How fast can Maker win in fair biased games? *

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Abstract

We study $(a : a)$ Maker-Breaker games played on the edge set of the complete graph on n vertices. In the following four games – perfect matching game, Hamilton cycle game, star factor game and path factor game, our goal is to determine the least number of moves which Maker needs in order to win these games. Moreover, for all games except for the star factor game, we show how Red can win in the strong version of these games.

1 Introduction

Let a and b be two positive integers, let X be a finite set and $\mathcal{F} \subseteq 2^X$ be a family of the subsets of X. In the $(a : b)$ *positional game* $(X, \mathcal{F})$, two players take turns in claiming a, respectively b, previously unclaimed elements of X, with one of them going first. The set X is referred to as the *board* of the game, while the elements of $\mathcal{F}$ are referred to as the *winning sets*. When there is no risk of confusion on which board the game is played, we just use $\mathcal{F}$ to denote the game. The integers a and b are referred to as *biases* of the players. When $a = b$, the game is said to be *fair*. If $a = b = 1$, the game is called *unbiased*. Otherwise, the game is called *biased*. If a player has a strategy to win against any strategy of the other player, this strategy is called *a winning strategy*. In the $(a : b)$ *Maker-Breaker* positional game $(X, \mathcal{F})$, the two players are called *Maker* and *Breaker*. Maker wins the game $\mathcal{F}$ at the moment she claims all the elements of some $F \subseteq \mathcal{F}$. If Maker did not win by the time all the elements of X are claimed by some player, then Breaker wins the game $\mathcal{F}$. In order to show that Maker wins the game as both first and second player, we will assume in this paper that Breaker starts the game (as being the first player can only be an advantage in Maker-Breaker games).

It is very natural to play Maker-Breaker games on the edge set of a given graph G (see e.g. [2, 8]). Here, we focus on the $(a : b)$ games played on the edge set of the complete graph on n vertices, K_n, where n is a sufficiently large integer. That is, in this case the board is $X = E(K_n)$.

For example: in the *connectivity game*, $\mathcal{T}_n$, the winning sets are all spanning trees of K_n; in the *perfect matching game*, $\mathcal{M}_n$, the winning sets are all independent edge sets of size $\lfloor n/2 \rfloor$ (note that in case n is odd, this matching covers all but one vertex in K_n); in the *Hamilton cycle game*, $\mathcal{H}_n$, the winning sets are all Hamilton cycles of K_n; in the k-vertex-connectivity game, $\mathcal{C}_n^k$, for $k \in \mathbb{N}$, the winning sets are all k-vertex-connected graphs on n vertices.

It is not very difficult to see that Maker wins all aforementioned unbiased games. Therefore, we can ask the following question: *How quickly can Maker win the game?* With parameter $\tau_{\mathcal{F}}(a : b)$ we denote the shortest *duration* of the $(a : b)$ Maker-Breaker game $\mathcal{F}$, i.e. the least number of moves

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t such that Maker has a strategy to win the $(a : b)$ game $\mathcal{F}$ within t moves. For completeness, we say that $\tau_{\mathcal{F}}(a : b) = \infty$ if Breaker has a winning strategy.

It was shown in [11] that, for $n \geq 4$, $\tau_{\mathcal{T}_n}(1 : 1) = n - 1$, which is optimal. In [7] it was proved that $\tau_{\mathcal{M}_n}(1 : 1) = n/2 + 1$, when n is even, and $\tau_{\mathcal{M}_n}(1 : 1) = \lceil n/2 \rceil$, when n is odd and also that $\tau_{\mathcal{H}_n}(1 : 1) \leq n + 2$ and $\tau_{\mathcal{C}_n^k}(1 : 1) = kn/2 + o(n)$. Hefetz and Stich in [10] showed that $\tau_{\mathcal{H}_n}(1 : 1) = n + 1$, and Ferber and Hefetz [4] recently showed that $\tau_{\mathcal{C}_n^k}(1 : 1) = \lfloor kn/2 \rfloor + 1$. Moreover, there are corresponding results when the mentioned games are played on graphs that are not complete, see e.g. [3].

In this paper, we are particularly interested in $(a : a)$ Maker-Breaker games on $E(K_n)$, for constant $a \geq 1$. Although these games are studied less than unbiased and $(1 : b)$ games, they are also significant. Just a slight change in the bias from $a = 1$ to $a = 2$ can completely change the outcome (and thus the course of the play) of some games (see [2]). One example is the *diameter-2* game (where the winning sets are all graphs with diameter at most 2). It was proved in [1] that Breaker wins the $(1 : 1)$ *diameter-2* game, but Maker wins the $(2 : 2)$ *diameter-2* game.

Not so much is known about fast winning strategies in fair $(a : a)$ Maker-Breaker games, where a can be greater than 1. From the results in [6, 9], we obtain that in the connectivity game $\tau_{\mathcal{T}_n}(a : a) = \lceil (n - 1)/a \rceil$.

Our research is concentrated on fast winning strategies in four $(a : a)$ Maker-Breaker games, for $a \in \mathbb{N}$. Firstly, we take a look at the $(a : a)$ perfect matching game, $\mathcal{M}_n$. The case $a = 1$ is already proved in [7], and we show the following theorem for all $a \geq 2$.

Theorem 1.1. *Let $a \in \mathbb{N}$. Then for every large enough n the following is true for the $(a : a)$ Maker-Breaker perfect matching game:*

$$\tau_{\mathcal{M}_n}(a : a) = \begin{cases} \frac{n}{2a} + 1, & \text{if } a = 1 \text{ and } n \text{ is even,} \\ \left\lceil \frac{n}{2a} \right\rceil - 1, & \text{if } 2a \mid n - 1 \\ \left\lceil \frac{n}{2a} \right\rceil, & \text{otherwise.} \end{cases}$$

Secondly, we analyse the $(a : a)$ Maker-Breaker Hamilton cycle game, $\mathcal{H}_n$, and prove the following result for $a \geq 2$. The case $a = 1$ is proved in [10].

Theorem 1.2. *Let $a \in \mathbb{N}$. Then for every large enough n the following is true for the $(a : a)$ Maker-Breaker Hamilton cycle game:*

$$\tau_{\mathcal{H}_n}(a : a) = \begin{cases} \frac{n}{a} + 1, & \text{if } a = 1 \text{ or } (a = 2 \text{ and } n \text{ is even}), \\ \left\lceil \frac{n}{a} \right\rceil, & \text{otherwise.} \end{cases}$$

We study two more $(a : a)$ Maker-Breaker games whose winning sets are spanning graphs. More precisely, we are interested in factoring the graph K_n with stars and paths. For fixed $k \geq 2$, let P_k denote a path with k vertices, and let S_k denote a star with $k - 1$ leaves. Now, for all large enough n, such that $k \mid n$, we are interested in finding winning strategies in the $(a : a)$ P_k-factor game, denoted by $\mathcal{P}_{k,n}$, and in the $(a : a)$ S_k-factor game, denoted by $\mathcal{S}_{k,n}$, where the winning sets are all path factors and star factors of K_n, respectively, on k vertices. We show the following.

Theorem 1.3. *Let $a \in \mathbb{N}$ and $k \in \mathbb{N}$. Then for every large enough n, such that $k \mid n$, the following is true for the $(a : a)$ Maker-Breaker P_k-factor game:*

$$\tau_{\mathcal{P}_{k,n}}(a : a) = \left\lceil \frac{(k-1)n}{ka} \right\rceil.$$

Theorem 1.4. *Let $a \geq 1$ and $k \geq 3$ be integers. Then for every large enough n, such that $k \mid n$, the following is true for the $(a:a)$ Maker-Breaker S_k-factor game:*

$$\tau_{\mathcal{S}_{k,n}}(a:a) \leq \begin{cases} \left\lceil \frac{(k-1)n}{ka} \right\rceil, & \text{if } a \nmid \frac{(k-1)n}{k}, \\ \frac{(k-1)n}{ka} + 1, & \text{otherwise.} \end{cases}$$

Strong games. We also look at another type of positional games. In the *strong* positional game $(X, \mathcal{F})$, the two players are called *Red* and *Blue*, and Red starts the game. The winner of the game is the *first* player who claims all the elements of one $F \in \mathcal{F}$. If none of the players manage to do that before all the elements of X are claimed, the game ends in a *draw*.
By the *strategy stealing argument* (see [2]), Blue cannot have a winning strategy in the strong game. So, in every strong game, either Red wins, or Blue has a drawing strategy. For the games where the draw is impossible, we know that Red wins. Unfortunately, the existence of Red's strategy tells us nothing about how Red should play in order to win. Finding explicit winning strategies for Red can be very difficult. The results in [4, 5] show that fast winning strategies for Maker in certain games can be used in order to describe the winning strategies for Red in the strong version of these games.
If Maker can win *perfectly fast* in the $(a:a)$ game $\mathcal{F}$, i.e. if the number of moves, t, she needs to win is equal to $\lceil \min(|F| : F \in \mathcal{F})/a \rceil$, that immediately implies Red's win in the strong game. Indeed, as Red starts the game, Blue has no chance to fully claim any winning set in less than t moves. Thus, Red can play according to the strategy of Maker, without worrying about Blue's moves, by which Red will claim a winning set in t moves, thus winning the game.
From Theorem 1.1, we see that Maker can win perfectly fast in the $(a:a)$ perfect matching game in all cases, but in case $a = 1$. Therefore, we immediately see that for $a \neq 1$, Red has a winning strategy for the corresponding strong game. For $a = 1$ the proof that Red wins the strong game appears in [4]. Similarly to the perfect matching game, from Theorem 1.2 we can immediately see that Red has a winning strategy for the strong $(a:a)$ Hamilton cycle game in all but two cases – the case $a = 1$ and the case $a = 2$ and n is even. The case $a = 1$ appears in [4], and for the remaining case, we prove the following theorem.

Theorem 1.5. *For every large enough even n the following is true: Red has a strategy for the $(2:2)$ Hamilton cycle game to win within $n/2 + 1$ rounds.*

Every P_k-factor of K_n, for given $k \in \mathbb{N}$ such that $k \mid n$, has to have $n(k-1)/k$ edges. Therefore, from Theorem 1.3, we obtain that Maker can win perfectly fast in $(a:a)$ game $\mathcal{P}_{k,n}$ and Red can use the winning strategy of Maker in this game to win in the corresponding strong game.

Notation and terminology. Our graph-theoretic notation is standard and mostly follows that of [13]. In particular, we use the following. We write $[n] := \{1, 2, \ldots, n\}$. For a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and $e_G(A, B) := |E_G(A, B)|$. Moreover, $E_G(A) := E_G(A, A)$ and $e_G(A) := |E_G(A)|$. For a vertex $x \in V(G)$ and a set $S \subseteq V(G)$, $N_G(x, S) = \{u \in S : ux \in E(G)\}$ denotes the set of *neighbours* of the vertex x in the set S with respect to (w.r.t.) G. We set $N_G(x) := N_G(x, V(G))$. Moreover, $d_G(x, S) := |N_G(x, S)|$ denotes the *degree* of x into S, while $d_G(x) = |N_G(x)|$ denotes the *degree* of x in the graph G. Whenever there is no risk of confusion, we omit the subscript G in the notation above. Given a graph $G = (V, E)$, we let $\overline{G} = (V, \overline{E})$ denote its *complement*, where $\overline{E} := \{xy \notin E : \ x, y \in V\}$. For $e \in E(G)$, we set $G - e := (V, E \setminus \{e\})$. For $S \subseteq V$, $G[S]$ denotes the subgraph of G *induced* by S, i.e. $G[S] = (S, E_S)$ where $E_S := \{xy \in E(G) : x, y \in S\}$. Given another graph H on the same vertex set as G, we let $G - H := (V, E(G) \setminus E(H))$. Given a path P in a graph G, we let $\text{End}(P)$ denote the set of its endpoints. For every family $\mathcal{P}$ of disjoint paths, we set $\text{End}(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} \text{End}(P)$ and $E(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} E(P)$. With P_n, S_n, K_n and $K_{n,n}$ we denote the path on n vertices, the star on n vertices (and $n-1$ edges), the complete graph on n vertices and the complete bipartite graph with vertex classes of size n each, respectively. A cycle in a graph G is called *Hamilton cycle* if it passes through every vertex of G; in case such a cycle exists, G is called *Hamiltonian.* A set of pairwise disjoint edges in a graph G is called a *matching*, and we call it a *perfect matching* if it covers every vertex (but at most one, in case $v(G)$ is odd).

Assume an $(a : a)$ Maker-Breaker game is in progress. By M we denote Maker's graph and by B we denote Breaker's graph. If an edge is unclaimed by any of the players we call it *free.* Each *round* (but maybe the very last one) consists of exactly one *move* of Breaker followed by a move of Maker, where the player claims a edges. Claiming only one of these edges is called a *step* in the game.

The rest of the paper is organized as follows. In Section 2 we start with some preliminaries. In Section 3, we prove Theorem 1.1, in Section 4 we prove Theorem 1.2, in Section 5 we prove Theorem 1.5 and in Section 6 we prove Theorem 1.3. Finally, in Section 7, we prove Theorem 1.4.

2 Preliminaries

In the strategy for the $(a : a)$ Hamilton cycle game, given later in Section 4, Maker tries to maintain a linear forest (a collection of vertex disjoint paths) as long as possible. We show that she can do so for the first $\lceil n/a \rceil - 1$ rounds. Within the following (at most) two rounds, motivated by the rotation techniques of Posá (see e.g. [12]), she then adds further edges to her graph and obtains a Hamilton cycle. In order to guarantee that this is possible, we prove the following statements.

Proposition 2.1. *Let P_1, P_2 be vertex disjoint paths in some graph G, and $d_{\overline{G}}(v) \le v(P_1)/2 - 1$ for every $v \in \bigcup_{i=1}^{2} End(P_i)$. Then there exists an edge $f \in E(P_1)$ and two edges $e_1, e_2 \in E(G)$ such that $((E(P_1) \cup E(P_2)) \setminus \{f\}) \cup \{e_1, e_2\}$ induces a path P of G with $V(P) = V(P_1) \cup V(P_2)$ and $End(P) \subseteq End(P_1) \cup End(P_2)$.*

Proof Let $\mathrm{End}(P_i) = \{x_i, y_i\}$. For a vertex $v \in V(P_1) \setminus \{x_1\}$ we define its left-neighbour v^- to be the unique neighbour of v on the subpath from x_1 to v. Similarly, for a vertex $v \in V(P_1) \setminus \{y_1\}$ we define its right-neighbour v^+ to be the unique neighbour of v on the subpath from v to y_1. Let $S \subseteq V(P_1)$ be some subset, then we set $S^- := \{v^- : x_1 \neq v \in S\}$. Now, let $S_1 := N_G(x_1) \cap V(P_1)$ and $S_2 := N_G(x_2) \cap V(P_1)$. Then, by assumption $|S_1^-| \ge v(P_1) - 1 - d_{\overline{G}}(x_1) \ge v(P_1)/2$ and $|S_2| \ge v(P_1) - d_{\overline{G}}(x_2) > v(P_1)/2$. Thus, $|S_1^-| + |S_2| > v(P_1)$ and therefore $S_1^- \cap S_2 \neq \emptyset$. Let $z \in S_1^- \cap S_2$. Then $((E(P_1) \cup E(P_2)) \setminus \{zz^+\}) \cup \{x_1 z^+, x_2 z\}$ induces a path P as claimed. □

Corollary 2.2. *Let G be a graph on n vertices; let $P_1, P_2, \ldots, P_t$ be pairwise disjoint paths in G such that $\bigcup_{i=1}^{t} V(P_i) = V(G)$. Let $d_{\overline{G}}(v) \le n/(2t) - 1$ for every $v \in \bigcup_{i=1}^{t} End(P_i)$. Then there exists a set $E^* \subseteq E(G)$ of $2t$ edges such that $\bigcup_{i=1}^{t} E(P_i) \cup E^*$ contains a Hamilton cycle of G.*

Proof W.l.o.g. let $v(P_1) = \max\{v(P_i) : 1 \le i \le t\} \ge n/t$ and therefore $d_{\overline{G}}(v) \le v(P_1)/2 - 1$ for every $v \in \bigcup_{i=1}^{t} \mathrm{End}(P_i)$. Then, applying Proposition 2.1 for P_1 and P_2, we can find two edges $e_1, e_1' \in E(G)$ such that $E(P_1) \cup E(P_2) \cup \{e_1, e_e'\}$ contains a path P_{12} with $V(P_{12}) = V(P_1) \cup V(P_2)$ and $\mathrm{End}(P_{12}) \subseteq \mathrm{End}(P_1) \cup \mathrm{End}(P_2)$. Now, set $P_1' = P_{12}, P_2' = P_3, P_3' = P_4, \ldots, P_{t-1}' = P_t$. We can repeat this argument for P_1' and P_2', thus reducing the number of paths again by 1, using at most two further edges from G. Doing this iteratively for $t-1$ iterations (using at most $2t-2$ edges) we finally end up with a path P such that $V(P) = \bigcup_{i=1}^{t} V(P_i) = V(G)$ and $\mathrm{End}(P) \subseteq \bigcup_{i=1}^{t} \mathrm{End}(P_i)$. Let $\mathrm{End}(P) = \{x, y\}$. By assumption, $d_G(x), d_G(y) \ge n/2$. Analogously to Bondy's proof of Dirac's Theorem for the existence of Hamilton cycles (see e.g. [13]), we can find (at most) two edges f_1, f_2 in G such that $E(P) \cup \{f_1, f_2\}$ contains a Hamilton cycle of G. □

3 Weak Perfect Matching Game

The main goal of this section is to prove Theorem 1.1. However, we prove a slightly stronger result which roughly says that Maker can claim a perfect matching rapidly even if we play on a nearly complete bipartite graph. This statement is used later in Section 7.

Proposition 3.1. *Let $a, C \in \mathbb{N}$ be constants with $a \geq 2$, then for every large enough n the following holds: Let $G \supseteq K_{n,n} - H$ be a graph such that $e(H) \leq C$. Playing an $(a : a)$ Maker-Breaker game on G, Maker has a strategy to gain a perfect matching of G within at most $\lfloor n/a \rfloor + 1$ rounds. In case $G = K_{2n}$, she can even win within $\lceil n/a \rceil$ rounds.*

Before we give the proof of this proposition, let us first see how Theorem 1.1 can be deduced.

Proof of Theorem 1.1. For $a = 1$ the proof is given in [7]. So, let $a \geq 2$. Assume first that $n = 2k$ is even. Since a perfect matching of K_n has $n/2$ edges, the game obviously lasts at least $\lceil n/(2a) \rceil$ rounds. By Proposition 3.1, Maker has a strategy to win within $\lceil k/a \rceil = \lceil n/(2a) \rceil$ rounds. Assume then n to be odd. According to the rules, Maker wins if she claims a matching covering all but one vertex, i.e. a matching of size $(n-1)/2$. This obviously takes at least

$$\left\lceil \frac{n-1}{2a} \right\rceil = \begin{cases} \lceil \frac{n}{2a} \rceil, & \text{if } 2a \nmid n-1 \\ \lceil \frac{n}{2a} \rceil - 1, & \text{if } 2a \mid n-1 \end{cases}$$

rounds. However, Maker for her strategy can consider to play on the graph $K_{n-1} \subseteq K_n$. By the previous argument this takes her at most $\lceil (n-1)/(2a) \rceil$ rounds, provided n is large enough. $\square$

Proof of Proposition 3.1. Assume that Breaker starts the game. Let $X \cup Y$ be the bipartition of $V(G) = V(K_{n,n})$. Further assume that H belongs to Breaker's graph.

Maker's strategy is split into two stages.

Stage I. The first stage lasts exactly $\lceil n/a \rceil - 1$ rounds. For $0 \leq i \leq \lceil n/a \rceil - 1$, Maker ensures that immediately after her i^{th} move her graph consists of a matching $M_i \subseteq E(X, Y)$ of size $i \cdot a$ and a set of isolated vertices $I_i = V \setminus V(M_i)$ such that the following properties hold:

$(P1)$ $e_B(I_i) \leq \max\{C - ia, 0\}$,

$(P2)$ $\forall v \in I_i : d_B(v) < n/8$.

Maker chooses the edges $e_1 = e_1^{(i)}, \ldots, e_a = e_a^{(i)}$ of her i^{th} move in the following way:
She sets $\delta = 0$ if $i \leq \lceil C/a \rceil$, and $\delta = 1$ otherwise. Then for every $1 \leq j \leq a - \delta$ she sets $I_{i-1}^{(j)} = I_{i-1} \setminus V(e_1, \ldots, e_{j-1})$ and chooses $e_j = x_j y_j \in E(I_{i-1}^{(j)}) \cap E(X, Y)$ such that

- x_j maximizes $d_B(z, I_{i-1}^{(j)})$ over all choices $z \in I_{i-1}^{(j)}$, and
- y_j maximizes $d_B(z, I_{i-1}^{(j)})$ over all choices $z \in I_{i-1}^{(j)}$ with $x_j z \in E(X, Y) \setminus E(B)$.

Afterwards, if $i > \lceil C/a \rceil$ (and so $\delta = 1$), she sets $E(I_{i-1}^{(a)}) = I_{i-1} \setminus V(e_1, \ldots, e_{a-1})$ and chooses an unclaimed edge $e_a = x_a y_a \in E(I_{i-1}^{(a)}) \cap E(X, Y)$ in such a way that x_a maximizes $d_B(z, V)$ over all choices $z \in I_{i-1}^{(a)}$. Finally, she sets $I_i = I_{i-1}^{(a+1)} := I_{i-1} \setminus V(e_1, \ldots, e_a)$.

Stage II. If $a \nmid n$ or $G = K_{2n}$, Maker plays one further round to complete a perfect matching of G. Otherwise, she does so within two more rounds. The details of how she can do this are given later in the proof.

It is evident that, if Maker can follow the strategy, she wins the game within the claimed number of rounds. Thus, it remains to show that Maker can follow the proposed strategy.

Stage I. We prove, by induction on i, that Maker can follow the strategy of Stage I and ensure the mentioned properties to hold immediately after her i^{th} move. For $i = 0$ there is nothing to prove. So, let $i > 0$. Assume that $(P1)$ and $(P2)$ were true immediately after Maker's $(i-1)^{\text{st}}$ move for the matching M_{i-1} and the set I_{i-1}. To show that Maker can follow the strategy for round i, we inductively prove the following claim.

Claim 3.2. *For every $1 \leq j \leq a - \delta$, Maker can claim e_j and ensure that immediately after claiming that edge, $e_B(I_{i-1}^{(j+1)}) \leq \max\{C - (i-2)a - 2j, a - 2j, 0\}$ holds.*

Indeed, if this claim is true, then Maker can claim the first $a - \delta$ edges, as described. If $\delta = 1$ and thus $i > \lceil C/a \rceil$, then after claiming the first $a - 1$ edges, we have $e_B(I_{i-1}^{(a)}) = 0$. That is, Breaker has no edges among the remaining isolated vertices, and thus Maker can claim e_a as described.

Proof of Claim 3.2. We apply induction on j, the number of steps in round i. At first, let $j = 1$. If $i \leq \lceil C/a \rceil$, then by (P1) after Maker's $(i-1)^{\text{st}}$ move and since Breaker has bias a, we have $e_B(I_{i-1}) \leq \max\{C - (i-2)a, a\} < n/2 < |I_{i-1} \cap X| = |I_{i-1} \cap Y|$ right before Maker's first step in round i. If otherwise $\lceil C/a \rceil < i \leq \lceil n/a \rceil - 1$, then analogously we have $e_B(I_{i-1}) \leq a < n - (i-1)a = |I_{i-1} \cap X| = |I_{i-1} \cap Y|$ right before Maker's first step in round i. So, in either case, looking at Breaker's graph, none of the vertices from $I_{i-1} \cap X$ can be adjacent to all vertices from $I_{i-1} \cap Y$, and vice versa. So, Maker can claim e_1 as given by the strategy. Now, if $e_B(I_{i-1}^{(1)}) \geq 2$, then by the choice of e_1 it easily follows that $d_B(x_1, I_{i-1}^{(1)}) + d_B(y_1, I_{i-1}^{(1)}) \geq 2$. But this gives $e_B(I_{i-1}^{(2)}) \leq e_B(I_{i-1}^{(1)}) - 2 \leq \max\{C - (i-2)a - 2, a - 2, 0\}$. Otherwise, if $e_B(I_{i-1}^{(1)}) \leq 1$, then e_1 is adjacent to all Breaker edges in $I_{i-1}^{(1)}$, ensuring $e_B(I_{i-1}^{(2)}) = 0$.

Now, let $j > 1$. After e_{j-1} is claimed, $e_B(I_{i-1}^{(j)}) \leq \max\{C - (i-2)a - 2(j-1), a - 2(j-1), 0\}$ holds by induction. If $i \leq \lceil C/a \rceil$, then $e_B(I_{i-1}^{(j)}) < n/2 < |I_{i-1}^{(j)} \cap X| = |I_{i-1}^{(j)} \cap Y|$. If $i > \lceil C/a \rceil$, then $e_B(I_{i-1}^{(j)}) \leq \max\{a - 2(j-1), 0\} < |I_{i-1} \cap Y| - (j-1) = |I_{i-1}^{(j)} \cap X| = |I_{i-1}^{(j)} \cap Y|$. So, when Maker wants to claim her edge e_j, none of the vertices from $I_{i-1}^{(j)} \cap X$ can be adjacent to all vertices from $I_{i-1}^{(j)} \cap Y$, and vice versa. So, as for the induction start, she can claim e_j and ensure $e_B(I_{i-1}^{(j+1)}) \leq \max\{e_B(I_{i-1}^{(j)}) - 2, 0\} \leq \max\{C - (i-2)a - 2j, a - 2j, 0\}$. □

When Maker claimed all the a edges, she has a matching in $E(X,Y)$ of size $|M_i| = |M_{i-1}| + a = ia$. Thus, to finish the discussion of Stage I, it remains to show that the mentioned properties are maintained. Using the claim, (P1) is given as follows: If $i \leq \lceil n/a \rceil$, then immediately after Maker's i^{th} move, $e_B(I_i) = e_B(I_{i-1}^{(a+1)}) \leq \max\{C - (i-2)a - 2a, a - 2a, 0\} = \max\{C - ia, 0\}$. If $i > \lceil n/a \rceil$, then immediately after claiming e_{a-1} we have $e_B(I_{i-1}^{(a)}) \leq \max\{C - (i-2)a - 2(a-1), a - 2(a-1), 0\} = 0$. In particular, $e_B(I_i) = 0$ follows then. Assume now that (P2) is violated. Then after Maker's move there needs to be a vertex $v_i \in I_i$ of degree at least $n/8$ in Breaker's graph. In particular, $i \geq n/(8a)$. However, Maker then in each of the last twenty rounds chose (with her last edge) an isolated vertex in her graph of maximum degree in Breaker's graph to be matched and therefore excluded from the set of isolated vertices. That means, if after round i there really was such a vertex v_i, then twenty rounds before there must have been at least twenty vertices of degree at least $n/8 - 20a$ in Breaker's graph, since otherwise v_i would have been matched earlier. But then, provided n is large enough, Breaker has claimed more than $2n$ edges, which is in contradiction to the number of rounds played so far.

Stage II. We need to show that Maker can complete a perfect matching within one round or two rounds, respectively. From now on, let $t = |I_{\lceil n/a \rceil - 1} \cap X| = |I_{\lceil n/a \rceil - 1} \cap Y|$ be the number of isolated vertices in X and Y, respectively, at the moment when Maker enters Stage II.

Case 1. $\mathbf{a \nmid n}$. Observe that $t = n - |M_{\lceil n/a \rceil - 1}| \leq a - 1$ holds in this case.
Case 1.1. Assume first that $t \leq a/2$. Maker then partitions the set of isolated vertices into t pairs $\{v_1, w_1\}, \ldots, \{v_t, w_t\}$ with $v_i \in Y$ and $w_i \in X$. By Property $(P2)$, she then finds distinct edges $m_1 = x_1 y_1, \ldots\ m_t = x_t y_t$ in $M_{\lceil n/a \rceil - 1}$ such that for each $1 \leq i \leq t$, $x_i \in X$, $y_i \in Y$ and the edges $v_i x_i$ and $w_i y_i$ do not belong to Breaker's graph. She then claims the edges $v_i x_i$ and $w_i y_i$, in total $2t \leq a$. By this, she creates a perfect matching of G.
Case 1.2. Assume then that $a/2 < t \leq a - 1$. Let F be the graph induced by all free edges between $I_{\lceil n/a \rceil - 1} \cap X$ and $I_{\lceil n/a \rceil - 1} \cap Y$. Since by (P2) Breaker can have at most a edges among all isolated vertices, then $e(F) \geq t^2 - a$. Thus, the smallest vertex cover in F is of size at least

$(t^2-a)/t = t - a/t > t-2$. Therefore, by the theorem of König-Egeváry (see e.g. [13]) F has a matching of size at least $t-1$. Maker claims this matching. This way, she creates a matching of G of size $n-1$, and two isolated vertices $v \in Y$ and $w \in X$. Again, using Property (P1), she finds an edge xy in her matching, with $x \in X$ and $y \in Y$, such that vx and wy are unclaimed. She claims these, and then she is done as before. In total, she claims at most $(t-1)+2 \leq a$ edges.
Case 2. a|n. Observe first that $t = a$. We now want to finish the perfect matching within one round if $G = K_{2n}$. Otherwise, it is enough to finish within two further rounds.
Case 2.1. Assume that $G = K_{2n}$. When Maker enters Stage II, by Property (P1), Breaker claims at most a edges in $G[I_{\lceil n/a\rceil -1}]$. So, there is a bipartite subgraph $G' \subset G[I_{\lceil n/a\rceil -1}]$ with classes of size a, such that Breaker claims less than a edges of G'. Set $F := G' \setminus B$. Then, analogously to Case 1.2, F has a matching of size at least $(t^2-(a-1))/t > a-1$. Thus, within one round, Maker can claim a matching of size a in F, which completes a perfect matching of G.
Case 2.1. Finally, assume $G \neq K_{2n}$. Let F be the graph induced by all free edges between $I_{\lceil n/a\rceil -1} \cap X$ and $I_{\lceil n/a\rceil -1} \cap Y$. Since by (P2) Breaker can have at most a edges among all isolated vertices, we analogously conclude that F contains a matching of size at least $a-1$. Maker claims such a matching in the first round of Stage II, and afterwards, she has a matching of G of size $n-1$, and two isolated vertices $v \in Y$ and $w \in X$. If vw is free, she claims it and wins. Otherwise, in the next round, analogously to Case 1.2, she finds an edge xy in her matching, with $x \in X$ and $y \in Y$, such that vx and wy are unclaimed. She then claims these two edges. □

4 Weak Hamilton Cycle Game

Proof of Theorem 1.2. For $a = 1$ the proof is given in [10]. So, let $a \geq 2$ and assume that Breaker starts the game. Since a Hamilton cycle has n edges, the game obviously lasts at least $\lceil n/a\rceil$ rounds. Moreover, one easily verifies that, if $a = 2$ and n is even, $\tau_{\mathcal{H}_n}(a:a) \geq \lceil n/a\rceil + 1$. Indeed, assume in this case that Maker had a strategy to create a Hamilton cycle within $n/2$ rounds. Then, after her $(n/2-1)^{\text{st}}$ round her graph would consist of two paths P_1 and P_2 (maybe one of length zero). In order to win in the next round, she would need to claim two edges between $\text{End}(P_1)$ and $\text{End}(P_2)$ in such a way that a Hamilton cycle is created. However, before this, Breaker can claim all edges of $E(x, \text{End}(P_2))$ for some $x \in \text{End}(P_1)$, therefore delaying Maker's win by at least one further round, in contradiction to the assumption.

Thus, it remains to prove that $\tau_{\mathcal{H}_n}(a:a) \leq \lceil n/a\rceil + 1$ if $a = 2$ and n is even, and $\tau_{\mathcal{H}_n}(a:a) \leq \lceil n/a\rceil$ otherwise (for large enough n depending on a).

Maker's strategy. The main idea of Maker's strategy is to create a linear forest, i.e. a graph which only consists of vertex disjoint paths. Her strategy is divided into three stages. In the first stage, she starts with a perfect matching, similarly to the strategy given for $a = 1$ in [7]. Then, in the second stage, she connects the edges of the matchings to create larger paths. Finally, in the third stage, when the number of paths is at most a, she completes a Hamilton cycle in at most two further rounds, making use of Proposition 2.1 and Corollary 2.2.

Assume the game is in progress. By $\mathcal{P}$ we denote the set of Maker's (maximal) paths, where an isolated vertex is seen as a path of length zero (empty path). Throughout the game this set is updated, meaning that whenever Maker connects the endpoints of two paths $P_1, P_2 \in \mathcal{P}$ by an edge e, we delete P_1 and P_2 from $\mathcal{P}$ and add the new path induced by $E(P_1) \cup E(P_2) \cup \{e\}$. By p_i we denote the size of $\mathcal{P}$ immediately after Maker's i^{th} move. Observe that, as long as Maker's graph is a linear forest, $p_i = n - ia$ holds.
Now, let e be some edge that is incident with the endpoints of two different paths from $\mathcal{P}$. We say that e is *good*, if it is free; otherwise we call it *bad*. By br_i we denote the number of bad edges right after Maker's i^{th} move. If an edge e is good, we set $D(e)$ to be the number of bad edges adjacent to e. Thus, $D(e)$ depends on the dynamic family $\mathcal{P}$ and on the considered round.

Stage I. Within $\lceil n/(2a)\rceil$ or $\lceil n/(2a)\rceil + 1$ rounds, Maker claims a collection of vertex disjoint paths, each of length at least 1, such that every vertex is incident with one of these paths, and there is no further Maker's edge. The details of how she can do this follow later in the proof. Afterwards, Maker proceeds with Stage II.

Stage II. Let $t_1 \in \{\lceil n/2a \rceil + 1, \lceil n/2a \rceil + 2\}$ be the round in which Maker enters Stage II. Let

$$t_2 := \lceil n/a \rceil - \lceil n/(6a^2) \rceil.$$

Maker now connects the paths of her collection $\mathcal{P}$. To be able to do so, she needs to guarantee that the number of bad edges does not become too large. For that reason, if a lot of bad edges exist, she claims good edges that are adjacent to many bad edges (part IIa). Moreover, similarly to the perfect matching game, she maintains some degree condition by caring about large degree vertices in Breaker's graph (part IIb). To be more precise:
For every $t_1 \leq i \leq \lceil n/a \rceil - 1$, in her i^{th}, Maker claims edges $e_1 = e_1^{(i)}, \ldots, e_a = e_a^{(i)}$ one after the other. The j^{th} edge e_j is claimed according to the following rules:

- In order to choose e_j she considers the following two cases.
  - **IIa.** If $i \leq t_2$ or $i \geq t_2 + 8$, Maker claims a good edge e_j such that $D(e_j)$ is maximal.
  - **IIb.** Otherwise, if $t_2 + 1 \leq i \leq t_2 + 7$, she chooses a vertex $x_j \in \text{End}(\mathcal{P})$ of maximal degree in Breaker's graph and then claims an arbitrary good edge $e_j = x_j y_j$.
- By claiming e_j, Maker connects two paths $P_{j,1},\ P_{j,2} \in \mathcal{P}$. Accordingly, she then deletes $P_{j,1},\ P_{j,2}$ from $\mathcal{P}$, and adds to $\mathcal{P}$ the path induced by $E(P_{j,1}) \cup E(P_{j,2}) \cup \{e_j\}$. She updates the sets of good and bad edges and the values $D(\cdot)$ before she proceeds with e_{j+1}.

Stage III. If $a = 2$ and n is even, Maker claims a Hamilton cycle within the next two rounds. Otherwise, she does so within one round. The details of how Maker can do this, follow later in the proof.

It is evident that, if Maker can follow the strategy, she wins the game within the desired number of rounds. Thus, it remains to show that Maker can follow the proposed strategy. Before that, let us prove the following propositions which bound the number of bad edges throughout the game.

Proposition 4.1. *At any point of the game, when $|\mathcal{P}| \geq 2$ and*

- *each $v \in End(\mathcal{P})$ is incident with at least one good edge,*
- *$D(e) \leq 1$ holds for all good edges e.*

Then the number of bad edges is at most 1.

Proof By assumption, each vertex in $\text{End}(\mathcal{P})$ is incident with at most one bad edge. If there were at least 2 bad edges, they would form a matching. Maker could then find a good edge e' that is adjacent to two of these edges, in contradiction to $D(e') \leq 1$. □

Proposition 4.2. *At any point of the game, when $|\mathcal{P}| \geq 4$ and*

- *each $v \in End(\mathcal{P})$ is incident with at least one good edge,*
- *$D(e) \leq 2$ holds for all good edges e.*

Then the number of bad edges is at most $|\mathcal{P}|$.

Proof By assumption, each vertex in $\text{End}(\mathcal{P})$ is incident with at most two bad edges. If each vertex is incident with at most one bad edge, then the bad edges form a matching on $\text{End}(\mathcal{P})$. Thus, there can be at most $|\mathcal{P}|$ such edges. So, assume that there is a vertex x incident with exactly two bad edges xy_1 and xy_2. Then the number of bad edges is at most 4, which can be seen as follows: Let z be the other endpoint of the path that x belongs to. If y_1 and y_2 belong to the same path in $\mathcal{P}$, then the only further edges that could be bad are zy_1 and zy_2. Indeed, if there was another endpoint $w \notin \{x, y_1, y_2, z\}$ incident with some bad edge, then $D(xw) \geq 3$, a contradiction. Otherwise, if y_1 and y_2 belong to different paths, then similarly one observes that y_1y_2 is the only edge that could be bad besides xy_1 and xy_2. □

Proposition 4.3. *Let br and $p = |\mathcal{P}| > 2$ be the numbers of bad edges and Maker's paths, respectively, immediately before Maker claims some edge e. If $br < 2p - 2$, then the following holds:*

i) Each vertex in $End(\mathcal{P})$ is incident with at least one good edge.

ii) If e is a good edge such that $D(e)$ is maximal (at the moment when e is chosen), and if br' and p' are the numbers of bad edges and Maker's paths immediately after Maker claimed e, then again $br' < 2p' - 2$. Moreover, $br' = 0$ if $D(e) \leq 1$.

Proof Part i) of the proposition holds, since for each vertex the number of incident good edges and the number of incident bad edges sums up to $2(p - 1)$.
So, consider part ii). If $D(e) \geq 2$, then Maker gets rid of at least two bad edges by claiming e which gives $br' \leq br - 2 < 2p - 4 = 2p' - 2$. Otherwise, if $D(e) \leq 1$, then $br \leq 1$, by Proposition 4.1. By the choice of e, it follows that $br' = 0$. □

The last proposition turns out to be very helpful for the discussion of Stage II. The reason is that whenever $br < 2p - 2$ holds, then it tells us that Maker can claim a good edge as asked by the strategy of Stage IIa. Moreover, after Maker claimed such an edge (and thus $br' < 2p' - 2$ holds), we can reapply this proposition, and continue this way until Maker's move in Stage IIa is over. With all the previous propositions in hand, let us now prove that Maker can follow the strategy.

Stage I. If n is even, Maker plays $\lceil n/(2a) \rceil$ rounds according to the strategy given for the perfect matching game. This produces a perfect matching (thus every vertex is covered) plus at most a further edges that, together with the matching edges, form a linear forest. In case this strategy stops in round $\lceil n/(2a) \rceil$ before Maker claimed exactly a edges, Maker claims further edges that maintain a linear forest. Note that this is possible since n is large enough and Breaker so far claimed at most $n/2$ edges. If n is odd, Maker plays $\lceil n/(2a) \rceil$ rounds on $K_{n-1} \subseteq K_n$, analogously occupying a family of paths of length at least 1, covering all vertices of K_{n-1}. In the next round, she connects the unique vertex $v \in V(K_n) \setminus V(K_{n-1})$ to an endpoint of one of her paths, and afterwards claims $a - 1$ further edges such that her graph remains a linear forest. Again, this is possible, since Breaker so far claimed at most $n/2 + 2a$ edges.

Stage II. Observe first that, when Maker enters Stage II, her collection $\mathcal{P}$ consists of $p_{t_1-1} = n - (t_1 - 1)a \geq n/2 - 2a$ paths. Moreover, immediately after her previous move the number of bad edges was $br_{t_1-1} \leq (t_1 - 1)a \leq n/2 + 3a \leq p_{t_1-1} + 5a$. The following claim splits Stage II naturally into three parts and ensures for each part that Maker can follow the proposed strategy.

Claim 4.4. *For Stage II the following is true.*

(a) For every $t_1 \leq i \leq t_2$, Maker can make her i^{th} move according to Stage IIa. After that move,

$$br_i \leq p_i - a \quad \text{or} \quad br_i - p_i < br_{i-1} - p_{i-1}. \tag{1}$$

In particular, $br_i \leq p_i + 5a$ for every $t_1 \leq i \leq t_2$, and $br_i \leq p_i - a$ for every $t_1 + 6a \leq i \leq t_2$.

(b) For every $t_2 + 1 \leq i \leq t_2 + 7$, Maker can follow her i^{th} move according to Stage IIb, and after that $br_i \leq p_i + 13a$.

(c) For every $t_2 + 8 \leq i \leq \lceil n/a \rceil - 1$, Maker can follow her i^{th} move according to Stage IIa. After that move,

$$br_i \leq \max\{p_i - a, 0\} \quad \text{or} \quad br_i - p_i < br_{i-1} - p_{i-1}. \tag{2}$$

In particular, $br_i \leq p_i + 13a$ always, and $br_i \leq \max\{p_i - a, 0\}$ for every $i \geq t_2 + 14a + 8$.

Proof

(a) We prove the statement by induction on i. When Maker has to make her i^{th} move, by induction hypothesis, she sees at most $br_{i-1} + a \leq p_{i-1} + 6a < 2p_{i-1} - 2$ bad edges on the board. Thus, by Proposition 4.3, Maker can follow her strategy and claim $e_1, \ldots, e_a$. Now, let D_j denote the value of $D(e_j)$ at the moment when e_j is chosen, and observe that $D_1 \geq D_2 \geq \ldots \geq D_a$.
Case 1. If $D_a \leq 1$, then by Proposition 4.3 *ii*), we obtain $br_i = 0$.
Case 2. If $D_1 \geq 3$ and $D_a \geq 2$, then Breaker in his i^{th} move created at most a bad edges, while Maker gets rid of $\sum_{j=1}^{a} D_j \geq 3 + 2(a-1)$ bad edges. We conclude that $br_i - p_i \leq br_{i-1} + a - (3 + 2(a-1)) - (p_{i-1} - a) < br_{i-1} - p_{i-1}$.
Case 3. If $D_1 = D_a = 2$, then after Breaker's i^{th} move there were at most p_{i-1} bad edges, as given by Proposition 4.2. Maker in her i^{th} move decreases the number of bad edges by $\sum_{i=1}^{a} D_i = 2a$, while the number of paths only decreases by a. This gives $br_i \leq p_i - a$.

Thus, in either case (1) holds. Finally, it follows that $br_i \leq p_i + 5a$ for all i and $br_i \leq p_i - a$ for all $i \geq t_1 + 6a$, since the difference $br_i - p_i$ decreases as long as it is larger than $-a$.

(b) If Maker can follow the strategy, then one verifies that $br_i \leq br_{t_2} + 7a \leq p_{t_2} + 6a = p_{t_2+7} + 13a \leq p_i + 13a$ for every $t_2 + 1 \leq i \leq t_2 + 7$. On the other hand, this inequality ensures that, when Maker has to make her i^{th} move, each vertex in End($\mathcal{P}$) is incident with at least $2(p_{i-1} - 1) - (p_{i-1} + 13a) \geq (n - (t_2 + 6)a) - 14a \geq n/(7a)$ good edges. Therefore, Maker can follow the proposed strategy for Stage IIb.

(c) Similarly to the proof of (a) we apply induction on i. Assume the statement was true until round $i-1$. If $i < t_2 + 8 + 14a$, then $br_{i-1} + a \leq p_{i-1} + 14a < 2p_{i-1} - 2$. If $i \geq t_2 + 8 + 14a$, then $p_{i-1} > a$ and, since by (2) the difference $br_i - p_i$ decreases as long as it is larger than $-a$, we obtain $br_{i-1} + a \leq p_{i-1} < 2p_{i-1} - 2$. So, in any case, when Maker starts her i^{th} move, she sees at most $br_{i-1} + a < 2p_{i-1} - 2$ bad edges on the board. Thus, applying Proposition 4.3 we know that Maker can follow the strategy for the current move. The proof of (2) is done similarly to the proof of (1) in (a). Indeed, Case 1 and 2 from that proof are handled analogously. Case 3 can be done as before, as long as Proposition 4.2 applies, i.e. as long as $p_{i-1} \geq 4$. Since $p_{i-1} > a$, the only time when this does not happen is when $a = 2$ and $p_{i-1} = p_{\lceil n/a \rceil - 2} = 3$. But then $i \geq t_2 + 8 + 14a$ and thus the number of bad edges is at most $br_{i-1} + a \leq 3 = p_{i-1}$, which is enough to handle Case 3 analogously. Finally, $br_i \leq \max\{p_i - a, 0\}$ for $i \geq t_2 + 14a + 8$ holds, since by (2) the difference $br_i - p_i$ decreases as long as it is larger than $-a$. □

Stage III. Let p be the size of $\mathcal{P}$ when Maker enters Stage III, and observe $p = p_{\lceil n/a \rceil - 1} \leq a$. In order to create a Hamilton cycle within 1 or 2 further rounds, we now make use of Proposition 2.1 and Corollary 2.2. Before doing that, we need the following claim.

Claim 4.5. *Right before Maker's first move in Stage III, the following properties hold:*

(H1) The number of bad edges is at most a,

(H2) $\forall\ v \in End(\mathcal{P}):\ d_B(v) < n/(3a)$.

Proof By Claim 4.4 (c) we have $br_{\lceil n/a \rceil - 1} \leq \max\{p_{\lceil n/a \rceil - 1} - a, 0\} = 0$. Breaker in his $\lceil n/a \rceil^{\text{th}}$ move creates at most a bad edges, proving (H1). Assume now that (H2) does not hold, i.e. there is a vertex $v \in$ End($\mathcal{P}$) with degree at least $n/(3a)$ in Breaker's graph, right after Breaker's $\lceil n/a \rceil^{\text{th}}$ move. Then, in round $t_2 + 1$, the degree of vertex v in Breaker's graph is at least $n/(3a) - a(\lceil n/a \rceil - t_2) \geq n/(6a) - a$. Now, Maker did not claim a good edge incident to v so far. Thus, whenever Maker claimed an edge in Stage IIb, one of its endpoints already had degree at least $n/(6a) - a$ in Breaker's graph. But, since Maker claims $7a$ independent edges throughout Stage IIb, this means that Breaker needs to have at least $7a$ vertices in his graph of degree at least $n/(6a) - a$, which gives the existence of more than n Breaker's edges, in contradiction to the number of rounds played so far. □

Finally, we show how Maker completes her Hamilton cycle by case distinction on p.

Case $\mathbf{p \leq a/2}$. Applying Corollary 2.2 (with $G = K_n \setminus B$; using Claim 4.5 (H2)), Maker can find (at most) $2p \leq a$ free edges to finish a Hamilton cycle. Maker claims these and is done.

Case $\mathbf{p = (a+1)/2}$. Observe that $a \geq 3$ and therefore, by Claim 4.5 (H1), the number of good edges is at least $4\binom{p}{2} - a > 0$. Maker at first claims one such good edge, thus reducing the number of paths to $p-1$. Afterwards, applying Corollary 2.2 as before, she can find $2(p-1)$ free edges finishing a Hamilton cycle. She claims these edges, which is possible since $2(p-1)+1 = a$.

Case $\mathbf{p = (a+2)/2}$ and $\mathbf{a > 4}$. By Claim 4.5 (H1), the number of good edges is at least $4\binom{p}{2} - a \geq 2a+2$. That is why we can find at least two good edges such that claiming them keeps Maker's graph being a linear forest. Maker at first claims these two good edges, thus reducing the number of paths to $p-2$. Afterwards, applying Corollary 2.2 as before, she can find $2(p-2)$ free edges finishing a Hamilton cycle. She claims these edges, which is possible since $2(p-2)+2 = a$.

Case $\mathbf{p = 3}$ and $\mathbf{a = 4}$. Similarly to the previous case, by Claim 4.5 (H1), the number of good edges is at least $4\binom{p}{2} - a = 8$. It is easily checked that we can find two good edges such that claiming them Maker's graph is a Hamilton path. Then, applying Corollary 2.2 and Claim 4.5 (H2), she can close this path into a Hamilton cycle by claiming at most two further edges.

Case $\mathbf{p = 2}$ and $\mathbf{a = 2}$. In this case, n is even, and we are allowed to play two further rounds. When Maker enters Stage III, her graph consists of two paths P_1 and P_2. Applying Proposition 2.1 (with $G = K_n \setminus B$; using Claim 4.5 (H2)), she can claim two edges to obtain a path P covering $V(P_1) \cup V(P_2) = V$ in the first round. Then similarly, applying Corollary 2.2, she can finish a Hamilton cycle in the next round.

Case $\mathbf{p \geq (a+3)/2}$. Observe that $a \geq p \geq 3$ and that, when Maker enters Stage III, the number of bad edges is at most $a < 2p-2$. Thus, by Proposition 4.3, Maker at first can claim $p-2$ edges as in Stage IIa. Afterwards, her graph consists of exactly two paths ($|\mathcal{P}| = 2$), while, by the same proposition, the number of bad edges is smaller than $2|\mathcal{P}| - 2 = 2$. Thus, one finds two good edges that finish a Hamilton cycle. Maker claims these, which is possible as $(p-2)+2 \leq a$. □

5 Strong Hamilton Cycle Game

Proof of Theorem 1.5. At first we give a short description of a strategy for Red, and then we show that Red indeed can follow that strategy and win the $(2:2)$ Hamilton cycle game in the desired number of rounds. As in the proof for the corresponding weak game, Red starts by maintaining a linear forest for all but a small constant number of rounds. Then she completes a Hamilton cycle in her graph, while blocking possible Hamilton cycles in Blue's graph.
Let $\mathcal{S}_{\mathcal{H}_n}$ be Maker's strategy given in the previous chapter for the $(2:2)$ Weak Hamilton cycle game. Assume that Red's graph is a collection $\mathcal{P}$ of paths. Again, an edge e between the endpoints of different paths from $\mathcal{P}$ is called *good* if it is unclaimed. Otherwise, it is called *bad*. For a good edge e we set $D(e)$ to be the number of bad edges adjacent to e. Red's strategy is divided into the following three stages.

Stage I. For the first $n/2-2$ rounds, Red follows the strategy $\mathcal{S}_{\mathcal{H}_n}$, thus creating a collection of 4 non-empty paths covering all vertices of K_n.

Stage II. At the very beginning of Stage II, let I_2 denote the set of isolated vertices in the graph of Blue, and let $\mathcal{P}_2$ be the collection of Red's paths (from Stage I). In round $n/2-1$, Red claims two good edges e_1 and e_2 such that the following properties hold:

(S1) $E(\mathcal{P}_2) \cup \{e_1\} \cup \{e_2\}$ induces a collection $\mathcal{P}_3$ of two non-empty paths. Moreover, immediately after Red's move in round $n/2-1$, there is no bad edge among the vertices of $\text{End}(\mathcal{P}_3)$.

(S2) If $\text{End}(\mathcal{P}_2) \cap I_2 \neq \emptyset$, then $(e_1 \cup e_2) \cap I_2 \neq \emptyset$. (That is, Red decreases the number of Blue's isolated vertices among the endpoints of her paths, if this number is not zero.)

The details of how she can claim her edges follow later in the proof.

Stage III. Within at most 2 further rounds, Red creates a Hamilton cycle. Moreover, in the meantime she prevents the same in Blue's graph. The details of how she can do this follow later.

It is evident that, if Red can follow the strategy, she wins the game within the desired number of rounds. Thus, it remains to show that Red indeed can follow the proposed strategy.

Stage I. We already saw that Red/Maker can follow the strategy $\mathcal{S}_{\mathcal{H}_n}$. So, Red creates a collection $\mathcal{P}_2$ of non-empty paths and, since she claims $2 \cdot (n/2-2) = n-4$ edges in total, we get $|\mathcal{P}_2| = 4$.

Stage II. Recall that for the strategy $\mathcal{S}_{\mathcal{H}_n}$, the parameters p_i and br_i were introduced to denote the number of Maker's/Red's paths and the number of bad edges immediately after Maker's/Red's i^{th} move, respectively. By Claim 4.4 (c) we then have that immediately after Red's last move in Stage I, the number of bad edges is at most $br_{n/2-2} \leq \max\{p_{n/2-2}-2, 0\} = 2$. Thus, when Red enters Stage II there can be at most $2+2=4$ bad edges. We distinguish between the three cases.

Case 1. The number of bad edges is at most 3. Then Red at first chooses a good edge e_1 such that $D(e_1)$ is maximal (w.r.t. $\mathcal{P}_2$) and creates a collection $\mathcal{P}_2'$ induced by $E(\mathcal{P}_2) \cup \{e_1\}$. Note that, at the moment when $\mathcal{P}_2'$ is created, the number of bad edges is at most 1. Indeed, if $D(e_1) \geq 2$, then at least two bad edges disappear. Otherwise, if $D(e_1) \leq 1$, then by Proposition 4.1, the number of bad edges was already at most 1. So, after e_1 is claimed, we have exactly three paths and at most one bad edge. Then, Red chooses an arbitrary edge e_2 which is good (w.r.t. the new collection $\mathcal{P}_2'$) such that the following holds: It is adjacent to the remaining bad edge if one exists; and it is incident with some vertex from I_2 if $I_2 \cap \text{End}(\mathcal{P}_2') \neq \emptyset$. It is easy to check that Red indeed can do so, and properties (S1) and (S2) hold then.

Case 2. The number of bad edges is 4, and there is some good edge e with $D(e) \geq 3$ (w.r.t. $\mathcal{P}_2$). Then Red chooses e_1 and e_2 as in Case 1. Just note that, when e_1 is chosen, the number of bad edges drops to at most 1. The rest follows analogously to Case 1.

Case 3. The number of bad edges is 4, and we have $D(e) \leq 2$ for every good edge e. Then the subgraph of Blue's graph induced on $\text{End}(\mathcal{P}_2)$ is either a matching, or it consists of a 4-cycle and 4 isolated vertices. Indeed, if we don't have a matching, then there needs to be some $x \in \text{End}(\mathcal{P}_2)$ which is incident with exactly two bad edges xy_1 and xy_2. Now, as in the proof of Proposition 4.2 either there is some endpoint $z \in \text{End}(\mathcal{P}_2)$ such that zy_1 and zy_2 can be the only further bad edges, or y_1y_2 is the only further edge that can be bad. The second case cannot happen, since we have 4 bad edges, and so there needs to be a 4-cycle (with vertices x, y_1, y_2, z).
Now, in either case, it is easy to see that Red can choose a good edge e_1 such that the following holds: e_1 is adjacent to exactly two bad edges; and e_1 is incident with some vertex from I_2 if $I_2 \cap \text{End}(\mathcal{P}_2) \neq \emptyset$. Afterwards, the collection $\mathcal{P}_2'$ induced by $E(\mathcal{P}_2) \cup \{e_1\}$ consists of three paths, while the number of bad edges is 2. Red then chooses e_2 to be good w.r.t. $\mathcal{P}_2'$, in such a way that e_2 intersects both of the remaining bad edges. Again, Red can easily do so, and by this ensure the properties (S1) and (S2) hold.

Stage III. When Red enters Stage III, her graph is the collection $\mathcal{P}_3$ with properties (S1) and (S2). Moreover, Property (H2) from Claim 4.5 holds again: If there was a vertex of degree at least $n/(3a)$ before Red's first move of Stage III (right after Blue's $(n/2-1)^{\text{st}}$ move), then analogously to the proof of Claim 4.5 in one of the previous rounds Blue must have had at least $7a$ vertices of degree at least $n/(6a) - a$ in his graph, a contradiction. Now, in the following we describe how Red finishes her Hamilton cycle while preventing such a cycle in Blue's graph.
For this, let B_2 denote Blue's graph right at the beginning of Stage II (i.e. after Blue's $(n/2-2)^{\text{nd}}$ move), and let B_3 be his graph at the beginning of Stage III.

Case 1. Assume that B_3 satisfies one of the following three properties:

- B_3 contains a cycle.
- B_3 has a vertex of degree at least 3.

- B_3 has at least 3 components.

Then, since $|E(B_3)| = n-2$, one needs to add at least 3 edges to B_3 in order to get a Hamilton cycle. That is, Blue cannot finish a winning set before round $n/2+1$. Therefore, Red wins, if she can finish a Hamilton cycle within the next two rounds (round $n/2$ and $n/2+1$). For this, just observe that with Property (H2) from Claim 4.5 in hand, Red can just follow Stage III of Maker's strategy from the $(2:2)$ Weak Hamilton cycle game.

Case 2. Assume that B_3 satisfies none of the three properties given in the first case. Then, B_3 is a collection of exactly 2 (maybe one-vertex) paths that cover all vertices. Therefore, B_2 is a collection of 4 (maybe one-vertex) paths and thus has at most 3 isolated vertices (i.e. $|I_2| \leq 3$).
Now, if at the beginning of Stage III there are two good edges in $\text{End}(\mathcal{P}_3)$ that finish a Hamilton cycle in Red's graph, then Red just claims these and wins the game.
So, we can assume that there is a vertex $w \in \text{End}(\mathcal{P}_3)$, which is incident with two bad edges $f_1 = wy_1$ and $f_2 = wy_2$ before Red's $(n/2)^{\text{th}}$ move, $y_1, y_2 \in \text{End}(\mathcal{P}_3)$. Since, by Property (S1), there was no bad edge among $\text{End}(\mathcal{P}_3)$ right after Red's $(n/2-1)^{\text{st}}$ move, we have that f_1 and f_2 were claimed in round $n/2-1$ (i.e. $f_1, f_2 \in B_3 \setminus B_2$). But then, since B_3 does not have a vertex of degree at least 3, we know that w must be isolated in B_2 (i.e. $w \in I_2$). Thus $|I_2| \geq 2$, since otherwise in Stage II (Property (S2)) we would ensure that $I_2 \cap \text{End}(\mathcal{P}_3) = \emptyset$, in contradiction to the existence of w. In particular, there is a vertex $x \in I_2$ with which Red claims an incident edge in Stage II, and thus $x \notin \{y_1, y_2\}$. As $|I_2| \leq 3$, it follows that at least one of the vertices y_1, y_2 does not belong to I_2, w.l.o.g. let $y_1 \notin I_2$. We are left with two cases.

Case 2.1. $|I_2| = 3$. Then B_2 consists of 3 isolated vertices (including w) and a path P_{B_2} with $n-3$ vertices. Then, the vertex $y_1 \notin I_2$ must be the endpoint of the path P_{B_2}, since $E(P_{B_2}) \cup f_1 \subseteq E(B_3)$ does not give a vertex of degree at least 3. Since $f_2 = wy_2 \in E(B_3)$ cannot create a cycle, we have $y_2 \in I_2$. But then, B_3 consists of one isolated vertex z, and a path P_{B_3} with $n-1$ vertices. Red's strategy is as follows: In the first move, Red takes one edge which is good (w.r.t. $\mathcal{P}_3$) and creates a Hamilton path. This is possible, since by (S1) there are only two bad edges between the two paths of $\mathcal{P}_3$. For the second edge, she chooses one edge between z and $\text{End}(P_{B_3})$. In his next move, Blue cannot close a Hamilton cycle, since this would need the two edges between z and $\text{End}(P_{B_3})$. In round $n/2+1$, using Claim 4.5 (H2) and Corollary 2.2, Red then completes a Hamilton cycle.

Case 2.2 $|I_2| = 2$. Then B_2 consists of 2 isolated vertices (including w) and two (non-empty) paths covering the remaining $n-2$ vertices. By Stage II (see Property (S2)), we ensured that $|I_2 \cap \text{End}(\mathcal{P}_3)| \leq 1$, and so w is the unique vertex in $I_2 \cap \text{End}(\mathcal{P}_3)$. In particular, $y_1, y_2 \notin I_2$. Since B_3 has no cycles and no vertex of degree at least 3, and since $f_1, f_2 \in B_3$, the vertices y_1 and y_2 must be the endpoints of different paths from B_2. But then, B_3 again has exactly one isolated vertex and one path with $n-1$ vertices. So, we can proceed as in Case 2.1. □

6 P_k-factor game

Proof of Theorem 1.3. If $k = 2$ and a is any constant, then $\mathcal{P}_{k,n}$ is a perfect matching and we can use the proof of Theorem 1.1. So, we let $k \geq 3$ and we fix some $\delta < 1/(8k)$. We first give a Maker's strategy. Then we prove that she can follow it and win within $\lceil (k-1)n/(ka) \rceil$ rounds.

Maker's strategy is to build n/k vertex disjoint paths of length $k-1$. During the course of the game, the collection of all paths in her graph is denoted by $\mathcal{P}$. Each path in $\mathcal{P}$ belongs to exactly one of the three classes: $\mathcal{P}_u$, which denotes the collection of *unfinished* paths (the paths of length at most $k-3$), $\mathcal{P}_f$, which denotes the collection of the *finished* paths (the paths of length exactly $k-2$) or $\mathcal{P}_c$, which denotes the collection of *complete* paths (the paths of length exactly $k-1$). Maker's strategy consists of three stages. In Stage I of her strategy, Maker makes sure that every unfinished path becomes (at least) a finished path, while in the following stages she aims for complete paths. The set of isolated vertices in Maker's graph is denoted by $U = V \setminus V(\mathcal{P})$.

By $\text{End}(\mathcal{P})$ we denote the set of endpoints of all paths. At the beginning, $\mathcal{P} := \mathcal{P}_u$ contains n/k arbitrarily chosen vertices; $\mathcal{P}_f$ and $\mathcal{P}_c$ are empty. If P is a path in Maker's graph, then v_1^P and v_2^P represent its endpoints.

Stage I. In this stage, Maker plays as follows: She gradually extends the unfinished paths with the vertices from U until they are finished. From time to time, we allow her to complete some of these paths in order to keep control on the distribution of Breaker's edges (as described by properties (Q1)–(Q3) in the following paragraph). After each step, the sets $\mathcal{P}_u, \mathcal{P}_f, \mathcal{P}_c$ and U are dynamically updated in the obvious way. That is, whenever Maker extends one of her paths, P, by some vertex $u \in U$, this vertex is removed from U and added to P, while P may be moved from $\mathcal{P}_u$ to $\mathcal{P}_f$ or from $\mathcal{P}_f$ to $\mathcal{P}_c$ according to its new length.

During Stage I, for a given graph G, we say that $(G, \mathcal{P})$ is *good* if the following properties hold:

$(Q1)$ $\forall u \in U : d_G(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) < \delta n$;

$(Q2)$ $G[U] = \emptyset$;

$(Q3)$ $\forall P \in \mathcal{P}_u \cup \mathcal{P}_f : d_G(v_1^P, U) + d_G(v_2^P, U) \leq 1$.

In each move during this stage, Maker claims a free edges between U and $\text{End}(\mathcal{P}_u \cup \mathcal{P}_f)$, so that after her move $(B, \mathcal{P})$ is good.
In her i^{th} move, Maker chooses the edges $e_1 = e_1^{(i)}, \ldots, e_a = e_a^{(i)}$ one after another. She makes sure that for every $t \in \{0, 1, \ldots, a\}$ the following holds:

$(Q4)$ Immediately after the edges $e_1, \ldots, e_t$ are claimed (and the paths are updated accordingly), there is a subgraph $H = H_t \subseteq B$ with $e(H) = a - t$ such $(B \setminus H, \mathcal{P})$ is good.

Assume $e_1, \ldots, e_t$ are already claimed and $\mathcal{P}$ is updated accordingly. Then, as next Maker chooses a free edge e_{t+1} according to the following rules:

R1. If there is $u \in U$ with $d_B(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) \geq \delta n$, then e_{t+1} is chosen such that it extends a path from $\mathcal{P}_u$ by the vertex u, if $|\mathcal{P}_u| \geq a+1$, or a path from $\mathcal{P}_f$ by the vertex u, otherwise.

R2. Otherwise, if there is a path $P \in \mathcal{P}_u$ with $d_B(v_1^P, U) + d_B(v_2^P, U) \geq 2$, then there is an edge $v_i^P x \in H_t$ with $x \in U$, $i \in [2]$. Maker claims an arbitrary free edge $e_{t+1} = v_i^P u$ with $u \in U$.

R3. Otherwise, if there is a path $P \in \mathcal{P}_f$ with $d_B(v_1^P, U) + d_B(v_2^P, U) \geq 2$, then there is an edge $v_i^P x \in H_t$ with $x \in U$, $i \in [2]$. Then

a) if there is a path $P_0 \in \mathcal{P}_u$, Maker claims an arbitrary free edge $e_{t+1} \in E(\text{End}(P_0), x)$,

b) otherwise, if $\mathcal{P}_u = \emptyset$, she claims an arbitrary free edge $e_{t+1} = v_i^P u$ with $u \in U$.

R4. Otherwise, if there is $uw \in E_B(U)$, w.l.o.g. $d_B(w, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) \leq d_B(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f))$, then Maker proceeds as follows.

a) If $d_B(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) = d_B(w, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) \geq \delta n - 1$, then Maker chooses a free edge $e_{t+1} = ux$ with $x \in N_B(w, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f))$. Its existence is proved later.

b) Otherwise, if $d_B(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) \geq \delta n - 1 > d_B(w, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f))$, then let $P \in \mathcal{P}_u \cup \mathcal{P}_f$ be a path with $e_B(\text{End}(P), u) = 0$ and $d_B(v_i^P, U) = 0$ for some $i \in [2]$. Its existence is proved later. Maker then sets $e_{t+1} = uv_{3-i}^P$.

c) Otherwise, if $d_B(u, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)), d_B(w, \text{End}(\mathcal{P}_u \cup \mathcal{P}_f)) < \delta n - 1$, let $P \in \mathcal{P}_u \cup P_f$, giving priority to unfinished paths, and let $d_B(v_i^P, U) \geq d_B(v_{3-i}^P, U)$ for some $i \in [2]$. Then Maker claims one of the edges $v_i^P u$, $v_i^P w$.

R5. Otherwise, in all the remaining cases, Maker extends a path which is not complete by some free edge e_{t+1}, arbitrarily, giving priority to unfinished paths.

Stage I ends when after Maker's move, her graph consists only of finished paths, complete paths and isolated vertices. At this point, $(n(k-2))/(ka) + T/a$ rounds are played with $T = |P_c|$ being the number of complete paths at the end of Stage I.

Stage II. In the following $\lceil (n/k - T)/a \rceil - 1$ rounds, Maker extends the finished paths. This time, Maker is interested in keeping the following property after each of her moves.

(F1) $\forall P \in \mathcal{P}_f : d_B(v_1^P, U) + d_B(v_2^P, U) \leq 1$.

To describe Maker's strategy, we introduce the following terminology: Let $e \in E(\text{End}(\mathcal{P}_f), U)$. Then e is called *good* if it is free; otherwise we call it *bad*. We say that $P \in \mathcal{P}_f$ is a *bad path* if $d_B(v_1^P, U) + d_B(v_2^P, U) \geq 2$ holds, and with $\mathcal{P}_b \subseteq \mathcal{P}_f$ we denote the dynamic set of all bad paths. Moreover, we introduce the *potential*

$$\varphi := \sum_{P \in \mathcal{P}_b} (d_B(v_1^P, U) + d_B(v_2^P, U) - 1),$$

which measures dynamically the number of edges that need to be deleted from B in order to reestablish Property (F1). Finally, with e_{end} we denote the very last edge claimed by Maker in Stage II.
Now, in every round i played in Stage II, Maker claims edges $e_1 = e_1^{(i)}, \ldots, e_a = e_a^{(i)}$, one after another. The j^{th} edge e_j is claimed according to the following rules:

- Maker chooses a good edge e_j between some vertex $u \in U$ and an endpoint of some path $P \in \mathcal{P}_f$ such that
  - (a) in case $\varphi > 0$, φ is decreased after e_j is claimed and all sets are updated,
  - (b) if $e_j \neq e_{end}$, then $\text{End}(P) \cup \{u\}$ contains a vertex of the largest degree in Breaker's graph among all vertices from $\text{End}(\mathcal{P}_f) \cup U$,
  - (c) if $e_j = e_a = e_{end}$, then after e_j is claimed and all sets are updated, there is a path $P \in \mathcal{P}_f$ with $e_B(\text{End}(P), U) = 0$.
- After e_j is claimed, Maker removes u from U and P from $\mathcal{P}_f$, and adds P to $\mathcal{P}_c$, before she proceeds with e_{j+1}.

The exact details of how Maker finds such an edge e_j will be given later in the proof.

Stage III. Within one round, Maker claims at most a free edges to complete a P_k-factor. The details are given later in the proof.

It is evident that if Maker can follow this strategy, she wins the game in the claimed number of rounds. For each of the stages above we show separately that Maker can follow her strategy.

Stage I. We start with the following useful claim.

Claim 6.1. *As long as Maker follows Stage I,* $|P_c| \leq a + 3/\delta < \delta n$.

Proof Following the strategy, Maker only creates complete paths in case there is a vertex of degree at least $\delta n - 1$ in Breaker's graph which is used to extend a finished path (cases R1, R4.a,b), or in case $\mathcal{P}_u = \emptyset$ (cases R3.b, R4.c, R5) holds. The first option happens less than $3/\delta$ times, as Breaker claims less than n edges throughout Stage I. The second option can only happen in the last round of Stage I, which cannot lead to more than a additional complete paths. □

Now, by induction on the number of rounds, i, we show that Maker can follow the proposed strategy of Stage I. We first observe that before the game starts, $(B, \mathcal{P})$ is good, as B is empty. Now, let us assume that she could follow the strategy for the first $i-1$ rounds and that immediately after her $(i-1)^{\text{st}}$ move, $(B, \mathcal{P})$ is good. In particular, this also means that in the next round, Property (Q4) is guaranteed for $t = 0$, by choosing $H = H_0$ to be the graph of all the a edges that Breaker claims in round i. By induction on the number of Maker's steps in round i, we prove that she can claim the edges $e_1, \dots, e_a$ as described, and that she ensures Property (Q4) for every $t \in \{0, \dots, a\}$. Setting $t = a$ then tells us that $(B, \mathcal{P})$ is good immediately after Maker's move, completing the induction on i.

Let us assume that Maker already claimed $e_1, \dots, e_t$ and that (Q4) holds after step t. Let H_t be the graph guaranteed by (Q4). We now look at the different cases for step $t+1$.

R1. In this case there must be an edge $g \in E(H_t)$ between u and $\mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)$, as $(B \setminus H_t, \mathcal{P})$ satisfies Property $(Q1)$ after step t. Now, if $|\mathcal{P}_u| \geq a+1$, then there needs to be a path $P \in \mathcal{P}_u$ with $d_B(v_1^P, U) + d_B(v_2^P, U) \leq 1$, as $(B \setminus H_t, \mathcal{P})$ satisfies (Q3) after step t and $e(H_t) \leq a$. Otherwise, Claim 6.1 ensures that $|\mathcal{P}_f| \geq a+1$ and thus there is a path $P \in P_f$ with the same property. In either case, Maker can extend P by u. Set $H_{t+1} := H_t - g$. Then, after the update, $g \in E(\mathrm{End}(\mathcal{P}))$ holds and therefore g has no influence on (Q1)–(Q3) anymore. Thus, using that $(B \setminus H_t, \mathcal{P})$ satisfied (Q2) after step t, we conclude that u has no edges towards U in $B \setminus H_{t+1}$ after step $t+1$. Now, one easily checks that $(B \setminus H_{t+1}, \mathcal{P})$ is good after step $t+1$.

R2. As $(B \setminus H_t, \mathcal{P})$ satisfies (Q3) after step t, there needs to exist an edge $v_i^P x$ as claimed. Moreover, we have $|U| \geq |\mathcal{P}_u \cup \mathcal{P}_f| = \frac{n}{k} - |P_c| > d_B(v_i^P, U)$ where the last inequality follows from Claim 6.1 and the fact that $(B \setminus H_t, \mathcal{P})$ satisfies (Q3) after step t. Thus, Maker can claim an edge $v_i^P u$ as proposed. Afterwards, we set $H_{t+1} := H_t - v_i^P x$. Then u has no edge towards U in $B \setminus H_{t+1}$ and the edge $v_i^P x$ has no influence on (Q1)–(Q3) anymore. We therefore conclude that $(B \setminus H_{t+1}, \mathcal{P})$ is good after step $t+1$.

R3. The existence of $v_i^P x$ is given as in case R2. If there is a path $P_0 \in \mathcal{P}_u$, then not both edges $v_1^{P_0} x, v_2^{P_0} x$ can be claimed by Breaker, as otherwise P_0 would force case R2. If otherwise $\mathcal{P}_u = \emptyset$, then analogously to the argument in case R2, we have $|U| > d_B(v_i^P, U)$. So, in either case, Maker can claim an edge as proposed by the strategy. After the update of $\mathcal{P}$, we obtain analogously to the previous case that $(B \setminus H_{t+1})$ is good with $H_{t+1} := H_t - v_i^P x$.

R4. As case R1 does not occur, we know that $d_B(u, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)), d_B(w, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)) < \delta n$. In particular, a)–c) cover all possible subcases. Moreover, as $(B \setminus H_t, \mathcal{P})$ satisfies (Q2) after step t, we must have $uw \in E(H_t)$.

a) Assume there is no edge ux as proposed by the strategy. Then there must be at least $\delta n - 1$ vertices in $\mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)$ which have degree at least $|\{u, w\}| = 2$ in B, contradicting the fact that $(B \setminus H_t, \mathcal{P})$ satisfies (Q3) after step t and $e(H_t) \leq a$. So, Maker can claim an edge ux as proposed. In case $xw \in E(H_t)$, we set $H_{t+1} := H_t - xw$. Then, after the update $xw \notin E(\mathrm{End}(\mathcal{P}) \cup U)$ holds, i.e. this edge has no influence on (Q1)–(Q3). Moreover, u has no edge towards U in $B \setminus H_{t+1}$ after step $t+1$. Otherwise, we set $H_{t+1} := H_t - uw$. To see that again $(B \setminus H_{t+1}, \mathcal{P})$ is good after step $t+1$, just observe the following: After step $t+1$, u has exactly one neighbour in U (namely w) in the graph $B \setminus H_{t+1}$. But, as $xw \in E(B \setminus H_t)$ and (Q3) was fulfilled by $B \setminus H_t$ after step t, we know that the other endpoint of the path P has no edges towards U in $B \setminus H_{t+1}$. Moreover, $d_{B \setminus H_{t+1}}(w, \mathrm{End}(\mathcal{P}_f \cup \mathcal{P}_u)) \leq d_B(w, \mathrm{End}(\mathcal{P}_f \cup \mathcal{P}_u)) < \delta n$ is maintained, as in B, w gains u and looses x as a neighbour in $\mathrm{End}(\mathcal{P}_f \cup \mathcal{P}_u)$.

b) By Claim 6.1 and since $(B \setminus H_t, \mathcal{P})$ is good after step t, we have that $|\mathcal{P}_u \cup \mathcal{P}_f| \geq \frac{n}{k} - \delta n > 3a + d_{B \setminus H_t}(u, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)) \geq 2a + d_B(u, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f))$. In particular, there are at least $2a$ paths $P \in \mathcal{P}_u \cup \mathcal{P}_f$ with $e_B(\mathrm{End}(P), u) = 0$. As $e(H_t) \leq a$ and since $(B \setminus H_t, \mathcal{P})$ satisfied (Q3) after step t, there needs to be such a path with $d_B(v_i^P, U) = 0$ for some $i \in [2]$. Obviously Maker can claim the edge $v_{3-i}^P u$, as $e_B(\mathrm{End}(P), u) = 0$. Now, set $H_{t+1} := H_t - uw$. Then to see that $(B \setminus H_{t+1}, \mathcal{P})$ is good after step $t+1$, just notice that u has only one edge, uw, towards U in $B \setminus H_{t+1}$, while $d_{B \setminus H_{t+1}}(v_i^P, U) = 0$.

Moreover, $d_B(w, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)) < \delta n$ is guaranteed by the assumption on w for this case and since w gains at most one new neighbour among $\mathrm{End}(\mathcal{P})$, namely u.

c) As the cases R2 and R3 do not occur, we have $d_B(v_{3-i}^P, U) = 0$ and $d_B(v_i^P, U) \leq 1$. In particular, one of the edges $v_i^P u, v_i^P w$ is free, so Maker can follow the proposed strategy. W.l.o.g. let Maker claim $v_i^P u$. Set $H_{t+1} := H_t - uw$. Then after step $t+1$, u has exactly one edge towards U (namely uw) in $B \setminus H_{t+1}$, while $d_{B\setminus H_{t+1}}(v_{3-i}^P, U) = 0$. Moreover, $d_B(w, \mathrm{End}(\mathcal{P}_u \cup \mathcal{P}_f)) < \delta n$ is guaranteed as in the previous case. Therefore, we conclude analogously that $(B \setminus H_{t+1}, \mathcal{P})$ is good after step $t+1$.

R5. As the cases R1–R4 do not occur, $(B, \mathcal{P})$ is good after step t. Therefore, Maker can easily follow the proposed strategy and afterwards $(B \setminus H_{t+1}, \mathcal{P})$ is good for every graph H_{t+1}.

So, in either case Maker can follow the proposed strategy for Stage I.

Stage II. At any moment throughout Stage II, let $p := |\mathcal{P}_f|$ and let br denote the number of bad edges. By the definition of φ, it always holds that $br \leq p + \varphi$. Moreover, before every move of Maker in this stage, we have $p \geq a+1$, and therefore $p \geq 2$ holds immediately before e_{end} is claimed.

By induction on the number of rounds, i, we now prove that Maker can follow the proposed strategy and always maintains (F1). We can assume that (F1) was already satisfied after Maker's last move in Stage I, using (Q3). Now, assume that Maker's i^{th} move happens in Stage II. As, by induction, (F1) was satisfied immediately after her previous move and as Breaker afterwards claimed only a edges in his previous move, we know that $\varphi \leq a < p$ before Maker's move, and therefore $br \leq p + \varphi < 2p$. We now observe that such a relation can be maintained as long as Maker can follow her strategy.

Claim 6.2. *Assume that Maker can follow her strategy. Then, after e_j is claimed and $\mathcal{P}_f, U$ are updated accordingly, $\varphi < p$ and $br < 2p$ hold.*

Proof For induction assume that $\varphi < p$ and $br < 2p$ hold immediately after e_{j-1} is claimed. When Maker claims e_j, two cases may occur. If $\varphi = 0$ holds, then after e_j is claimed, we still have $\varphi = 0$ and $br \leq p + \varphi < 2p$. Otherwise, we have $\varphi > 0$, in which case Maker claims an edge that decreases the value of φ. As p decreases by one within one step of Maker, we thus obtain that $\varphi < p$ and $br \leq p + \varphi < 2p$ are satisfied after all updates. □

With this claim in hand, we can deduce that Maker can always follow the proposed strategy.
Consider first that $e_j \neq e_{end}$. We note that each path in $\mathcal{P}_f$ and each vertex in U is incident with $2p > br$ edges from $E(\mathrm{End}(\mathcal{P}_f), U)$. Thus each such path and each such vertex intersects at least one good edge. Let v be a vertex of the largest Breaker's degree among all vertices in $\mathrm{End}(\mathcal{P}_f) \cup U$. Assume first that $v \in \mathrm{End}(P)$ for some $P \in \mathcal{P}_f$. If $\varphi = 0$ or $P \in P_b$, then Maker claims an arbitrary good edge $e_j \in E(\mathrm{End}(P), U)$ which exists as explained above. Just note that in case $P \in P_b$, the value of φ will be decreased, as P gets removed from $\mathcal{P}_b$. Moreover, v is contained in the path P (after the update). Otherwise, if $\varphi > 0$ and $P \notin P_b$, we find some bad path $P_0 \neq P$ and some bad edge $u_0 v_0$ with $u_0 \in U$ and $v_0 \in \mathrm{End}(P_0)$. Then Maker claims an edge $e_j = x u_0$, with $x \in \mathrm{End}(P)$ and $d_B(x, U) = 0$. Such a vertex exists, as we assumed $P \notin P_b$. Again this decreases φ by (at least) one, as $d_B(v_1^{P_0}, U) + d_B(v_2^{P_0}, U)$ decreases by (at least) one after u_0 is removed from U; and again v is contained in the updated path P.
Assume then that $v \in U$. If $\varphi = 0$, then Maker can choose e_j to extend an arbitrary path in $\mathcal{P}_f$ by the vertex v, which is possible as there are no bad paths. So, assume that there is a path $P \in P_b$. If there is a good edge in $E(v, \mathrm{End}(P))$, Maker claims such an edge and then φ decreases as P is removed from $\mathcal{P}_f$, and v again is contained in the updated path P. If there is no such good edge, then Maker claims an arbitrary good edge between v and some path $P_0 \in \mathcal{P}_f \setminus \{P\}$. Then, φ decreases as $e_B(v, \mathrm{End}(P)) = 2$ and v gets removed from U, and v finally belongs to the updated path P_0.
Consider then that $e_j = e_a = e_{end}$, and recall that $p \geq 2$ before Maker claims e_{end}. We know

that $\varphi \leq a$ holds before Maker's first step in round i. As Maker decreased the value of φ by at least one, in case $\varphi > 0$, with every previous edge in this round, we know that $\varphi \leq 1$ immediately before she wants to claim e_{end}. W.l.o.g. let $\varphi = 1$, and let P_0 be the unique bad path. Note that then $e_B(\text{End}(P_0), U) = 2$. Moreover, let $P \in \mathcal{P}_f \setminus \{P_0\}$. If $e_B(\text{End}(P), U) = 0$, then Maker extends P_0 by an arbitrary edge e_{end} (which is possible as $br < 2p$). Then, after the update, $\varphi = 0$ holds as P_0 is removed from $\mathcal{P}_f$, and P satisfies $e_B(\text{End}(P), U) = 0$. Otherwise, we have $e_B(\text{End}(P), U) = 1$ as $P \notin \mathcal{P}_b$, and thus there is a unique vertex u such that $e_B(\text{End}(P), u) = 1$. If $e_B(\text{End}(P_0), u) \leq 1$ holds, then Maker claims an edge $e_{end} = uv_i^{P_0}$ with $i \in [2]$. Afterwards, P satisfies $e_B(\text{End}(P), U) = 0$ as u gets removed from U; $\varphi = 0$ holds as P_0 is removed from $\mathcal{P}_b$. Otherwise, we have $e_B(u, \text{End}(P_0)) = 2 = e_B(U, \text{End}(P_0))$ as $\varphi = 1$. In this case Maker just claims a free edge $e_{end} = uv_i^P$ with $i \in [2]$ (which is possible as $e_B(\text{End}(P), u) = 1$). Then, P_0 satisfies $e_B(\text{End}(P_0), U) = 0$ after u is removed from U, and as P_0 is not bad anymore, i.e. $\varphi = 0$.

In total, we see that in either case Maker can claim the edges e_j as proposed. Finally, Property (F1) always holds immediately after e_a is claimed. For this just recall that $\varphi \leq a$ holds immediately before a Maker's move, and that Maker reduces φ by at least one in each step as long as $\varphi > 0$ holds. That is, we obtain $\varphi = 0$ at the end of her move, which makes Property (F1) hold.

Stage III. Finally, we prove that Maker can finish a P_k-factor within one additional round. We start with the following claim.

Claim 6.3. *Before Maker's move in Stage III, $d_B(v) < 2\delta n$ holds for every $v \in (U \cup End(\mathcal{P}_f))$.*

Proof Suppose that the statement does not hold, i.e. there exists a vertex $v \in U \cup End(\mathcal{P}_f)$ such that $d_B(v) \geq 2\delta n$. Then, during the last $\lfloor 3/\delta \rfloor$ rounds of Stage II, $d_B(v) > \delta n$. However, in each step of these rounds (except when claiming e_{end}), Maker included a vertex w into some complete path for which $d_B(w) \geq d_B(v) > \delta n$ was satisfied (see (b)). But then, Breaker would have claimed more than n edges, a contradiction to the number of edges he could claim in all rounds so far. □

With this claim in hand, we are able to describe how to finish a P_k-factor within one further round. We distinguish between the following cases depending on the size of U.

Case 1: $\mathbf{0 < |U| \leq a/2}$. We denote the isolated vertices in U by $u_1, u_2, \ldots, u_t$, and the paths in $\mathcal{P}_f$ by $P_1 \ldots, P_t$. Using Claim 6.3, for every $i \in [t]$, we find at least $|\mathcal{P}_c| - 4\delta n > n/(4k)$ paths $R \in \mathcal{P}_c$ such that $u_i v_1^R, v_2^R v_1^{P_i}$ are free. We thus can fix t distinct paths $R_1, \ldots, R_t \in \mathcal{P}_c$ such that, for every $i \in [t]$, the edges $u_i v_1^{R_i}, v_2^{R_i} v_1^{P_i}$ are free. Maker claims these edges, in total at most a, and by this completes a $\mathcal{P}_{k,n}$, as $V(R_i) \cup V(P_i) \cup \{u_i\}$ contains a copy of P_{2k} for every $i \in [t]$.
Case 2: $\mathbf{a/2 < |U| = a}$. Let $\mathcal{G} = (\mathcal{P}_f \cup U, E(\mathcal{G}))$ be the bipartite graph where two vertices $u \in U$ and $P \in \mathcal{P}_f$ form an edge if and only if $d_B(u, \text{End}(P)) \leq 1$. Then $e(\overline{\mathcal{G}}) \leq a - 1$ holds, as after Maker's last move in Stage II we had $e_B(\text{End}(\mathcal{P}_f), U) \leq a - 1$ (by (F1) and (c)), while Breaker afterwards claimed at most a bad edges. By the theorem of König-Egervary (see e.g. [13]) we thus obtain that $\mathcal{G}$ contains a matching of size at least

$$\frac{|U|^2 - e(\overline{\mathcal{G}})}{|U|} > \begin{cases} |U| - 1, & \text{if } |U| = a \\ |U| - 2, & \text{if } a/2 < |U| < a. \end{cases}$$

So, in case $|U| = a$, we have that $\mathcal{G}$ contains a perfect matching, say $u_1 P_1, u_2 P_2, \ldots, u_a P_a$. Then Maker claims a good edge in $E(u_i, \text{End}(P_i))$ for every $i \in [a]$, and by this creates a copy of $\mathcal{P}_{k,n}$. Otherwise, in case $a/2 < |U| \leq a - 1$, we find a matching of size $|U| - 1$, say $u_1 P_1, u_2 P_2, \ldots, u_{|U|-1} P_{|U|-1}$. Maker then claims a good edge in $E(u_i, \text{End}(P_i))$ for every $i \in [|U| - 1]$, in total at most $a - 2$ edges. For the remaining (unique) vertices $u \in U$ and $P \in P_f$ that are not covered by the matching, we proceed as in Case 1: we find a path $R \in \mathcal{P}_c$ such that the edges $uv_1^R, v_2^R v_1^P$ are free, which then Maker claims to complete a P_k-factor. □

7 Weak S_k-factor game

Proof of Theorem 1.4. In the following we give a strategy for Maker in the $\mathcal{S}_{k,n}$ game. Afterwards, we prove that she can follow that strategy and win in the claimed number of rounds.

Maker's strategy. Maker makes the k-star factor by gradually increasing the size of n/k stars, so that at any point of the game, no two disjoint stars differ in size by more than one. Before the game starts, she splits the vertex set into three sets C, R and F, that are dynamically maintained. C represents the centres of the stars in the star factor, F contains the endpoints of the current stars, and R are the remaining isolated vertices in Maker's graph. At the beginning of the game, $F := \emptyset$, C contains n/k arbitrary chosen vertices and $V = C \cup R$. All the star centres have degree 0 at the beginning of the game. Maker's strategy consists of the following two stages.

Stage I. Maker divides this stage into phases $1, 2, \ldots, k-1$. The game starts in phase 1, when all vertices in C have degree 0 in Maker's graph. In phase i, $1 \le i \le k-1$, she makes the vertices in C get degree exactly i in her graph. The phase i, $1 \le i \le k-2$ finishes (and the phase $i+1$ starts) immediately after the step in which the last vertex of C reached degree i. So, it might happen that Maker switches from phase i to phase $i+1$ between two steps of the same move. In particular, she will play exactly n/k consecutive steps (from consecutive rounds) in each of her first $k-2$ phases. Finally, Stage I ends immediately after the Maker's move in which phase $k-2$ ended, i.e. when all the vertices in C have degree at least $k-2$ in Maker's graph. (Note that it might happen that phase $k-1$ consists of zero steps.)

To describe Maker's strategy more precisely, we let C_A always denote the subset of C containing the vertices of smallest degree in her graph. That is, in phase i, C_A contains those vertices that are centres of stars of size $i-1$. At the beginning of each phase, $C_A = C$. Moreover, we call a Breaker's edge e *bad* if $e \in E(C, R)$.

Assume now, Maker wants to make her j^{th} move in Stage I. Let t be the number of elements in C_A right at the beginning of her move. Maker iteratively chooses the edges $e_1 = e_1^{(j)}, \ldots, e_a = e_a^{(j)}$ of her j^{th} move in the following way: For every $1 \le s \le a$, she first sets $t := |C_A|$. Then,

(1) if there is a free edge $e_F \in E(C_A, R)$ such that $\emptyset \ne e_F \cap e_B \in R$ for some bad edge e_B, then Maker chooses e_s to be such an edge e_F. Let $x_s \in C_A$ and $y_s \in R$ be the vertices of e_s.

(2) Otherwise, she chooses $e_s = x_s y_s$ arbitrarily with $x_s \in C_A$ and $y_s \in R$.

(3) Afterwards, she updates $R := R \setminus \{y_s\}$, $F := F \cup \{y_s\}$ and $C_A := C_A \setminus \{x_s\}$ if $t \ne 1$, or $C_A := C$ if $t = 1$, before she proceeds with e_{s+1}.

This stage lasts $\lceil (k-2)n/(ak) \rceil$ rounds.

Stage II. When Maker enters Stage II, her graph consists of stars of size $k-2$ and $k-1$. Moreover,

$$|C_A| = |R| = \frac{(k-1)n}{k} - a \left\lceil \frac{(k-2)n}{ak} \right\rceil =: N.$$

Maker now completes her k-star factor, by claiming a perfect matching between C_A and R in the following $\lfloor N/a \rfloor + 1$ rounds. The details follow later in the proof.

It is easy to see that if Maker can follow the strategy, she wins the $\mathcal{S}_{k,n}$ within the claimed number of rounds. Indeed, the number of rounds Stage I and II last together is

$$\left\lceil \frac{(k-2)n}{ak} \right\rceil + \lfloor N/a \rfloor + 1 = \left\lfloor \frac{(k-1)n}{ak} \right\rfloor + 1 = \begin{cases} \lceil \frac{(k-1)n}{ak} \rceil, & \text{if } ak \nmid (k-1)n \\ \lceil \frac{(k-1)n}{ak} \rceil + 1, & \text{otherwise.} \end{cases}$$

So, to finish the proof, we show separately for each stage that Maker can follow the proposed strategy.

Stage I. In order to show that Maker can follow her strategy in this stage, we first prove that as long as she can follow the strategy, the number $e_B(C, R)$ of bad edges cannot be too large. Based on this, we then conclude that Maker indeed can follow her strategy.
Before this, it is useful to observe the following: If $|C_A| > e_B(C, R)$ holds before Maker claims an edge, we know that each vertex in R has a free neighbour in C_A. Thus, Maker can easily claim an edge according to (1) if there exists some bad edge. Moreover, this bad edge then disappears from the set of bad edges when F and R are updated. Thus, the number of bad edges decreases, and again $|C_A| > e_B(C, R)$ holds. In particular, we can continue this way. So, if $|C_A| > e_B(C, R) =: b'$ holds at the beginning of a Maker's move, then Maker in her whole move decreases the number of bad edges by at least $\min\{a, b'\}$. Using this, we prove our first claim.

Claim 7.1. *Assume that Maker can follow the proposed strategy. Then, for each $1 \leq i \leq k-2$, there exists a constant upper bound $c = c(a, i)$ for the number of bad edges throughout the phase i.*

Proof The proof goes by induction on i. Set $c(a, 0) := 0$. There can be at most $c(a, i-1) + a$ bad edges before the first Maker's move that happens completely in phase i, either by induction hypothesis (when $i > 1$) or as Breaker claims a edges in his first move (when $i = 1$). Now, by the observation above, we know that as long as $|C_A| > e_B(C, R)$, Maker can reduce the number of bad edges by (at least) $\min\{e_B(C, R), a\}$ in each round, while Breaker can increase this number by at most a. Thus, as long as $|C_A| > c(a, i-1) + a$ holds before some Maker's move, her strategy ensures that after such a move, $e_B(C, R) \leq c(a, i-1)$ holds. Just when $|C_A| \leq c(a, i-1) + a$ holds before a move of Maker, it might happen that Maker cannot reduce the number of bad edges anymore. But then, the number of remaining steps in phase i is bounded by a constant, and Breaker can add only another constant number of bad edges until phase i ends, giving some constant bound $c(a, i)$. Thus, before any Maker's move in phase i there cannot be more than $c(a, i)$ bad edges, which completes the claim. $\square$

Claim 7.2. *For large n, Maker can follow the strategy of Stage I.*

Proof Assume that Maker could follow the strategy for the first j rounds. Our goal is to show that she can do so in round $j + 1$ as well. For this, observe that before Maker's move, $|R| \geq n/k$ holds. Moreover, the number of bad edges is bounded by a constant, according to the previous claim. So, provided n is large enough, there exist more than a vertices in R that are not incident with bad edges. In particular, in each step of her $(j+1)^{\text{st}}$ move, Maker can claim an edge according to (2). Thus, she can follow the strategy. $\square$

Stage II. Observe that when Maker enters Stage II, the number of bad edges is at most $c(a, k-2) + a$, according to the Claim 7.1. Moreover, $N = |C_A| = |R| \geq n/k - a$. Provided that n is large enough, Proposition 3.1 (with G being the bipartite graph induced by the free edges between C_A and R) now ensures that Maker has a strategy to create the desired matching within $\lfloor N/a \rfloor + 1$ rounds. $\square$

8 Conculding remarks and open problems

Star factor game. Theorem 1.4 tells us that $\tau_{\mathcal{S}_{k,n}}(a : a) \in \{(k-1)n/(ka), (k-1)n/(ka) + 1\}$ for large enough n, in case $ak|(k-1)n$. In fact, in can be checked that there are pairs (a, k) were the first value occurs, while there exist pairs (a, k) for which it does not. It would be interesting to describe all the pairs (a, k) for which Maker cannot win the $(a : a)$ $\mathcal{S}_{k,n}$-game perfectly fast and to determine a winning strategy for the first player in the corresponding strong version in these cases.

***H*-factors.** More generally, it seems to be challenging to describe $\tau_{\mathcal{F}}(a : a)$ in case $\mathcal{F}$ is the family of H-factors, for any given graph H not being a forest. Even in the case $a = 1$ not so much is known. In all the games $\mathcal{F}$ we studied here, it happens that $\tau_{\mathcal{F}}(a : a) = (1 + o(1))\tau_{\mathcal{F}}(1 : 1)/a$. We wonder whether there exist families $\mathcal{F}$ of spanning subgraphs of K_n, where such a relation does not hold.

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