{\omega}_W)$ is a directed open walk.*

Proof. Parts (a) and (b) are by definition of the projection.

For part (c), let $W = v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$ be a walk in Σ , lifted to a walk $\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \cdots \tilde{e}_n v_n^{\alpha_n}$ in $\tilde{\Sigma}$. Since $\sigma(e_i) = \alpha_{i-1} \alpha_i$ by (2.3),

$$\sigma(W) = \prod_{i=1}^n \sigma(e_i) = \prod_{i=1}^n \alpha_{i-1} \alpha_i = \alpha_0 \alpha_n.$$

Thus, $\sigma(W) = +1$ if and only if $\alpha_n = \alpha_0$. It follows that the walk \widetilde{W} is closed if and only if W has positive sign. \square

3. FLOWS

3.1. Flows on Signed Graphs.

The *incidence matrix* [14] of an oriented signed graph (Σ, ω) is the $V \times E$ matrix $\mathbf{M} = \mathbf{M}(\Sigma, \omega) = [\mathbf{m}(v, e)]$, where \mathbf{m} is a function $\mathbf{m} : V \times E \rightarrow \mathbb{Z}$ defined by

$$\mathbf{m}(v, e) = \sum_{v \in \text{End}(e)} \omega(v, e) = \begin{cases} \omega(v, e) & \text{if } e \text{ is a link,} \\ 2\omega(v, e) & \text{if } e \text{ is a negative loop,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

When e is a loop with $\text{End}(e) = \{v, v\}$ we add $\omega(v, e)$ for each end of e , so the sum is 0 when e is a positive loop and is $2\omega(v, e)$ when e is a negative loop.

An *integral flow* on an oriented signed graph (Σ, ω) is a function $f : E(\Sigma) \rightarrow \mathbb{Z}$ which is *conservative* at every vertex, meaning that the net contribution to each vertex is zero. The *boundary operator* of (Σ, ω) , $\partial : \mathbb{Z}^{E(\Sigma)} \rightarrow \mathbb{Z}^{V(\Sigma)}$, is defined by

$$\partial f(v) = \sum_{e \in E} \mathbf{m}(v, e) f(e) = \sum_{(v, e) \in \text{End}(v)} \omega(v, e) f(e). \quad (3.2)$$

Thus a function $f : E(\Sigma) \rightarrow \mathbb{Z}$ is a flow if and only if ∂f is identically zero.

The set of all integral flows on (Σ, ω) forms a \mathbb{Z} -module, called the *flow lattice* by Chen and Wang, who developed its basic theory in [5]. One can define flows with values in an arbitrary abelian group, for example, the additive group of real numbers and finitely generated abelian groups. Many of the following remarks are applicable for such flows. We omit the word “integral” when mentioning integral flows.

The theory of flows on signed graphs depends essentially on the graph and the sign function but not on the orientation, since only the notation changes when edges are reoriented. Specifically, a flow f on Σ with respect to an orientation ω represents the same flow on Σ as $[\omega, \rho]f$ on Σ with respect to another orientation ρ . (Therefore it is correct to speak of “flows on signed graphs”.)

The *support* of a function $f : E(\Sigma) \rightarrow \mathbb{Z}$ is the set of edges e such that $f(e) \neq 0$; it is denoted by $\text{supp } f$. We denote by $\Sigma(f)$ the subgraph of Σ whose edge set is $\text{supp } f$ and whose vertex set consists of vertices incident with edges in $\text{supp } f$. The *zero flow* is the flow that is zero on all edges. Flows other than the zero flow are referred to as *nonzero flows*. A *circuit flow* of a signed graph (as defined in [5]) is a flow whose support is a signed-graph circuit, having values ± 1 on the edges of the circles and ± 2 on the edges of the circuit path (for Type III circuits). See Figure 3 for circuit flows of Type II and Type III.

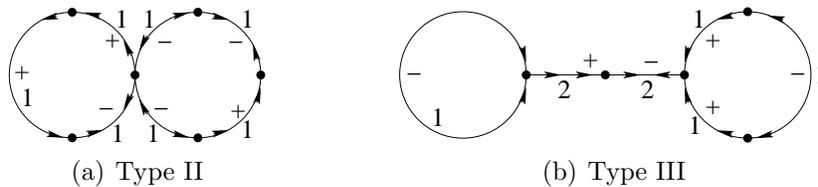


FIGURE 3. Signed-graph circuit flows of Types II and III.

An integral flow f_1 *conforms to the sign pattern of f* if $\text{supp } f_1 \subseteq \text{supp } f$ and $f_1(e)$ has the same sign as $f(e)$ for all edges e in $\text{supp } f_1$.

An integral flow f on (Σ, ω) lifts to a flow on the oriented double covering graph $(\tilde{\Sigma}, \tilde{\omega})$, possibly in more than one way. The best way to see existence of a lift is through the correspondence between integral flows and walks (when $\Sigma(f)$ is connected).

A directed closed, positive walk (W, ω_W) on (Σ, ω) corresponds to a unique integral flow $f_{(W, \omega_W)}$, defined by

$$f_{(W, \omega_W)}(e) = \sum_{e_i \in W, e_i = e} [\omega, \omega_W](e_i), \quad (3.3)$$

where W is viewed as a multiset $\{e_1, \dots, e_n\}$ of edges if $W = v_0 e_1 v_1 \dots e_n v_n$ (see [5]). Clearly, $f_{(W, \omega_W)}(e)$ is the number of times W traverses the edge e with ω_W and ω agreeing, minus the number of times W traverses e while ω_W disagrees with ω . In case the direction ω_W is the same as ω restricted to W , we simply write $f_{(W, \omega_W)}$ as f_W and we have

$$f_W(e) = |\{e_i \in W : e_i = e\}| \quad (\text{as a multiset}).$$

To see why $f_{(W, \omega_W)}$ is a flow, consider the contribution to $f_{(W, \omega_W)}$ of a pair of consecutive edges, $e_i v_i e_{i+1}$, at the intervening vertex v_i . Since (W, ω_W) is coherent at v_i , the contribution of these edges to $\partial f_{(W, \omega_W)}(v_i)$ is 0. This same argument applies to the initial vertex if we take subscripts modulo the length of W .

We can apply the same definition of $f_{(W, \omega_W)}$ to any directed walk (W, ω_W) , not necessarily closed or positive, but then the result may no longer be a flow. In fact, we have the following lemma.

Lemma 3.1. *Let (W, ω_W) be a directed walk with $W = v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$, $n > 0$. Then the function $f_{(W, \omega_W)}$ is conservative everywhere except possibly at v_0 and v_n ; that is, $\partial f_{(W, \omega_W)}(v) = 0$ for all $v \neq v_0, v_n$. If W is closed, then*

$$\partial f_{(W, \omega_W)}(v_0) = \begin{cases} 0 & \text{when } W \text{ is positive,} \\ 2\omega_W(v_0, e_1) & \text{when } W \text{ is negative.} \end{cases}$$

If W is open, then

$$\begin{aligned} \partial f_{(W, \omega_W)}(v_0) &= \omega_W(v_0, e_1), \\ \partial f_{(W, \omega_W)}(v_n) &= \omega_W(v_n, e_n) = -\sigma(W) \omega_W(v_0, e_1). \end{aligned}$$

Proof. Incoherence of (W, ω_W) can only possibly occur at v_0 and v_n . Fix a vertex v . We have

$$\begin{aligned}
\partial f_{(W, \omega_W)}(v) &= \sum_{e \in E} \mathbf{m}(v, e) \sum_{e_i \in W, e_i = e} [\omega, \omega_W](e_i) \\
&= \sum_{e_i \in W} \mathbf{m}(v, e_i) [\omega, \omega_W](e_i) \\
&= \sum_{e_i \in W, v_{i-1} = v} \omega(v_{i-1}, e_i) [\omega, \omega_W](e_i) + \sum_{e_i \in W, v_i = v} \omega(v_i, e_i) [\omega, \omega_W](e_i) \\
&= \sum_{e_i \in W, v_{i-1} = v} \omega_W(v_{i-1}, e_i) + \sum_{e_i \in W, v_i = v} \omega_W(v_i, e_i) \\
&= \sum_{e_i \in W} \mathbf{m}(v, e_i).
\end{aligned}$$

The second to last equality follows from the definition of coupling.

Let v appear in the vertex-edge sequence of W as $v_{l_1}, v_{l_2}, \dots, v_{l_m}$. If $l_i \neq 0, n$, then $\omega_W(v_{l_i}, e_{l_i}) + \omega_W(v_{l_i}, e_{l_i+1}) = 0$. Thus

$$\partial f_{(W, \omega_W)}(v) = \begin{cases} \omega_W(v_0, e_1) + \omega_W(v_n, e_n) & \text{if } v_0 = v_n = v, \\ \omega_W(v_0, e_1) & \text{if } v_0 = v \neq v_n, \\ \omega_W(v_n, e_n) & \text{if } v_n = v \neq v_0, \\ 0 & \text{if } v_0, v_n \neq v, \end{cases}$$

regardless of whether W closed or open.

In the case that W is closed and $v = v_0 = v_n$, then

$$\partial f_{(W, \omega_W)}(v) = \omega_W(v, e_1) [1 - \sigma(W)]$$

by Lemma 2.1. If $\sigma(W)$ is positive, then $\partial f_{(W, \omega_W)}(v_0) = 0$. If $\sigma(W)$ is negative, then $\partial f_{(W, \omega_W)}(v_0) = 2\omega_W(v_0, e_1) = 2\omega_W(v_n, e_n)$.

In the case that W is an open walk, namely, $v_0 \neq v_n$, we have $\partial f_{(W, \omega_W)}(v_0) = \omega_W(v_0, e_1)$ and $\partial f_{(W, \omega_W)}(v_n) = \omega_W(v_n, e_n) = -\sigma(W)\omega_W(v_0, e_1)$. \square

Conversely, directed closed walks can be constructed (though usually not uniquely) from integral flows.

Proposition 3.2. *If f is a nonnegative, nonzero integral flow on (Σ, ω) such that $\Sigma(f)$ is connected, then there exists a directed closed, positive walk (W, ω) on $\Sigma(f)$ such that $f_W = f$.*

Proof. We apply induction on the total weight of f ,

$$\|f\| := \sum_{e \in E(\Sigma)} |f(e)|.$$

To begin, choose a vertex v_0 and an edge e_1 incident with v_0 in $\Sigma(f)$. Let v_1 be the other endpoint of e_1 ($v_1 = v_0$ if e_1 is a loop). This gives a walk $W_1 = v_0 e_1 v_1$ of length 1. Clearly, $f \geq f_{W_1} \geq 0$.

Assume that we have constructed a partial walk $W_k = v_0 e_1 v_1 \cdots e_k v_k$ on $\Sigma(f)$ with $k \geq 1$, and that $f \geq f_{W_k} \geq 0$. If W_k is not closed and positive, then by Lemma 3.1 the function f_{W_k} is not a flow, for it is not conservative at v_k ; indeed, $\partial f_{W_k}(v_k) = \omega(v_k, e_k)$ (or $2\omega(v_k, e_k)$ if $v_0 = v_k$). Since f is conservative at v_k , $\partial(f - f_{W_k}) = -\omega(v_k, e_k)$ or $-2\omega(v_k, e_k)$. Since $f \geq 0$,

there exists an edge e_{k+1} incident with v_k in $\Sigma(f - f_{W_k})$ such that $\omega(v_k, x_{k+1}) = -\omega(v_k, e_k)$. Let v_{k+1} denote the other endpoint of e_{k+1} and extend W_k to a walk $W_{k+1} := W_k e_{k+1} v_{k+1}$ on $\Sigma(f)$. We have $f \geq f_{W_{k+1}} \geq 0$ by the construction. Continuing this procedure as long as W_{k+1} is not a closed, positive walk, we finally obtain a directed closed, positive walk (W_n, ω) . Then $f' := f - f_{W_n}$ is a nonnegative integral flow and $\|f'\| < \|f\|$.

Let $\Sigma(f')$ have connected components $\Sigma_1, \dots, \Sigma_m$. Set $f'_i = f'|_{\Sigma_i}$, $i = 1, \dots, m$. Note that $f' = \sum_{i=1}^m f'_i$ and $\text{supp } f'_i = E(\Sigma_i) \subseteq \Sigma(f)$, $1 \leq i \leq m$. Each f'_i is a nonnegative, nonzero integral flow on (Σ, ω) and satisfies $\|f'_i\| < \|f\|$. By induction there exists a directed closed, positive walk (W'_i, ω) on Σ_i such that $f'_i = f_{W'_i}$. The union of $(W'_1, \omega), \dots, (W'_m, \omega)$ and (W_n, ω) is connected. One can construct a single directed closed, positive walk (W, ω) by rearranging the initial and terminal vertices of W'_1, \dots, W'_m and W_n , and connecting them properly at some of their intersections. Then W is a walk on $\Sigma(f)$ and $f_W = f$. \square

Let f be an integral flow on (Σ, ω) . Associated with f is an orientation ω_f on Σ defined by

$$\omega_f(v, e) = \begin{cases} \omega(v, e) & \text{if } f(e) \geq 0, \\ -\omega(v, e) & \text{if } f(e) < 0, \end{cases} \quad (3.4)$$

for each edge e and endpoint $v \in \text{End}(e)$. The *absolute function* $|f|$ is defined by

$$|f|(e) = |f(e)| \quad \text{for } e \in E(\Sigma).$$

Since $|f| = [\omega, \omega_f]f$, $|f|$ is a nonnegative, nonzero integral flow on (Σ, ω_f) .

Corollary 3.3. *Let f be a nonzero integral flow on (Σ, ω) .*

- (a) *If $\Sigma(f)$ is connected, then there exists a directed closed, positive walk (W, ω_f) on $\Sigma(f)$ such that $f = f_{(W, \omega_f)}$.*
- (b) *There exists a directed closed, positive walk (W, ω_W) such that $f = f_{(W, \omega_W)}$.*

Proof. (a) Note that $|f|$ is a nonnegative, nonzero integral flow on (Σ, ω_f) and $\Sigma(|f|) = \Sigma(f)$. According to Proposition 3.2, there exists a directed closed, positive walk (W, ω_f) on $\Sigma(|f|)$ such that $f_W = |f|$ within the oriented signed graph (Σ, ω_f) , where

$$f_W(e) = \sum_{e_i \in W, e_i = e} [\omega_f, \omega_f](e_i) = \sum_{e_i \in W, e_i = e} 1$$

for each $e \in E(\Sigma)$. For the same directed closed, positive walk (W, ω_f) within (Σ, ω) , we have

$$\begin{aligned} f_{(W, \omega_f)}(e) &= \sum_{e_i \in W, e_i = e} [\omega, \omega_f](e_i) = [\omega, \omega_f](e) \sum_{e_i \in W, e_i = e} 1 \\ &= [\omega, \omega_f](e) f_W(e) \end{aligned}$$

for each $e \in E(\Sigma)$. Since $f = [\omega, \omega_f]|f|$ and $|f| = f_W$, it follows that $f_{(W, \omega_f)} = f$.

(b) Let $\Sigma(f)$ have components Σ_i , $1 \leq i \leq m$, where $m > 1$. Since each $\Sigma(f_i)$ is connected, there are directed closed, positive walks (W_i, ω_{f_i}) on Σ_i such that $f_i = f_{(W_i, \omega_{f_i})}$. Let v_i be the initial and terminal vertex of W_i . Let (P_i, ω_{P_i}) be a directed path from v_1 to v_i , where $i = 2, \dots, m$. Then

$$(W, \omega_W) := (W_1 P_2 W_2 P_2^{-1} \cdots P_m W_m P_m^{-1}, \omega_{W_1} \omega_{P_2} \omega_{W_2} \omega_{P_2}^{-1} \cdots \omega_{P_m} \omega_{W_m} \omega_{P_m}^{-1})$$

is a directed closed, positive walk such that $f = f_{(W, \omega_W)}$. \square

3.2. Lifted Flows.

Consider a function $\tilde{f} : E(\tilde{\Sigma}) \rightarrow \mathbb{Z}$ defined on the edge set of the double covering graph $\tilde{\Sigma}$. The *projection* of \tilde{f} is the function $\pi(\tilde{f}) : E(\Sigma) \rightarrow \mathbb{Z}$ defined by

$$\pi(\tilde{f})(e) = \tilde{f}(\tilde{e}) + \tilde{f}(\tilde{e}^*), \quad e \in E(\Sigma).$$

Let $\tilde{\mathbf{M}} = \mathbf{M}(\tilde{\Sigma}, \tilde{\omega}) = [\mathbf{m}(v^\alpha, \tilde{e})]$ denote the $V(\tilde{\Sigma}) \times E(\tilde{\Sigma})$ incidence matrix of $(\tilde{\Sigma}, \tilde{\omega})$; it is defined as in (3.1) but now every edge is positive. Since only positive loops of Σ are lifted to loops in $\tilde{\Sigma}$, we have

$$\mathbf{m}(v^\alpha, \tilde{e}) = \sum_{v^\alpha \in \text{End}(\tilde{e})} \tilde{\omega}(v^\alpha, \tilde{e}) = \begin{cases} 0 & \text{if } e \text{ is a positive loop,} \\ \alpha \omega(v, e) & \text{otherwise,} \end{cases}$$

where $e = \pi(\tilde{e})$, as usual (and the sum is over two edge ends if \tilde{e} is a loop at v^α). Also as usual, the boundary operator $\partial : \mathbb{Z}^{E(\tilde{\Sigma})} \rightarrow \mathbb{Z}^{V(\tilde{\Sigma})}$ is defined by

$$\partial \tilde{f}(v^\alpha) = \sum_{\tilde{e} \in E(\tilde{\Sigma})} \mathbf{m}(v^\alpha, \tilde{e}) \tilde{f}(\tilde{e}) = \sum_{(v^\alpha, \tilde{e}) \in \text{End}(v^\alpha)} \tilde{\omega}(v^\alpha, \tilde{e}) \tilde{f}(\tilde{e}).$$

Lemma 3.4. *Let \tilde{f} be a function defined on $E(\tilde{\Sigma})$. Then $\partial \pi(\tilde{f})$ is given by*

$$\partial \pi(\tilde{f})(v) = \partial \tilde{f}(v^+) - \partial \tilde{f}(v^-), \quad v \in V. \quad (3.5)$$

If \tilde{f} is a flow on $(\tilde{\Sigma}, \tilde{\omega})$, so is $\pi(\tilde{f})$ on (Σ, ω) .

Proof. Fix a vertex v in $V(\Sigma)$. Note that π acts as a bijection between $\text{End}(v^\alpha)$ in $\tilde{\Sigma}$ and $\text{End}(v)$ in Σ , for $\alpha \in \{+1, -1\}$, and recall that $\tilde{\omega}(v^\alpha, \tilde{e}) = \alpha \omega(v, e)$. Then

$$\begin{aligned} \partial \pi(\tilde{f})(v) &= \sum_{(v, e) \in \text{End}(v)} \omega(v, e) \pi(\tilde{f})(e) \\ &= \sum_{(v, e) \in \text{End}(v)} \omega(v, e) [\tilde{f}(\tilde{e}) + \tilde{f}(\tilde{e}^*)] \\ &= \sum_{(v, e) \in \text{End}(v)} \omega(v, e) \tilde{f}(\tilde{e}) + \sum_{(v, e) \in \text{End}(v)} \omega(v, e) \tilde{f}(\tilde{e}^*) \\ &= \sum_{(v, e) \in \text{End}(v)} \tilde{\omega}(v^+, \tilde{e}) \tilde{f}(\tilde{e}) - \sum_{(v, e) \in \text{End}(v)} \tilde{\omega}(v^-, \tilde{e}^*) \tilde{f}(\tilde{e}^*) \\ &= \sum_{(v^+, \tilde{e}) \in \text{End}(v^+)} \tilde{\omega}(v^+, \tilde{e}) \tilde{f}(\tilde{e}) - \sum_{(v^-, \tilde{e}^*) \in \text{End}(v^-)} \tilde{\omega}(v^-, \tilde{e}^*) \tilde{f}(\tilde{e}^*) \\ &= \partial \tilde{f}(v^+) - \partial \tilde{f}(v^-). \end{aligned}$$

When \tilde{f} is a flow on $(\tilde{\Sigma}, \tilde{\omega})$, then $\partial \tilde{f}(v^+) = \partial \tilde{f}(v^-) = 0$. Thus $\partial \pi(\tilde{f})(v) = 0$ by (3.5), so $\pi(\tilde{f})$ is a flow on (Σ, ω) . \square

A *lift* of an integral flow f of (Σ, ω) to $\tilde{\Sigma}$ is an integral flow \tilde{f} of $(\tilde{\Sigma}, \tilde{\omega})$ such that $\pi(\tilde{f}) = f$.

Proposition 3.5. (a) *Let $(\tilde{W}, \tilde{\omega}_W)$ be a lift of a directed walk (W, ω_W) . If W is closed and has positive sign, then $(\tilde{W}, \tilde{\omega}_W)$ is a directed closed walk, and*

$$\pi(f_{(\tilde{W}, \tilde{\omega}_W)}) = f_{(W, \omega_W)}. \quad (3.6)$$

(b) Let f be an integral flow on (Σ, ω) such that $f = f_{(W, \omega_f)}$, where (W, ω_f) is a directed closed positive walk. If (W, ω_f) is lifted to a directed closed walk $(\widetilde{W}, \widetilde{\omega}_f)$ in $\widetilde{\Sigma}$, then $f_{(\widetilde{W}, \widetilde{\omega}_f)}$ is a flow on $(\widetilde{\Sigma}, \widetilde{\omega})$ lifted from f . Moreover, if $f \geq 0$, then $f_{(\widetilde{W}, \widetilde{\omega}_f)} \geq 0$.

Proof. (a) It follows from Lemma 2.2 that $(\widetilde{W}, \widetilde{\omega}_W)$ is a directed closed walk. To see that $\pi(f_{(\widetilde{W}, \widetilde{\omega}_W)}) = f_{(W, \omega_W)}$, it suffices to show that $f_{(W, \omega_W)}(e) = f_{(\widetilde{W}, \widetilde{\omega}_W)}(e^+) + f_{(\widetilde{W}, \widetilde{\omega}_W)}(e^-)$. Since lifting of orientations preserves the coupling, by (2.2) we have

$$\begin{aligned} \pi(f_{(\widetilde{W}, \widetilde{\omega}_W)})(e) &= f_{(\widetilde{W}, \widetilde{\omega}_W)}(e^+) + f_{(\widetilde{W}, \widetilde{\omega}_W)}(e^-) \\ &= \sum_{\tilde{e}_i \in \widetilde{W}, \tilde{e}_i \in \{e^+, e^-\}} [\widetilde{\omega}, \widetilde{\omega}_W](\tilde{e}_i) \\ &= \sum_{e_i \in W, e_i = e} [\omega, \omega_W](e_i) \\ &= f_{(W, \omega_W)}(e). \end{aligned}$$

(b) Since ω_f is an orientation on Σ , $\widetilde{\omega}_f$ is an orientation on $\widetilde{\Sigma}$. Viewing ω_f as a direction ω_W on W , $\widetilde{\omega}_W$ is a direction of \widetilde{W} . Note that the lift of ω_W is the same as the lift of ω_f , so we can view $\widetilde{\omega}_f$ as the direction $\widetilde{\omega}_W$ of \widetilde{W} .

Since $f = f_{(W, \omega_f)}$, it follows from (3.6) and Lemma 3.4 that $f_{(\widetilde{W}, \widetilde{\omega}_f)}$ is a flow on $(\widetilde{\Sigma}, \widetilde{\omega})$ and a lift of f . If $f \geq 0$, then $\omega_f = \omega$; consequently, $\widetilde{\omega}_f = \widetilde{\omega}$. Thus $f_{(\widetilde{W}, \widetilde{\omega}_f)} \geq 0$ by the definition in (3.3). \square

3.3. Decomposability.

An integral flow f is *conformally decomposable* if it is nonzero and can be represented as a sum of two other integral flows, $f = f_1 + f_2$, each of which is nonzero and conforms to the sign pattern of f , that is, $f_1(e)f_2(e) \geq 0$ for all edges e ; this means that $f_1(e)$ and $f_2(e)$ have the same sign when they are nonzero. An integral flow is said to be *conformally indecomposable* if it is nonzero and not conformally decomposable. It is well known and easy to see that conformally indecomposable flows on an unsigned graph are just graphic circuit flows.

A nonnegative, nonzero integral flow f is *minimal* provided that if g is a nonnegative, nonzero integral flow on (Σ, ω) such that $g(e) \leq f(e)$ for all edges e , then $g = f$.

If an integral flow f is nonnegative, then its minimality is equivalent to its conformal indecomposability. In fact, if f is conformally decomposed into $f = f_1 + f_2$, then f_1 and f_2 must be nonnegative, nonzero integral flows such that $f_1 \leq f$ and $f_1 \neq f$; this means that f is not minimal. Conversely, if f is not minimal, then there is a nonnegative, nonzero integral flow g on (Σ, ω) such that $g \leq f$ but $g \neq f$. Now $f - g$ is nonzero and nonnegative, and f decomposes conformally into g and $f - g$.

The following proposition shows that conformal indecomposability of a nonzero integral flow f on (Σ, ω) is equivalent to the minimality of the flow $|f|$ on (Σ, ω_f) , where $|f|$ is the absolute value function of f and ω_f is the orientation given by (3.4).

Proposition 3.6. *Let f be a nonzero integral flow on (Σ, ω) . Then the following properties are equivalent.*

- (a) f is a conformally indecomposable flow on (Σ, ω) .
- (b) $|f|$ is a conformally indecomposable flow on (Σ, ω_f) .

(c) $|f|$ is a minimal flow on (Σ, ω_f) .

Proof. Applying the boundary operator (3.2), it is clear that f is a flow on (Σ, ω) if and only if $|f| = [\omega, \omega_f] f$ is a flow on (Σ, ω_f) . Since $|f|$ is nonnegative, its minimality is equivalent to its conformal indecomposability, so we already have (b) \Leftrightarrow (c) by the argument above.

(a) \Rightarrow (c): Suppose $|f|$ is not minimal, that is, $|f| = g_1 + g_2$, where g_1 and g_2 are nonnegative, nonzero integral flows on (Σ, ω_f) . Setting $f_i = [\omega, \omega_f] g_i$ yields nonzero integral flows on (Σ, ω) , $i = 1, 2$. Thus

$$f = [\omega, \omega_f] |f| = [\omega, \omega_f] g_1 + [\omega, \omega_f] g_2 = f_1 + f_2,$$

and $f_1 f_2 = g_1 g_2 \geq 0$, meaning that f is conformally decomposable. This is a contradiction.

(b) \Rightarrow (a): Suppose f is conformally decomposable, that is, $f = f_1 + f_2$, where f_1 and f_2 are nonzero integral flows on (Σ, ω) such that $f_1 f_2 \geq 0$. Setting $g_i = [\omega, \omega_f] f_i$ yields nonzero integral flows on (Σ, ω_f) , $i = 1, 2$. For each edge e , if $f_i(e) > 0$, we must have $f(e) > 0$ and $[\omega, \omega_f](e) = 1$ by the definition of ω_f ; if $f_i(e) < 0$, we must have $f(e) < 0$ and $[\omega, \omega_f](e) = -1$; thus $g_i(e) \geq 0$. Hence

$$|f| = [\omega, \omega_f] f = [\omega, \omega_f] f_1 + [\omega, \omega_f] f_2 = g_1 + g_2,$$

meaning that $|f|$ is conformally decomposable. This is a contradiction. \square

4. INDECOMPOSABLE FLOWS

A signed graph with nonempty edge set is called *sesqui-Eulerian* if there exists a directed closed, positive walk that uses every edge at least once but at most twice, and whose direction has the same orientation on each pair of repeated edges. A sesqui-Eulerian signed graph is *prime* if no such directed closed, positive walk properly contains any directed closed, positive subwalks. A sesqui-Eulerian signed graph is *minimal* if it does not properly contain any sesqui-Eulerian signed subgraphs. It is clear that minimal sesqui-Eulerian signed graphs must be prime.

Definition 4.1. A signed graph T with nonempty edge set is called a *circle-tree* if it satisfies the following conditions:

- (a) T is connected.
- (b) Each block of T is either a circle or an edge.
- (c) Each end block of T is a circle.
- (d) Each cut-vertex is incident with exactly two blocks.

The blocks that are circles are called the *circle blocks* of T . The paths (of possibly zero length) between pairs of circle blocks are called the *block paths* of T . The *length* of T is

$$l(T) := \sum_i l(C_i) + 2 \sum_j l(P_j),$$

where the C_i are the circle blocks and the P_j are the block paths.

A circle-tree is said to be *sesqui-Eulerian* if it further satisfies

- (e) *Parity Condition:* The sign of a circle block equals $(-1)^p$, where p is the number of cut-vertices of T on the circle.

We shall see that prime sesqui-Eulerian signed graphs are sesqui-Eulerian circle-trees (Proposition 4.7) and that minimal sesqui-Eulerian signed graphs are signed-graph circuits, i.e., circuits of Types I, II, and III (Corollary 4.8).

A circle-tree can be viewed as a tree-like graph whose “vertices” are the circle blocks and whose “edges” are the block paths. The endpoints of block paths are cut-vertices. A block path of length zero is a common cut-vertex of two circle blocks. If each circle block is contracted to a point, the resulting graph is a tree.

We may also think of a sesqui-Eulerian circle-tree as a “tree” whose “vertices” are its vertex-disjoint maximal Eulerian subgraphs and whose “edges” are the paths (of positive length) between the maximal Eulerian subgraphs, where each such maximal Eulerian subgraph is also a tree-like structure whose “vertices” are edge-disjoint circles and “edges” are the intersection vertices between pairs of circles.

Let T be a sesqui-Eulerian circle-tree. A *direction* of T is an orientation ω_T on the signed graph T such that (T, ω_T) has neither a sink nor a source, and for each circle block C the subgraph (C, ω_T) has either a sink or a source at each cut-vertex of T on C . It is easy to see that there exist exactly two (opposite) directions on T . Figure 4 exhibits a sesqui-Eulerian circle-tree with a direction.

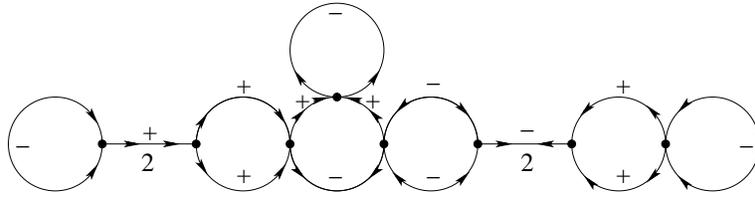


FIGURE 4. A sesqui-Eulerian circle-tree with a direction.

For unsigned graphs, since all edges are positive, sesqui-Eulerian circle-trees are just circles, and directed sesqui-Eulerian circle-trees are directed circles.

When a circle-tree T is contained in a signed graph Σ , the *indicator function* of T is the function $I_T : E(\Sigma) \rightarrow \mathbb{Z}$ defined by

$$I_T(e) = \begin{cases} 1 & \text{if } e \text{ belongs to a circle block,} \\ 2 & \text{if } e \text{ belongs to a block path,} \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic vector* of (T, ω_T) within the oriented signed graph (Σ, ω) is the function $[\omega, \omega_T] I_T$.

A *minimal tour* on T is a closed walk that uses every edge of T and has minimum length. A subgraph of T is a *circle-subtree* if it is a circle-tree and its circle blocks and block paths are circle blocks and block paths of T .

Proposition 4.2 (Existence and Uniqueness of Direction on a Sesqui-Eulerian Circle-Tree). *If T is a sesqui-Eulerian circle-tree, then:*

- (a) *There exists a closed walk W on T such that*
 - (i) *W uses each edge of a circle block once and each edge of a block path twice, and*
 - (ii) *whenever W meets a cut-vertex, it crosses from one block to another block.**Moreover, each such closed walk W is a minimal tour on T with length $l(W) = l(T)$.*
- (b) *Each minimal tour W on T satisfies the conditions (i), (ii), and $l(W) = l(T)$.*
- (c) *There exists a unique direction ω_T of T (up to negation of ω_T) such that (W, ω_T) is coherent for each minimal tour W on T . The direction ω_T satisfies*

$$f_{(W, \omega_T)} = [\omega, \omega_T] I_T.$$

- (d) If v is a cut-vertex of T and $W = W_1W_2$, where W_1, W_2 are closed sub-walks having initial and terminal vertices at v , then both W_1 and W_2 are negative and both (W_1, ω_T) and (W_2, ω_T) are incoherent at v and coherent elsewhere.

Proof. (a) We prove it by the covering graph method, rather than working directly on T as in [7]. Let \widetilde{T} be the double covering graph of T . We claim that there exists a directed circle $(\widetilde{W}, \omega_{\widetilde{W}})$ such that (i') \widetilde{W} covers each edge of a circle block once and each edge of a block path twice, (ii') the orientations on each edge of T induced from $(\widetilde{W}, \omega_{\widetilde{W}})$ by the projection are identical, and (iii') the induced orientation ω_T from $\omega_{\widetilde{W}}$ is a direction of T .

If T is a circle block C , the circle C must be positive, since there is no cut-vertex. Choose a direction ω_C of C . By Lemma 2.2, (C, ω_C) lifts to a directed closed walk $(\widetilde{C}, \widetilde{\omega}_C)$ in \widetilde{T} that covers C once. The closed walk \widetilde{C} must be a circle since a self-intersection of \widetilde{C} implies a self-intersection of the circle C , which is impossible.

Now we describe how to lift a circle-tree T that contains at least two circle blocks to a circle \widetilde{W} in \widetilde{T} .

In a circle block $C = v_0x_1 \cdots x_nv_n$ ($v_n = v_0$), call an *arc* a part of the circle that connects consecutive cut-vertices on C and let p be the number of cut-vertices in C , which is also the number of arcs. There are two ways to lift each arc, which are determined by the lift of its initial vertex v to either v^+ or v^- . If we lift an arc $A = v_0x_1 \cdots x_kv_k$ so that v_0 lifts to v_0^α , then v_k lifts to $v_k^{\alpha\sigma(A)}$. For the next arc, $A' = v_kx_{k+1} \cdots x_lv_l$, the initial vertex v_k lifts to $v_k^{-\alpha\sigma(A)}$, and the terminal vertex v_l lifts to $v_l^{-\alpha\sigma(AA')}$. Thus \widetilde{A} jumps to \widetilde{A}' since they have no vertex in common. We continue this process through all the arcs of C . The last lifted arc \widetilde{A}'' ends at vertex $v_0^{(-1)^{p-1}\alpha\sigma(C)}$ because there are $p - 1$ jumps from the first arc to the last of the p arcs in C ; since $\sigma(C) = (-1)^p$, \widetilde{A}'' ends at $v_0^{-\alpha}$, leaving a jump to the initial vertex v_0^α of \widetilde{A} . For the lift of C to use in \widetilde{W} we arbitrarily choose one of the two possible lifts described by this rule.

A block path P has two lifts that are vertex-disjoint paths $\widetilde{P}, \widetilde{P}^*$. We use both of them in \widetilde{W} .

We now prove \widetilde{W} is a circle. If there exists any sesqui-Eulerian circle-tree for which \widetilde{W} is not a circle, let T be one with the fewest edges. We say two circle blocks are *attached* if they have a common vertex. A *stout block* B is a circle block C_B of T together with the (necessarily negative) loops that are attached to it. An *end stout block* is a stout block such that C_B is an end block in T with loops (other than C_B) deleted. The important fact about a stout block B is that each attached loop, when lifted to \widetilde{T} , connects two consecutive lifted arcs of C_B into a path. It follows that \widetilde{W} is a circle if T has only one stout block. Thus we may assume T has more than one stout block. Let B be an end stout block, joined by a block path P to another stout block B_1 at the vertex u of B_1 . The lift of $B \cup P$ is a path in \widetilde{T} connecting u^+ to u^- . Replace $B \cup P$ by a single negative loop at u , forming a new circle-tree T' with fewer edges, whose lifted walk \widetilde{W}' is therefore a circle. Thus \widetilde{W} is a circle.

The circle \widetilde{W} covers each edge of a circle block once and each edge of a block path twice. We give it a direction $\omega_{\widetilde{W}}$. Let e be an edge of a block path and let $\text{End}(e) = \{u, v\}$; then \widetilde{W} contains $e^+ = u^+v^{\sigma(e)}$ and $e^- = u^-v^{-\sigma(e)}$. Then we may write $\widetilde{W} = \widetilde{W}_1u^+e^+v^{\sigma(e)}\widetilde{W}_2v^{-\sigma(e)}e^-u^-$. It follows that $\omega_{\widetilde{W}}(u^+, e^+) = -\omega_{\widetilde{W}}(u^+, \tilde{x})$ (by coherence of $\omega_{\widetilde{W}}$) = $\omega_{\widetilde{W}}(u^-, \tilde{y})$ (by Lemma 2.1

and positivity of the edges in $\tilde{\Sigma}) = -\omega_{\tilde{W}}(u^-, e^-)$, where \tilde{x} is the first edge in \tilde{W}_1 and \tilde{y} is its last edge. This implies that the projected orientations on e from e^+ and e^- are identical.

We have obtained a directed circle $(\tilde{W}, \omega_{\tilde{W}})$ in \tilde{T} that satisfies the conditions (i')–(iii'). We define the closed walk W by $W = \pi(\tilde{W})$ and its direction ω_W by $\omega_W = \pi(\omega_{\tilde{W}})$. Since π is locally an incidence-preserving bijection, it is clear that W satisfies conditions (i) and (ii) and is a minimal tour on T .

(b) A minimal tour W on T must traverse each edge in a block path at least twice, since each such edge is a cut-edge. So W has length at least $l(T)$. Since W has minimum length, it follows that $l(W) = l(T)$. The minimality of $l(W)$ obliges W to satisfy conditions (i) and (ii).

(c) Let ω_T be the direction of T obtained by projecting the direction $\omega_{\tilde{W}}$. For an arbitrary minimal tour W on T , the fact that ω_T is a direction forces (W, ω_W) to be a directed closed walk. The uniqueness of ω_T up to sign follows from the tree-like structure of T . Properties (i) and (ii) imply that I_T is a flow on (T, ω_T) . Thus $f_{(W, \omega_T)} = [\omega, \omega_T] I_T$.

(d) To prove negativity of W_1 , consider that each block path implies two minus signs in $\sigma(W)$, one for each circle block with which it is incident. Hence $\sigma(W) = +$ and $\sigma(W_1) = \sigma(W_2)$. If the block path contains the cut-vertex that separates W_1 and W_2 , then one incident circle is in W_1 and contributes a minus sign to $\sigma(W_1)$ while the other incident circle is in W_2 and does not contribute to $\sigma(W_1)$. No other block path affects the sign of W_1 . \square

The proof shows that the number of directed minimal tours on a sesqui-Eulerian circle-tree T (up to changing the initial vertex and reversing the direction) is 2^q , where q is the number of circle blocks in T .

Lemma 4.3 (Minimality of Sesqui-Eulerian Circle-Trees). *No directed sesqui-Eulerian circle-tree properly contains another directed sesqui-Eulerian circle-tree.*

Proof. Let (T, ω_T) be a directed sesqui-Eulerian circle-tree that has a proper subgraph T' which is also a sesqui-Eulerian circle-tree with the direction $\omega_T|_{T'}$, the restriction of ω_T to T' . The circle blocks of T' are certainly circle blocks of T . There exist a circle block C of T' and a vertex u of C such that u is not a cut-vertex in T' but is a cut-vertex in T . It follows that (C, ω_T) must be coherent at u when C is considered as a circle block in T' but must be incoherent (be either a sink or a source) at u when C is considered as a circle block in T . This is a contradiction. \square

Lemma 4.4. *Every flow f on (Σ, ω) can be lifted to a flow \tilde{f} on $(\tilde{\Sigma}, \tilde{\omega})$.*

Proof. This follows from (b) of Corollary 3.3 and Part (a) of Proposition 3.5. \square

Theorem 4.5 (Resolution of Indecomposable Flows). *If f is a conformally indecomposable flow on (Σ, ω) , then there exists a directed closed, positive walk (W, ω_f) on $\Sigma(f)$ such that $f = f_{(W, \omega_f)}$.*

Furthermore, if $(\tilde{W}, \tilde{\omega}_f)$ is a lift of (W, ω_f) , then \tilde{W} is a circle, and $f_{(\tilde{W}, \tilde{\omega}_f)}$ is a lift of f and is a conformally indecomposable flow on $(\tilde{\Sigma}, \tilde{\omega})$.

Proof. The conformal indecomposability of f implies that $\Sigma(f)$ is connected. By (b) of Corollary 3.3, there exists a directed closed, positive walk (W, ω_f) on $\Sigma(f)$ such that $f = f_{(W, \omega_f)}$. Lift (W, ω_f) to a directed closed walk $(\tilde{W}, \tilde{\omega}_f)$ in $\tilde{\Sigma}$. Now f is lifted to a flow $f_{(\tilde{W}, \tilde{\omega}_f)}$ of $(\tilde{\Sigma}, \tilde{\omega})$ by Proposition 3.5(a), denoted by $\tilde{f} = f_{(\tilde{W}, \tilde{\omega}_f)}$.

Suppose \tilde{f} is decomposed as $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, where \tilde{f}_i are nonzero integral flows and $\tilde{f}_1 \tilde{f}_2 \geq 0$. Notice that

$$f = [\omega, \omega_f] |f|, \quad \tilde{f} = [\tilde{\omega}, \tilde{\omega}_f] |\tilde{f}|, \quad \tilde{f}_i = [\tilde{\omega}, \tilde{\omega}_f] |\tilde{f}_i|.$$

Let $f_i = \pi(\tilde{f}_i)$, which are nonzero flows on (Σ, ω) . For each edge e of Σ ,

$$\begin{aligned} f_i(e) &= [\tilde{\omega}, \tilde{\omega}_f](e^+) |\tilde{f}_i|(e^+) + [\tilde{\omega}, \tilde{\omega}_f](e^-) |\tilde{f}_i|(e^-) \\ &= [\omega, \omega_f](e) (|\tilde{f}_i|(e^+) + |\tilde{f}_i|(e^-)) \\ &= [\omega, \omega_f](e) \pi(|\tilde{f}_i|)(e). \end{aligned}$$

Since $\pi(|\tilde{f}_i|) \geq 0$, it follows that $f_i = [\omega, \omega_f] |f_i|$. Thus $f = f_1 + f_2$ by linearity of π , and $f_1 f_2 = |f_1| |f_2| \geq 0$, meaning that f is conformally decomposable. This is a contradiction.

The conformal indecomposability of $f_{(\tilde{W}, \tilde{\omega}_f)}$ implies that it is a graphic circuit flow. So \tilde{W} is a circle. \square

Remark. The projection of a flow is a flow, but the projection of a conformally indecomposable flow is not necessarily a conformally indecomposable flow.

Theorem 4.6 (Classification of Conformally Indecomposable Flows). *Let f be a flow on (Σ, ω) . Then f is conformally indecomposable if and only if $\Sigma(f)$ is a sesqui-Eulerian circle-tree with a direction ω_f and*

$$f = [\omega, \omega_f] I_{\Sigma(f)}.$$

Proof. \Rightarrow : We assume f is conformally indecomposable. Recall the directed closed, positive walks (W, ω_f) on $\Sigma(f)$ and $(\tilde{W}, \tilde{\omega}_f)$ on $\tilde{\Sigma}$ in Theorem 4.5, where \tilde{W} is a lift of W . Note that $f = f_{(W, \omega_f)}$, $|f| = [\omega, \omega_f] f$, and \tilde{W} is a circle. Let $W = v_0 e_1 v_1 \cdots e_n v_n$ and $\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \tilde{e}_2 \cdots \tilde{e}_n v_n^{\alpha_n}$. Since $\tilde{\Sigma}$ is a double covering of Σ and $\pi(\tilde{W}) = W$, it follows that vertices and edges appear in W at most twice.

If W has no double vertices, that is, W has no self-intersections, then (W, ω_f) is a directed circle and W is a positive circle since it is a one-to-one image of a circle in $\tilde{\Sigma}$. Then W is a sesqui-Eulerian circle-tree and $f = [\omega, \omega_f] I_{\Sigma(f)}$.

Assume that W has some self-intersections. Let v be a double vertex of W . Rewrite W as $W_1 W_2$, where each W_i is a closed walk with the initial and terminal vertices at v . More specifically, $W_1 = v_0 e_1 v_1 \cdots e_m v_m$ and $W_2 = v_m e_{m+1} v_{m+1} \cdots e_n v_n$, where $v_0 = v_m = v_n = v$. We claim that each W_i is negative, each (W_i, ω_f) is incoherent at v and coherent elsewhere, and v is a cut-vertex of $\Sigma(W)$.

Each (W_i, ω_f) is coherent everywhere except at v . Suppose (W_1, ω_f) is coherent at v . That forces (W_2, ω_f) to also be coherent at v . Then each (W_i, ω_f) is a directed closed, positive walk. We thus have $f_W = f_{W_1} + f_{W_2}$ within (Σ, ω_f) , meaning that $|f|$ is conformally decomposable; this is a contradiction by Proposition 3.6. Hence (W_i, ω_f) must be incoherent at v . Lemma 3.1 implies that the closed walks W_i are negative. Write \tilde{W} as $\tilde{W}_1 \tilde{W}_2$, where

$$\tilde{W}_1 = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \cdots \tilde{e}_m v_m^{\alpha_m} \quad \text{and} \quad \tilde{W}_2 = v_m^{\alpha_m} \tilde{e}_{m+1} v_{m+1}^{\alpha_{m+1}} \cdots \tilde{e}_n v_n^{\alpha_n}$$

are paths. We have $\alpha_0 = \alpha_n = -\alpha_m$ by Lemma 2.2.

Suppose v is not a cut-vertex, that is, W_1 and W_2 meet at a vertex u other than v . Let u occur as v_k in W_1 and as v_h in W_2 , that is, $u = v_k = v_h$, where $1 \leq k \leq m-1$ and

$m + 1 \leq h \leq n - 1$. Since \widetilde{W} is a circle, we have $v_k^{\alpha_k} \neq v_h^{\alpha_h}$, consequently, $\alpha_k = -\alpha_h$. Consider the closed walk

$$\widetilde{W}' = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \cdots v_{k-1}^{\alpha_{k-1}} \tilde{e}_k v_k^{\alpha_k} \tilde{e}_h^* v_{h-1}^{-\alpha_{h-1}} \cdots v_{m+1}^{-\alpha_{m+1}} \tilde{e}_{m+1}^* v_m^{-\alpha_m} = \widetilde{W}'_1 \widetilde{W}'_2,$$

where $v_k^{\alpha_k} = v_h^{-\alpha_h}$, $v_m^{-\alpha_m} = v_0^{\alpha_0}$, and

$$\widetilde{W}'_1 = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \cdots v_{k-1}^{\alpha_{k-1}} \tilde{e}_k v_k^{\alpha_k} \quad \text{and} \quad \widetilde{W}'_2 = v_h^{-\alpha_h} \tilde{e}_h^* v_{h-1}^{-\alpha_{h-1}} \cdots v_{m+1}^{-\alpha_{m+1}} \tilde{e}_{m+1}^* v_m^{-\alpha_m}.$$

Let $s = \tilde{\omega}_f(v_0^{\alpha_0}, \tilde{e}_1) = \alpha_0 \omega_f(v_0, e_1)$, where the second equality follows from the definition (2.1). Now $\tilde{\omega}_f(v_k^{\alpha_k}, \tilde{e}_k) = -s$, since as an open walk $(\widetilde{W}_1, \tilde{\omega}_f)$ is directed. Similarly, $s = \tilde{\omega}_f(v_m^{\alpha_m}, \tilde{e}_{m+1}) = \alpha_m \omega_f(v_m, e_{m+1})$. Hence $\tilde{\omega}_f(v_m^{-\alpha_m}, \tilde{e}_{m+1}^*) = -\alpha_m \omega_f(v_m, e_{m+1}) = -s$, that is, $(\widetilde{W}', \tilde{\omega}_f)$ is coherent at $v_0^{\alpha_0}$. Thus $(\widetilde{W}', \tilde{\omega}_f)$ is a directed closed walk on $\widetilde{\Sigma}(f)$. Let $W' = \pi(\widetilde{W}')$. It follows from Lemma 2.2 that (W', ω_f) is a directed closed, positive walk on $\Sigma(f)$. Moreover, for a repeated edge e in W' : if e is the projection of two edges in \widetilde{W}'_1 or two in \widetilde{W}'_2 , then e is the projection of two edges in \widetilde{W}_1 or two in \widetilde{W}_2 ; if e is the projection of one edge in \widetilde{W}'_1 and another in \widetilde{W}'_2 , then e is the projection of one edge in \widetilde{W}_1 and one edge in \widetilde{W}_2 ; it follows that e is a repeated edge in W . Thus $E(W')$ is a proper sub-multiset of $E(W)$ with the same edge orientations. Consequently $f_{W'} \leq f_W$ and $f_{W'} \neq f_W$. This is contradictory to the minimality of f_W . It follows that a repeated vertex in W is a cut-vertex of $\Sigma(W)$.

The subgraph $\Sigma(f)$ is obtained from the circle \widetilde{W} by projection, which identifies each pair v^+, v^- of vertices and each pair \tilde{e}, \tilde{e}^* of edges of which both appear in \widetilde{W} . Since each identified vertex v is a cut-vertex of $\Sigma(f)$, each identified edge e is a cut-edge. Removing all cut-edges from $\Sigma(f)$, every vertex has degree 2 or 4; in the latter case it is a cut-vertex that separates two of its incident edges from the other two. It follows that in $\Sigma(f)$ with cut edges removed, every block is a circle. The connected components of the identified vertices and edges form block paths P_j (possibly of zero length) joining some pairs of the circles C_i . Thus $\Sigma(f)$ satisfies (a, b, d) of Definition 4.1. It satisfies (c) because an end block that is an edge would have a vertex of degree 1 in $\Sigma(f)$, which is impossible.

Recall the incoherence of (W_i, ω_f) at the double vertex v . It follows that each (C_i, ω_f) is incoherent at the cut-vertices of $\Sigma(f)$ on C_i and coherent elsewhere. Thus the circle C_i has sign $(-1)^p$ by Lemma 2.1, where p is the number of cut-vertices on C_i .

We have proved that $\Sigma(f)$ is a sesqui-Eulerian circle-tree, ω_f is a direction of $\Sigma(f)$, and (W, ω_f) is a directed minimal tour. Under these conditions, each edge in a circle block appears once in W and each edge in a block path appears twice in W , so $f_W = I_{\Sigma(f)}$ within (Σ, ω_f) . Therefore $f = f_{(W, \omega_f)} = [\omega, \omega_f] I_{\Sigma(f)}$.

\Leftarrow : We assume $\Sigma(f)$ is a sesqui-Eulerian circle-tree with direction ω_f and $f = [\omega, \omega_f] I_{\Sigma(f)}$. Let us write $f = \sum_{i=1}^k f_i$, where the f_i are conformally indecomposable flows on (Σ, ω) that conform to f , and $k \geq 1$. Let (Σ_i, ω_i) be sesqui-Eulerian circle-trees such that $f_i = [\omega, \omega_i] I_{\Sigma_i}$. Since each f_i conforms to the sign pattern of f , it follows that Σ_i is a subgraph of $\Sigma(f)$ and ω_{f_i} is the restriction of ω_f to Σ_i . The circle blocks of Σ_i are certainly circle blocks of $\Sigma(f)$. Consider a block path P of Σ_i . If $l(P) = 0$, then P is the intersection of two circle blocks C_1 and C_2 of Σ_i . Since $C_1, C_2 \subseteq \Sigma(f)$, P is a block path in $\Sigma(f)$. If $l(P) > 0$, then $f_i(e) = \pm 2$ for all edges e of P . Since f_i conforms to the sign pattern of f and $|f_i| \leq |f| = I_{\Sigma(f)}$, it must be that $f(e) = \pm 2$ for every edge e of P . This means that P is a block path of $\Sigma(f)$. Hence (Σ_i, ω_i) is a directed sesqui-Eulerian circle-tree contained in $(\Sigma(f), \omega_f)$. It follows by Lemma 4.3 that $\Sigma(f) = \Sigma_i$. Therefore $k = 1$, i.e., f is conformally indecomposable. \square

Proposition 4.7. *A signed graph Σ is prime sesqui-Eulerian if and only if Σ is a sesqui-Eulerian circle-tree.*

Proof. If Σ is a sesqui-Eulerian circle-tree, then by Proposition 4.2 there exist an orientation ω on Σ and a closed walk W that uses every edge of Σ once or twice such that (W, ω) is a directed closed, positive walk. If (W, ω) did properly contain a directed closed, positive subwalk (W', ω') , then (Σ, ω) would properly contain a directed sesqui-Eulerian signed graph (Σ', ω') , where Σ' consists of the vertices and edges of W' ; but that would contradict Lemma 4.3. Thus Σ is a prime sesqui-Eulerian signed graph.

Conversely, if Σ is a prime sesqui-Eulerian signed graph, then by definition there exist an orientation ω on Σ and a closed positive walk W that uses every edge of Σ once or twice, such that (W, ω) is a directed walk. The flow f_W on (Σ, ω) can be conformally decomposed into conformally indecomposable flows f_i so that $f_W = \sum_{i=1}^m f_i \geq 0$. By Theorem 4.6, for each f_i there exist a sesqui-Eulerian circle-tree T_i and its direction ω_{T_i} such that $f_i = [\omega, \omega_{T_i}] I_{T_i}$. Hence $\omega_{T_i} = \omega$ on T_i . By Theorem 4.2, there exists a directed closed, positive walk (W_i, ω_{T_i}) on each T_i . Since Σ is connected, we can construct a directed closed positive walk (W', ω) that uses every edge of Σ exactly as often as does (W, ω) , by concatenating the W_i in a suitable order after changing their initial/terminal vertices as necessary. Since Σ is prime and (W', ω) contains the directed closed, positive subwalk (W_1, ω_{T_1}) , Σ must be the sesqui-Eulerian circle-tree T_1 and $W' = W_1$. \square

Corollary 4.8. *A signed graph is a minimal sesqui-Eulerian signed graph if and only if it is a signed-graph circuit.*

Proof. Let Σ be a minimal sesqui-Eulerian signed graph. Since it is prime, it is a sesqui-Eulerian circle-tree by Proposition 4.7. If it has only one circle block, it is a positive circle, i.e., a circuit of Type I. If it has more than one circle block, it has two (or more) end blocks C_1, C_2 , each of which is a negative circle, and it contains a path P (possibly of length 0) connecting those blocks. Then $C_1 \cup P \cup C_2$ is a circuit of Type II or III.

Conversely, a signed-graph circuit is obviously a sesqui-Eulerian circle-tree and minimal. \square

Theorem 4.9 (Half-Integer Decomposition). *Let T be a sesqui-Eulerian circle-tree with a direction ω_T . Either T is a signed-graph circuit, or there exists a closed, positive walk W on T ,*

$$W = P_1 C_1 P_2 \cdots P_n C_n, \quad n \geq 2,$$

satisfying the following conditions:

- (a) *The C_i , $1 \leq i \leq n$, are the end blocks of T , and the P_i are paths of positive length.*
- (b) *Each edge of a non-end circle block appears in exactly one of the paths P_i , and each edge of a block path appears in exactly two of the paths P_i .*
- (c) *Each $(C_i P_{i+1} C_{i+1}, \omega_T)$, where $C_{n+1} = C_1$, is a directed circuit of Type III.*
- (d) *T is a conformal half-integral linear combination of signed-graph circuits; more precisely,*

$$I_T = \frac{1}{2} \sum_{i=1}^n I_{T_i},$$

where T_i is the restriction of T to the subgraph $C_i \cup P_{i+1} \cup C_{i+1}$.

Proof. Let \tilde{W} be the circle in \tilde{T} that covers T , constructed in the proof of Proposition 4.2. This circle has the form $\tilde{P}_1 \tilde{C}_1 \tilde{P}_2 \tilde{C}_2 \cdots \tilde{P}_n \tilde{C}_n$, where the \tilde{C}_i are paths that are lifts of the

positive integral linear combination of conformally indecomposable flows. It is also true that each conformally indecomposable flow can be further decomposed into an integral linear combination of signed-graph circuit flows without conforming sign patterns. Thus each integral flow on a signed graph is an integral linear combination of circuit flows. This fact is already explicitly given in terms of a maximal independent edge set (that is, a matroid basis) in [5, Eq. (4.7) of Theorem 4.9, p. 275]. For instance, the signed graph in Figure 6 is a maximal independent set of the sesqui-Eulerian circle-tree in Figure 4. The conformally indecomposable flow in Figure 4 is further decomposed into circuit flows in Figure 7 without conforming signs. We summarize these observations in the following Corollary 4.10.

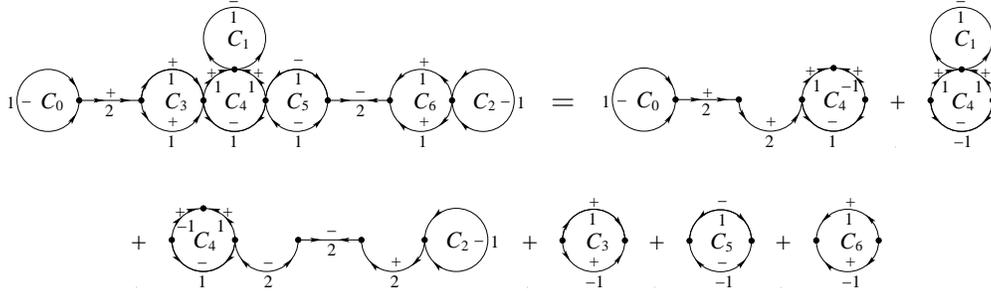


FIGURE 7. A conformally indecomposable flow is decomposed non-conformally but integrally into circuit flows.

- Corollary 4.10.** (a) *If f is a nonzero integral flow on a signed graph, then $2f$ can be conformally decomposed into a positive integral linear combination of signed-graph circuit flows.*
- (b) *Every nonzero integral flow on a signed graph can be decomposed (conformally or non-conformally) into a positive integral linear combination of signed-graph circuit flows.*

The following proposition is elementary but helps explain why there are exactly three natural types of circuit for signed graphs, as introduced by Zaslavsky [11].

Proposition 4.11. *For a signed graph Σ , the following statements are equivalent.*

- (a) *Σ is a minimal sesqui-Eulerian signed graph.*
- (b) *Σ is a minimal prime sesqui-Eulerian signed graph.*
- (c) *Σ is a minimal sesqui-Eulerian circle-tree.*
- (d) *Σ is a signed-graph circuit.*

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