# Hall sets, Lazard sets and comma-free codes 

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#### Abstract

We investigate the relationship between two constructions of maximal comma-free codes described respectively by Eastman and by Scholtz and the notions of Hall sets and Lazard sets introduced in connection with factorizations of free monoids and bases of free Lie algebras.


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## 1 Introduction

The notion of comma-free code has been introduced by Golomb, Gordon and Welsh [ 4 ] after the mention of their possible role in molecular genetics [Z]. These codes are defined by a property of nonoverlap which makes the decoding very simple. The definition implies that the words of a comma-free code are primitiye and that it cannot contain two distinct conjugate words. It was conjectured in [U] that for every odd integer $n$ there exists a comma-free code which is a system of representatives of the conjugacy classes of words of length $n$.

This conjecture was proved by Eastman $\mid$ Fastman 1965 . This conjecture was proved by Eastman R3and later Scholtz [8] gave a different construction. In II and in $\mathbb{7} 7$, the construction of Scholtz was described using concepts related with factorizations of free monoids and bases of free Lie algebras, namely Hall sets and Lazard sets. These concepts introduced initially by Schützenberger form a remarkable interaction of notions from classical algebra, such as free Lie algebras and notions from information theory such as comma-free codes. They were studied extensively by Viennot 9$]$ who introduced the terms of Lazard and Hall sets.

Recently Knuth has incorporated the problem of constructing comma-free codes as an example involving techniques important for backtrack programming. He gives in particular a simplified description of Eastman's construction in Exercise 32 .

The aim of this article is to show that Eastman construction is also related to Hall and Lazard sets. We prove that there exist a Lazard set of of words such that for any odd integer $n$, its words of length $n$ form the comma-free codes obtained by Eastman's construction (Theorem (25).

Eastman's construction has an advantage over Schotz construction: it gives directly for a comma-free code $X$ constructed by his method a polynomial time algorithm to find the conjugate of a given primitive word which belongs to $X$. This allows to perform the decoding in polunomial time. We will show here that this is also the case for Scholtz construction using an algorithm due to Melançon to find the conjugate of a primitive word which belongs to a given Hall set.

The paper is organized as follows. In Section we recall the definition and basic properties of Hall sets of trees and words and of Lazard sets of words.

In Section we recall some definitions and properties of codes. We define circular codes and the subfamily of comma-free codes.

In Section ${ }^{50 c t i o n b i p s}$, duced by Knuth hitit describe Eastmn's algorithm.

In Section we describe Eastman's algorithm in terms of the notions introduced in the preceding section and prove that it gives a maximal comma-free code of length $n$ for each odd integer $n$ (Proposition $\frac{\text { orrope }}{10}$.

In Section we show the existence of a Lazard set $Z$ such that for every odd integer $n$ the set $Z \cap A^{n}$ is the result of Eastman's construction.

In the last section, we describe the algorithm of Melancon (see R Rutenauer 1993 allows to find the conjugate of a primitive word which belongs to a Lazard set $Z$. It can be applied to find the conjugate of a primitive word which belongs to a comma-free code of the form $Z \cap A^{n}$. In particular it gives such an algorithm for the code obtained by Scholtz construction.

## 2 Hall sets and Lazard sets

We begin by recalling the notions of Hall and Lazard sets. These sets were introduced in connexion with the description of bases of free Lie algebras (see ligutenauer 1993 for the historical background).

### 2.1 Hall sets

The free magma $M(A)$ on the alphabet $A$ is the set of all terms containing the letters and closed under the binary operation $x, y \mapsto(x, y)$. It can be identified with the set of complete binary trees with leaves labeled by $A$.

A totally ordered subset $H$ of $M(A)$ containing $A$ is called a Hall set (of trees) if for any $x, y \in H$, one has $(x, y) \in H$ if and only if $x<y$ and either $y \in A$ or $y=(z, t)$ with $z \leq x$. Moreover, in this case $x<(x, y)$. An element of $H$ is called a Hall tree.

We warn the reader that we follow here the notation of Viennot in Viennot1978 notation used in R Reutenauer 1993 , following Lothaire 篤, is symmetrical and a Hall set $H$ in Rutenauer1993 is such that for any $x, y \in H$, one has $(x, y) \in H$ if and only $x<y$ and either $x \in A$ or $x=(v, w)$ with $y \leq w$. Moreover, in this case $(x, y)<y$. To recover the above definition, one has to reverse the order and to take the mirror image.

The foliage of an element $z$ of $M(A)$ is the word $f(z) \in A^{*}$ defined by $f(a)=a$ if $a \in A$ and $f(x, y)=f(x) f(y)$. Thus the foliage of $z$ is obtained by erasing the parentheses when $z$ is viewed as a term and by following the frontier of the tree if $z$ is viewed as a binary tree.

A Hall set of words is the foliage of a Hall set tree. Its elements are called Hall words

Fix a Hall set. By Reutenauer1993 Corollary 4.5], each Hall word is the foliage of a unique Hall tree.

Two words $x, y$ are conjugate if they have the form $x=u v$ and $y=v u$. Since a conjugate of a word is just cyclic shift, conjugacy is an equivalence on words. A word $x$ is primitive if it is not a power of another word. The conjugacy class of a primitive word is made of brimitive words and a primitive word has $|x|$ distinct conjugates (see [亩 for a more detailed presentation of these properties).

A Lyndon word is a primitive word which is minimal in its conjucacy class.
Example 1 The set of Lyndon words on $A$ is a Hall set. Indeed, for each Lyndon word $x$ which is not a letter, its standard factorization is the pair $(y, z)$ such that $x=y z$ where $y$ is the longest proper prefix of $x$ which is a Lyndon word. We then associate to any Lyndon word $x$ a tree $\pi(x)$ by $\pi(a)=a$ if $a \in A$ and $\pi(x)=(\pi(y), \pi(z))$ if $(y, z)$ is the standard factorization of $x$ One may then verify the condition defining a Hall set (see $\frac{1}{9}$ p. 15 or $[1$, Exercise 8.1.4). The set of Lyndon words of length 4 on $A=\left\{\begin{array}{c}\text { Ifigure } a, ~ b y, ~ i s ~\end{array} a a a a b, a a b b, a b b b\right\}$. The corresponding trees are represented in Figure $\mathbb{\pi}$


Figure 1: The 3 Lyndon trees of degree 4
figureLyndon4

### 2.2 Lazard sets

We denote $A^{\leq n}=\varepsilon \cup A \cup \ldots \cup A^{n}$ the set of words of length at most $n$.
A totally ordered set $Z$ of words on the alphabet $A$ is called a Lazard set if the following holds. For any integer $n \geq 1$, denote the set $Z \cap A \leq n=\left\{z_{1}, z_{2}, \ldots, z_{k}\right\}$ with

$$
\begin{equation*}
z_{1}<z_{2}<\ldots<z_{k} \tag{1}
\end{equation*}
$$

## equationz_i

For $1 \leq i \leq k$, let $Z_{i}$ be the sequence of sets defined by $Z_{1}=A$ and for $1 \leq i \leq k$,

$$
\begin{equation*}
Z_{i+1}=z_{i}^{*}\left(Z_{i} \backslash z_{i}\right) \tag{2}
\end{equation*}
$$

equationZ_i
Then $z_{i} \in Z_{i}$ for $1 \leq i \leq k$ and

$$
\begin{equation*}
Z_{k+1} \cap A^{\leq n}=\emptyset . \tag{3}
\end{equation*}
$$

equationZ_ $\{\mathrm{k}+1\}$
The same remark on the choice of right or left made for Hall sets holds for Lazard sets. We follow here the choice of Viennot $\bar{y}$ and of 1 . The choice made by the second author in is symmetrical. The sets $Z_{i}$ are defined by $Z_{i+1}=\left(Z_{i} \backslash z_{i}\right) z_{i}^{*}$.

By a result due to Viennot [ [ T , Hall sennots of words and Lazard sets coincide (see [7, Therem 4.18]).

The Hall set of trees corresponding to a given Lazard set $Z$ is obtained via the mapping $\pi: Z \rightarrow M(A)$ defined as follows. First $\pi(a)=a$ for any letter $a \in A$.
 $Z_{1}, \ldots, Z_{n}$ be the sequence of prefix codes defined by (畆). Let $z \in Z \cap A^{\leq n}$. If $z \in A$, we set $\pi(z)=z$. Otherwise, let $i$ be the least index such that $z \in Z_{i+1}$. One has then $z=z_{i} y$ with $y \in Z$ and we set $\pi(z)=\left(\pi\left(z_{i}\right), \pi(y)\right)$.

Example 2 Let us verify that the set $L$ of Lyndon words satisfies the condition of the definition of Lazard sets with $n=5$. One has $L \cap A^{[5]}=$ $\{a, a a a a b, a a a b, a a a b b, a a b, a a b a b, a a b b, a a b b b, a b, a b a b b, a b b, a b b b, a b b b b, b\}$. The
corresponding sequence of prefix codes is

$$
\begin{aligned}
Z_{1} & =\{a, b\}, \\
Z_{2} \cap A^{\leq 5} & =\{a a a a b, a a a b, a a b, a b, b\}, \\
Z_{3} \cap A^{\leq 5} & =\{a a a b, a a b, a b, b\}, \\
Z_{4} \cap A^{\leq 5} & =\{a a a b b, a a b, a b, b\}, \\
Z_{5} \cap A^{\leq 5} & =\{a a b, a b, b\}, \\
Z_{6} \cap A^{\leq 5} & =\{a a b a b, a a b b, a b, b\}, \\
Z_{7} \cap A^{\leq 5} & =\{a a b b, a b, b\}, \\
Z_{8} \cap A^{\leq 5} & =\{a a b b b, a b, b\}, \\
Z_{9} \cap A^{\leq 5} & =\{a b, b\}, \\
Z_{10} \cap A^{\leq 5} & =\{a b a b b, a b b, b\}, \\
Z_{11} \cap A^{\leq 5} & =\{a b b, b\}, \\
Z_{12} \cap A^{\leq 5} & =\{a b b b, b\}, \\
Z_{13} \cap A^{\leq 5} & =\{a b b b b, b\}, \\
Z_{14} \cap A^{\leq 5} & =\{b\},
\end{aligned}
$$

The following result is analogous with Proposition 4.1 in $\frac{\text { Reutenauer } 1993}{\text { which is stated }}$ for Hall sets (and requires a total order on $M(A)$ ).
propositionLazard Proposition 3 Assume that $A^{*}$ is totally ordered with an order such that for any $u, v \in A^{*}$, if $u<v$, then $u<u v$. Then there is a unique Lazard set $Z$ ordered by the restriction to $Z$ of this order.

Proof. We show for each $n \geq 1$ the existence and uniqueness of a sequence $\left(z_{i}, Z_{i}\right)$ of pairs of a word $z z_{i}$ and $\operatorname{set}^{2} Z_{i z_{i}} \operatorname{such}_{k+1} \ddagger$ hat $z_{i} \in Z_{i}$ and satisfying for $1 \leq i \leq k$ conditions (III), (LI) and (II).

Let $n \geq 1$. Starting with $Z_{1}=A$, we define for $i \geq 1$ a pair $\left(z_{i}, Z_{i}\right)$ of a word $z_{i}$ and a maximal prefix code $Z_{i}$ as follows. We choose $z_{i} \in Z_{i}$ as the minimal element of $Z_{i}$ of length at most $n$ and set $Z_{i+1}=z_{i}^{*}\left(Z_{i} \backslash z_{i}\right)$. We stop when $Z_{i} \cap A^{\leq n}=\emptyset$.

Let $k$ be the first index such that $Z_{k+1} \cap A \leq n=\emptyset$. Then, $z_{1}<z_{2}<\ldots<z_{k}$. Indeed, let $1 \leq i \leq k-1$ and set $z_{i+1}=z_{i}^{p} z$ with $z \in Z_{i} \backslash z_{i}$. Let us show by induction on $0 \leq q \leq p$ that $z_{i}<z_{i}^{q} z$. It is true for $q=0$ since $z_{i}$ is minimal in $Z_{i}$. Next, for $q \geq 1$, set $u=z_{i}$ and $v=z_{i}^{q-1} z$. Then $u<v$ by induction hypothesis and thus $u<u v$ by the hypothesis made on the order. Thus $z_{i}<z_{i}^{q} z$.

The sequence defined as above for each $n$ is clearly an initial part of the sequence defined for $n+1$. Thus there is a unique set $Z$ such that $Z \cap A^{\leq n}=$ $\left\{z_{1}, \ldots, z_{k}\right\}$.

## 3 Comma-free codes

 more detailed presentation). Let $A$ be an alphabet. A set $X \subset A^{+}$is a code if every word in $A^{*}$ has at most one factorization in words of $X$. Formally, for any $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots y_{m}$ in $X$ one has

$$
x_{1} \cdots x_{n}=y_{1} \cdots y_{m}
$$

only if $n=m$ and $x_{i}=y_{i}$ for $i=1, \ldots, n$.
A prefix code is a set $X \subset A^{+}$such that no word of $X$ is a prefix of another word of $X$.

A circular code is a set $X \subset A^{+}$such that any word written on a circle has at most one factorization in words of $X$. Formally, for any $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots y_{m}$ in $X$ and $p \in A^{*}, s \in A^{+}$, one has

$$
s x_{2} \cdots x_{n} p=y_{1} \cdots y_{m}, x_{1}=p s
$$

only if $n=m, p=\varepsilon$, and $x_{i}=y_{i}$ for $i=1, \ldots, n$.
It can be shown that a code $X$ is circular if and only if the submonoid $X^{*}$ satisfies the following condition. For any $u, v \in A^{*}$, one has

$$
u v, v u \in X^{*} \Rightarrow u, v \in X^{*}
$$

Let $n \geq 1$. A set $X \subset A^{n}$ is a comma-free code if the following property holds. For any $x, y, z \in X$, if $x$ is a factor of $y z$, then $x=y$ or $x=z$.

One may verify that $X \subset A^{n}$ is comma-free if and only if for any $x \in X^{+}$ and $u, v \in A^{*}$,

$$
\begin{equation*}
u x v \in X^{*} \Rightarrow u, v \in X^{*} \tag{4}
\end{equation*}
$$

It is clear that a comma-free code is circular. In particular a comma-free code of length $n$ contains only primitive words and at most one element of each conjugacy class of primitive words of length $n$.

Denote by $\ell_{n}(k)$ the number of primitive conjugacy classes of primitive words of length $n$ on an alphabet with $k$ elements.
theoremEastman Theorem 4 (Eastman) For any alphabet $A$ with $k$ letters and for any odd integer $n \geq 1$, there is a comma-free code $X \subset A^{n}$ with $\ell_{n}(k)$ elements.

## 4 Scholtz algorithm

We reproduce here the construction of Scholtz giving a proof of Theorem 苗 (see lif for more detalls).

We say that a sequence $\left(x_{i}, X_{i}\right)_{i \geq 1}$ of pais of a word $x_{i}$ and a set $X_{i}$ containing $x_{i}$ is a Hall sequence if it is obtained as follows. Let $X_{1}=A$ and let $x_{1}$ be an arbitrary letter. If $x_{i}$ and $X_{i}$ are already defined, then $X_{i+1}$ is defined by

$$
\begin{equation*}
X_{i+1}=x_{i}^{*}\left(X_{i} \backslash x_{i}\right) \tag{5}
\end{equation*}
$$

eqHallSequence
and $x_{i+1}$ is an element of $X_{i+1}$ such that $\left|x_{i+1}\right| \geq\left|x_{i}\right|$.
To describe Scholtz construction, we build a particular Hall sequence. For each $i \geq 1$, assuming $\left(x_{1}, X_{1}\right), \ldots\left(x_{i}, X_{i}\right)$ already build, we choose $x_{\text {sed }}$ in $X_{i}$ as a word of minimal odd length in $X_{i}$ and we define $X$ dit by ( 15 )

The following is given in $[\mathbf{I}]$ as a proof of Theorem $\mathbb{4}$ Set $k=\operatorname{Card}(A)$.
Theorem 5 For every odd integer $n$, the set of words of length $n$ in the union of the $X_{i}$ is a comma-free code with $\ell_{n}(k)$ elements.

We now relate Scholtz construction with Hall sets.
Let ( $X$ Renperninde be a Hall sequence and let $U=\bigcup_{n \geq 1} X_{n}$. We use for $u \in U$, as in 11, Section 7.3], the notation

$$
\begin{aligned}
\nu(u) & =\min \left\{i \geq 1 \mid u \in X_{i}\right\}-1 \\
\delta(u) & =\sup \left\{i \geq 1 \mid u \in X_{i}\right\}
\end{aligned}
$$

We define a parenthesis map $\pi: U \rightarrow M(A)$ by $\pi(a)=a$ for $a \in A$ and then inductively $\pi(x)=\left(\pi\left(x_{n}\right), \pi(y)\right)$ if $\nu(x)=n$ and $x=x_{n} y$.

Proposition 6 The set $\pi(U)$ is a Hall set for the order defined by $\pi(u)<\pi(v)$ if $|u|$ is odd and $|v|$ even or if $|u|,|v|$ have the same parity and $u<v$ for the radix order.

Actually, this follows from the genereral result that Hall sets coincide with Lazard sets, as seen above.

This can be verified directly as follows. For any $x, y \in U$, one has

$$
(\pi(x), \pi(y)) \in \pi(U) \Leftrightarrow \nu(y) \leq \delta(x)<\delta(y)
$$

and if $\pi(y)=(\pi(z), \pi(t))$, one has $\nu(y)=\delta(z)$. Thus $\pi(U)$ is a Hall set.

## 5 Dips and superdips

We now come to the descrition of Eastman's construction, as presented in $\frac{\boxed{K}, \overline{5},}{}$ Exercise 32].

Let $A$ be a totally ordered alphabet with at least two elements. Consider the set

$$
D(A)=\left\{a_{1} a_{2} \cdots a_{n} \mid a_{i} \in A, n \geq 2, a_{1} \geq a_{2} \geq \ldots \geq a_{n-1}<a_{n}\right\}
$$

The elements of $D(A)$ are called dips. Let $S(A)$ be the words formed of a dip of odd length followed by a (possibly empty) sequence of dips of even length. The elements of $S(A)$ are called superdips.

Example 7 For $A=\{a, b, c\}$ and $a<b<c$, we have $D(A)=c^{*} b^{*} a^{*}(a b+a c)+$ $c^{*} b^{*} b c$.

Let $X \subset A^{*}$ be a maximal prefix code. A word $x \in X^{*}$ is synchronizing if for any $u \in A^{*}$, one has $u x \in X^{*}$.

Proposition 8 The set $D(A)$ is a maximal prefix code such that each word of $D(A)$ of length at least 3 is synchronizing.

Proof. It is clear that $D(A)$ is a prefix code. To show that it is maximal, assume that $w$ has no prefix in $D(A)$. Then $w=a_{1} a_{2} \cdots a_{k}$ with $a_{1} \geq \ldots \geq a_{k}$. If $a_{k}$ is not the largest letter, then $w b \in D(A)$ for a letter $b>a_{k}$. Otherwise, wab is in $D(A)$ for $a_{k}>a<b$ ( $a, b$ exist since $A$ has at least two elements).

Let $u \in A^{*}$ and let $x \in D(A)$ be of length at least 3 . We show that $u x \in D(A)^{*}$, which will imply that $x$ is a synchronizing word. Set $x=p b a c$ with $p \in A^{*}$ and $a, b, c \in A$ and $b \geq a<c$. Set $u x=y q$ with $y \in D(A)^{*}$ and $q$ a proper prefix of $D(A)$. Since $a<c$, the word $a c$ is not an internal factor of $D(A)$ nor a proper prefix of $D(A)$. This implies that $q=c$ or $q=1$. The first case is not possible because $y$ would end with $b a$ which is not a suffix of $D(A)$. Thus $u x \in D(A)^{*}$.

Note that a word of $D(A)$ of length 2 may be not synchronizing. Indeed, for $a<b<c$, we have $b c \in D(A)$ although $a a b c \notin D(A)^{*}$ and thus $b c$ is not synchronizing.

Proposition 9 The set $S(A)$ is a code on the alphabet $A$.
Proof. The set $S(A)$ is a suffix code on the alphabet $D(A)$ and $D(A)$ is a prefix code on the alphabet $A$.

Note that $S(A)$ is actually a maximal code on the alphabet $A$. Indeed, $D(A)$ is a thin maximal prefix code on the alphabet $A$ and $S(A)$ is a thin maximal suffix code on the alphabet $D(A)$ Thus $S(A)$ is a thin and complete code on the alphabet $A$ by II Proposition 2.6.13].

Proposition 10 Any primitive word $w$ of odd length $m \geq 3$ has a conjugate in $D(A)^{*}$ and thus in $S(A)^{*}$. More precisely, the conjugate of $w$ starting after the shortest prefix of $w^{2}$ which ends with a dip of length at least 3 is in $D(A)^{*}$.

Proof. We first show that $w$ has a conjugate in $D(A)^{*}$. Let $p$ be the shortest prefix of $w^{2}$ which ends with a dip of length at least 3 .

To show the existence of $p$, we consider a factorization $w=u a v$ with $a$ the largest letter among the letters occurring in $w$. The word $v u a$ has at least some ascent (that is, a factor $b c$ with $b<c$ ), since $a$ is its largest letter, and since vua is of length at least 3 and not a power of $a$. Hence vua has a prefix $q$ which is a dip of length at least 2 . Thus $w^{2}$ has the prefix $p=u a q$ which ends with a dip of length at least 3 .

Assume first that $p$ is a prefix of $w$. Set $w=p s$. Since $p$ is synchronizing, we have $s p \in D(A)^{*}$, whence the conclusion. Assume next that $w$ is a prefix of
$p$. Set $p=w r$ and $w=r s$. Since $p$ is synchronizing, we have $p, w p \in D(A)^{*}$. Since $w p=w w r=w r s r=p s r$, we have $s r \in D(A)^{*}$.

We may now assume that $w \in D(A)^{*}$. Since $w$ has odd length, at least one of the dips forming $w$ has odd length. The conjugate starting before this dip is in $S(A)^{*}$.
exampleAbracadabra Example 11 We illustrate Proposition oropositionConjugate by finding the conjugate of the word $w=$ abracadabra which is in $S(A)^{*}$.

The shortest prefix of $w^{2}$ which ends with a dip of length at least 3 is $p=a b r a c$. The conjugate of $w$ starting after $p$ is $a d a b r a a b r a c$ which factorizes in dips as indicated. Its conjugate rac ad ab raab is in $S(A)$
propositionOverlap Proposition 12 No word of $S(A)$ overlaps nontrivially the product of two words in $S(A)$, in the sense that for $x, y, z \in S(A)$, if $x=x_{1} x_{2}$ with $y, x_{1}$ (resp. $z, x_{2}$ ) comparable for the suffix (resp. prefix) order, then $x_{1}$ or $x_{2}$ is empty.

## Proof.

Let $a_{1} a_{2} \cdots a_{k}$ be the last dip of $y$ and let $b_{1} b_{2} \cdots b_{\ell}$ be the first dip of $z$. Note that since $z \in S(A)$, $\ell$ is odd. We assume that $x_{1}, x_{2}$ are nonempty.

Assume first that $x_{1}$ is a letter. Then $x_{1}=a_{k}$. If $a_{k}<b_{1}$, the first dip of $x$ is $a_{k} b_{1}$ and has even length, which is impossible. Otherwise, the first dip of $x$ is $a_{k} b_{1} b_{2} \cdots b_{\ell}$ and thus has also even length, a contradiction.

Assume next that $x$ has length 2. Then $x_{1}=a_{k-1} a_{k}$. But then the first dip of $x$ has length 2 , which is again impossible.

Thus $x_{1}$ has length at least 3 . We distinguish two cases.
Case 1. Assume that $x_{1}$ is shorter than $y$. Set $y=u x_{1}$ (see Figure figureSuperdip left). Since $x_{1}$ ends with $a_{k-1} a_{k}$ which is not an internal factor of $D(A)$, the first dip of $x$ is a prefix of $x_{1}$. Set $x_{1}=t s$ where $t$ is the first dip of $x$. Since $t$ has odd length, its length is at least 3. Since a dip of length at least 3 is synchronizing, and since $y=u x_{1}=u t s$, we have $u t, s \in D(A)^{*}$ and thus also $x_{1} \in D(A)^{*}$ since $t, s \in D(A)^{*}$. This implies that $x_{2} \in D(A)^{*}$ a contradiction, since the first dip of $x_{2}$ is equal to the first dip of $z$ which has odd length and since $x$ is a superdip.


Figure 2: The two cases

Case 2. Assume now that $x_{1}$ is longer than $y$. Set $x_{1}=v y$ (see Figure $\frac{\text { figureSuperdip }}{\text { on }}$ the right). Let $t$ be the first dip of $y$. Since $t$ is synchronizing as in Case 1, we have $t, v t \in D(A)^{*}$. Set $y=t s$. Since $y, t \in D(A)^{*}$, we have $s \in D(A)^{*}$. Thus $x_{2} \in D(A)^{*}$ a contradiction, as in Case 1 .

The following corollary shows that the code $S(A)$ satisfies Condition (III).

## corollaryComma

Corollary 13 For any $x \in S(A)^{+}$and $u, v \in A^{*}$ such that uxv $\in S(A)^{*}$, one has $u, v \in S(A)^{*}$.

Proof. Set $x=x_{1} \cdots x_{k}$ and $u x v=y_{1} \cdots y_{\ell}$ with $y_{i} \in S(A)$. We assume that $u, v$ are nonempty. We may assume that $u$ is a prefix of $y_{1}$ since otherwise we may simplify by $y_{1}$ on the left. Next, $u x_{1}$ cannot be a proper prefix of $y_{1}$ because we would have $u, v \in D(A)^{*}$ and all dips of $x_{1}$ would be even, a contradiction. Thus $x$ overlaps $y_{1} y_{2}$ which implies that $u=y_{1}$.

## 6 Eastman algorithm

Consider now the radix order on $A^{*}$, defined by $u<v$ if $|u|<|v|$ or if $|u|=|v|$ and $u$ precedes $v$ in lexicographic order.

Let $\left(S_{n}(A)\right)_{n>0}$ be the sequence of maximal codes on the alphabet $A$ obtained as follows. We start with $S_{0}(A)=A$. For $n \geq 1$, let $D_{n}(A)=$ $D\left(S_{n-1}(A)\right)$ and $S_{n}(A)=S\left(S_{n-1}(A)\right)$ where $D, S$ are defined in the previous exercise with $S_{n-1}(A)$ instead of $A$ as an alphabet, using the order on $S_{n-1}(A)$ induced by the radix order on $A^{*}$.

Note that each $S_{n}(A)$ is formed of words of odd length. Indeed, this is true for $n=0$ and assuming that it is true for $n-1$, we obtain the property for $D_{n}(A)$ using the following lemma whose proof is left to the reader. A graded alphabet is a set $A$ with a map $\operatorname{deg}: A \rightarrow \mathbb{N}$ associating to each letter its degree. The degree of a word is the sum of the degrees of its letters.

Lemma 14 Let $A$ be a graded alphabet such that $\operatorname{deg}(a)$ is odd for every $a \in A$. Then every word of odd length has odd degree.

The elements of $D_{n}(A)$ are called $n$-dips and those of $S_{n}(A)$ are called $n$ superdips. Recognizing if a word $w$ is in $D_{n}(A)^{*}$ or $S_{n}(A) *$ can be done operating for increasing values of $i=1, \ldots, n$. Assume that $w \in S_{i-1}(A)^{*}$. Then one may use a left to right scan of $w$ to write $w=x p$ with $x \in D_{i}(A)^{*}$ and $p$ a proper prefix of $D_{i}(A)^{*}$. Set $x=x_{1} \cdots x_{k}$ with $x_{j} \in D_{i}(A)$. Selecting the blocks beginning with an odd $i$-dip, we obtain $x=q y_{1} \cdots y_{\ell}$ with $y_{j} \in S_{i}(A)$. Then $w \in S_{i}(A)^{*}$ if and only if $p=q=\varepsilon$.

Consider the following algorithm to compute a conjugate of a primitive word $w \in A^{*}$ of odd length $m \geq 3$ which is in $S_{n}(A)$. For successive values of $n \geq 1$, we perform the following steps.

1. Let $p$ be the shortest prefix of $w^{2}$ which ends with an $n$-dip of length at least 3 on the alphabet $S_{n-1}(A)$ (such a prefix exists because the length of $w$ on the alphabet $S_{n-1}(A)$ is at least 3). Replace $w$ by its conjugate starting after $p$. Now $w \in D_{n}(A)^{*}$ (see below).
2. Let $q$ be the first dip of odd length of $w$ (it exists because $w$ has odd length). Replace $w$ by its conjugate starting before $q$. Now $w \in S_{n}(A)^{*}$.
We stop when $w$ is in $S_{n}(A)$. It follows from Proposition $\frac{\text { roropositionConjugate }}{100} D_{n}(A)=$ $D\left(S_{n-1}(A)\right)$, that the conjugate of $w$ chosen as in step 1 of the algorithm is in $D_{n}(A)^{*}$.

Example 15 We perform the above algoritm on the word abracadabra (already considered in Example III.

Write periodically the word abracadabra and factorize it in dips. We obtain
abracadabra abracadabra $\cdots=a b$ rac ad ab raab rac ad ab raab...
We thus replace abracadabra by its conjugate

$$
\text { racadabraab }=\text { rac ad ab raab. }
$$

Since it is a superdip (a dip of length 3 followed by dips of lengths 2, 2, 4), the algorithm stops.

We conclude from the above that one has the following result.

Proposition 16 For every odd integer $m$, the set of words of length $m$ which belongs to some $S_{n}(A)$ is a comma-free code which meets all conjugacy classes of primitive words of length $m$.

Proof. Set $U=\cup_{n \geq 0} S_{n}(A)$. The set $U \cap A^{m}$ meets every conjugacy class of of primitive words of length $m$ because the algorithm above gives this conjugate. To verify that it is comma-free, we prove by induction on $|x|+|y|+|z|$ that a word $x$ in $U$ does not overlap nontrivially a product $y z$ of words $y, z \in U$ in the sense that if $x=x_{1} x_{2}$ with $x_{1}$ a proper suffix of $y$ and $x_{2}$ a proper prefix of $z$, then $x_{1}$ or $x_{2}$ is empty.

The property is true if one of $x, y, z$ is a letter.
Otherwise, we have $x, y, z \in S_{1}(A)^{*}$. Set $y=u x_{1}$ and $z=x_{2} v$. By Corollary IT, we have $u, v \in S_{1}(A)^{*}$ and thus also $x_{1}, x_{2} \in S_{1}(A)^{*}$. Since $S_{n}(A)=$ $S_{n-1}\left(S_{1}(A)\right)$ for all $n \geq 1$, we may apply the induction hypothesis to the words $x^{\prime}, y^{\prime}, z^{\prime}$ obtained from $x, y, z$ by considering $S_{1}(A)$ as a new alphabet, obtaining the conclusion.

The following example shows that the comma-free codes obtained by Eastman algorithm are not the same as those obtained by Scholtz algorithm.
exampleDiff Example 17 Let $A=\{a, b, c\}$, the word $a^{12} c a^{10} c a^{2} c a^{2} b a^{2} c$ is in $S_{2}(A)$ since all words $a^{12} c, a^{10} c, a^{2} c, a^{2} b$ are odd length dips and $a^{12} c \geq a^{10} c \geq a^{2} c \geq$ $a^{2} b<a^{2} c$. Thus it belongs to the comma-free code of length 33 obtained by Eastman algorithm. But it is not in the code obtained by Scholtz algorithm which contains the conjugate $a^{10} c a^{2} c a^{2} b a^{2} c a^{12} c$.

## 7 A new Lazard set

The aim of this section is to show that the comma-free code obtained by Eastman algorithm is a Lazard set and thus can be obtained by an elimination method, as in Scholtz algorithm, although using a different order (which does not respect length and thus in not defined by a Hall sequence).

Let $D_{n}(A)$ be the set of $n$-dips as defined in the previous section and let $P_{n}(A)$ denote the set of proper prefixes of $D_{n}(A)$. One has

$$
A^{*} \supset D_{1}(A)^{+} P_{1}(A) \supset \cdots \supset D_{n}(A)^{+} P_{n}(A) \supset \cdots
$$

For $w \in A^{*}$, the index of $w$, denoted $\operatorname{index}(w)$, is the largest integer $n$ such that $w \in D_{n}(A)^{+} P_{n}(A)$.

Example 18 The index of aabaa is 1 .
Note that if $x \in S_{n}(A)$, then $\operatorname{index}(x)=n$. Indeed, $x \in D_{n}(A)^{+}$and $x$ cannot have a prefix in $D_{n+1}(A)$ since the words of $D_{n+1}(A)$ have length at least 2 on the alphabet $S_{n}(A)$.

We define the following order on $A^{*}$. For $x, y \in A^{*}$, we define $x \prec y$ if
(i) $x$ has odd length and $y$ has even length, or
(ii) the lengths of $x$ and $y$ the same parity and
(a) $\operatorname{index}(x)<\operatorname{index}(y)$ or
(b) $\operatorname{index}(x)=\operatorname{index}(y)$ and $x<y$ for the radix order.

With the objective of applying Proposition bropositionLazard we prove the following property of this order.

Proposition 19 For $x, y \in A^{*}$, if $x \prec y$, then $x \prec x y$.
Proof. Assume first that $y$ has even length. Then the lengths of $x, x y$ have the same parity. Assume that $\operatorname{index}(x)=n$. Set $x=z p$ with $z \in D_{n}(A)^{+}$and $p \in P_{n}(A)$. Since $D_{n}(A)$ is a maximal prefix code, we have $p y \in D_{n}(A)^{*} P_{n}(A)$. Thus $x y \in D_{n}(A)^{+} P_{n}(A)$ showing that index $(x y) \geq n$. If index $(x y)>n$, then $x \prec x y$. Otherwise, $x$ is shorter than $x y$ and thus $x \prec x y$.

Next, if $y$ has odd length, then $x$ is also of odd length and $x y$ has even length, which implies $x \prec x y$.

Let $Z$ be the Lazard set corresponding to the order $\prec$, (which exists and is unique by Proposition

Let $N \geq 1$ be an integer and set $Z \cap A^{\leq N}=\left\{z_{1}, z_{2}, \ldots, z_{M}\right\}$ with $z_{1} \prec z_{2} \prec$ $\ldots \prec z_{M}$. Let $Z_{1}, \ldots, Z_{M+1}$ be the sequence of maximal prefix codes defined, starting with $Z_{1}=A$, by $Z_{n+1}=z_{n}^{*}\left(Z_{n} \backslash z_{n}\right)$.

The standard factorization of a word $z$ of $Z \cap A^{\leq N}$ which is not a letter is the pair $\left(z_{n}, y\right)$ such that $z=z_{n} y$ with $1 \leq n<M$ and $y \in z_{n}^{*}\left(Z_{n} \backslash z_{n}\right)$. In this way the Hall tree corresponding to $z$ is $\pi(z)=\left(\pi\left(z_{n}\right), \pi(y)\right)$.

We need another definition. For some $k \geq 0$, an element $y \in Z$ is consistently in $D_{k+1}(A)$ if $y=y_{1} \cdots y_{s}$ with $y_{i} \in S_{k}(A) \cap Z, y_{1} \geq \ldots \geq y_{s-1}<y_{s}$, and $\pi(y)=\left(\pi\left(y_{1}\right), \pi\left(y_{2} \cdots y_{s}\right)\right)$. Thus, when $y$ is consistently in $D_{k+1}(A)$, its standard factorization is given by its $(k+1)$-dip structure.

Note that any element of $D_{1}(A)$ is in $Z$ (for large enough $N$ ) and is consistently in $D_{1}(A)$. Indeed, if $y=a_{1} \cdots a_{k}$ with $a_{i} \in A$ and $a_{1} \geq \ldots \geq a_{k-1}<a_{k}$, then $\pi(y)=\left(a_{1} \ldots\left(a_{k-1}, a_{k}\right)\right)$.

Note also that there can be elements of $Z \cap D_{k+1}(A)$ which are not consistently in $D_{k+1}(A)$, as shown in the example below.

Example 20 The word $z=$ aadaabaacab is in $D_{2}(A)$ since aad, aab, aacab $\in$ $S_{1}(A)$ and $a a d>a a b<a a c a b$. We have also $z \in Z$ with the decomposition $\pi(y)=((\pi(a a d),(\pi(a a b), \pi(a q c))) \pi(a b))$. But the two decompositions do not coincide (see Figure where the tree is only partially developped).


Figure 3: The tree $\pi(y)$.

We will use two lemmas concerning these notions. The first one describes a situation where the structure of the Lazard set coincides with that of the dip and superdip structure.
lemma2 Lemma 21 Let $k \geq 0$ and $n \geq 1$ be such that $x \in S_{k}(A)$ and $y \in Z_{n+1}$ with $|y|$ odd. If $y$ is consistently in $D_{k+1}(A)$, then $z_{n} y$ is in consistently in $D_{k+1}(A)$.

Proof. Set $y=y_{1} \cdots y_{s}$ with $y_{i} \in S_{k}(A) \cap Z$ and $y_{1} \geq \ldots \geq y_{s-1}<y_{s}$. By the hypothesis, $y_{1}=z_{m}$ with $m<n$ and thus $y_{1} \prec z_{n}$. Thus $z_{n} \geq y_{1} \geq \ldots \geq$ $y_{s-1}<y_{s}$, which shows that $x y$ is in $D_{k+1}(A)$. It is consistently in $D_{k+1}(A)$ since $\pi\left(z_{n} y\right)=(\pi(x), \pi(y))$.

The next lemma describes a situation where the two structures are distinct.
lemma3 Lemma 22 For $k \geq 0$ and $n \geq 1$, if $x \in S_{k}(A)$ and $y \in D_{m}(A)$ with $|y|$ even and $m \leq k$, then $x y \in S_{k}(A)$.

Proof. We prove the statement by induction on $k-m$. It is true if $m=k$ by definition of $S_{k}(A)$. Assume $m<k$ and set $x=u_{1} \cdots u_{s}$ with $u_{i} \in D_{k}(A)$. Set $u_{s}=v_{1} \cdots v_{t}$ with $v_{i} \in S_{k-1}(A)$ and $v_{1} \geq \ldots \geq v_{t-1}<v_{t}$. We have $v_{t} y \in S_{k-1}(A)$ by induction hypothesis. Since $v_{t}<v_{t} y$ in the radix order, this implies $u_{s} y=v_{1} \cdots v_{t-1}\left(v_{t} y\right) \in D_{k}(A)$. Since $|y|$ is even, the lengths of $u_{s}$ and $u_{s} y$ have the same parity. This implies that $x y=u_{1} \cdots u_{s-1}\left(u_{s} y\right)$ is in $S_{k}(A)$.

Example 23 Set $x=$ aadaabaac and $y=a b$ as in Example lexampleNotConsistent have seen, $x y \in D_{2}(A)$ but the decomposition of $x y$ in 1-dips is (aad, aab, aacab) although $\pi(x y)=((\pi(a a d),(\pi(a a b), \pi(a a c))), \pi(a b))$.

Let $\Delta=\cup_{n \geq 1} D_{n}(A)$ and $\Sigma=\cup_{n \geq 0} S_{n}(A)$.
Proposition 24 The words of odd length in $Z$ are in $\Sigma$.
Proof. For a given integer $k \geq 0$, we define a set $\mathcal{P}_{k}$ which is formed of the words $y \in Z$ such that the following holds.
(i) If $|y|$ is odd, then $y$ is in $S_{k}(A)$ or consistently in $D_{k+1}(A)$.
(ii) If $|y|$ is even, then $y$ is consistently in $D_{k+1}(A)$ or in $D_{m}(A)$ with $m \leq k$.

We prove by induction on $n \geq 1$ that if $n=1$ or if $z_{n-1}$ has odd length, one has $Z_{n} \subset \Delta \cup \Sigma$, and more precisely that, for some integer $k \geq 0$, all words of $Z_{n}$ are in $\mathcal{P}_{k}$.

This is true for $n=1$ with $k=0$. Indeed, $Z_{1}=A=S_{0}(A)$. Thus $Z_{1} \subset \mathcal{P}_{0}$.
Assume now that $n \geq 2$ and that $z_{n-1}$ has odd length. Then by the induction hypothesis $Z_{n-1} \subset \mathcal{P}_{k-1}$ for some $k \geq 1$. Since $z_{n-1}$ has odd length it is either in $S_{k-1}(A)$ or in $D_{k}(A)$.

Case 1 Assume first that $z_{n-1} \in S_{k-1}(A)$. We show that in this case $Z_{n} \subset$ $\mathcal{P}_{k-1}$. We have to prove that for $y \in Z_{n} \cap \mathcal{P}_{k-1}$, one has $z_{n-1} y \in \mathcal{P}_{k-1}$.
1.1 Suppose that $y$ has odd length. Then $y$ is in $S_{k-1}(A)$ or consistently in $D_{k}(A)$.
1.1.1 If $y \in S_{k-1}(A)$, then $z_{n-1} y$ is consistently in $D_{k}(A)$ and thus $z_{n-1} y \in$ $\mathcal{P}_{k-1}$
1.1.2 If $y \in D_{k}(A)$, then $z_{n-1} y$ is consistently in $D_{k}(A)$ by Lemma and thus $z_{n-1} y \in \mathcal{P}_{k-1}$.
1.2 Suppose now that $y$ has even length. Then $y$ is either consistently in $D_{k}(A)$ or in $D_{m}(A)$ for $m<k-1$.
1.2.1 If $y$ is consistently in $D_{k}(A)$, then $z_{n-1} y$ is consistently in $D_{k}(A)$ by Lemma
1.2.2 Otherwise, it is in in $D_{m}(A)$ for $m<k-1$ and thus $z_{n-1} y$ is in $S_{k-1}(A)$ by Lemma

In both cases, $z_{n-1} y \in \mathcal{P}_{k-1}$.
Case 2 Assume now that $z_{n-1}$ is in $D_{k}(A)$. We show that in this case $Z_{n} \subset$ $\mathcal{P}_{k}$. Since $z_{n-1} \prec z$ for every $z \in Z_{n-1} \backslash z_{n-1}$, the words of odd length in $Z_{n-1}$ cannot be in $S_{k-1}(A)$. Thus $Z_{n-1} \subset \mathcal{P}_{k}$ and we have to prove that for any $y \in Z_{n} \cap \mathcal{P}_{k}$, we have $z_{n-1} y \in \mathcal{P}_{k}$. The proof is the same as in Case 1 with $k$ instead of $k-1$.

Using Proposition wropositionInduction
Theorem 25 For any odd integer $m$, the set of words of $\Sigma$ of length $m$ is equal to the set of words of length $m$ in $Z$.

Proof. We have shown that $Z 2 A^{m}$ is contained in $\Sigma \cap A^{m}$. Since $Z$ is a Lazard set, by TIT Propositron of $A^{*}$. Thus, by IT Corollary 8.1.7], it is a set of representatives of the primitive conjugacy classes. Since $\Sigma \cap A^{m}$ is comma-free by Proposition likp this forces $Z \cap A^{m}=\Sigma \cap A^{m}$.

## 8 The Melançon algorithm

The last part, taken from Reutenauer 1993 , describes an algorithm due to Melançon to find the conjugate of a primitive word which belongs to a Lazard set. We include it here because it gives an algorithm to compute the conjugate of a word of even length which belongs to the comma-free code obtained by Scholtz algorithm.

Let $Z$ be a Lazard set. Consider the following algorithm starting with a primitive word $w=a_{1} a_{2} \cdots a_{m}$ and operating on a sequence $s=\left(s_{1}, \ldots, s_{n}\right)$ of $n$ elements of $Z$, not all equal. Initially, $s=\left(a_{1}, a_{2}, \ldots, a_{m}\right)$. The main step transformes $s$ as follows. Since not all $s_{i}$ are equal, there is an index $i$ with $1 \leq i \leq n$ such that $s_{i}<s_{i+1}$ (taking the indices cyclically) with $s_{i}$ minimal among $s_{1}, \ldots, s_{n}$. If $i<n$, change $s$ into $\left(s_{1}, \ldots, s_{i-1}, s_{i} s_{i+1}, s_{i+2}, \ldots, s_{n}\right)$. If $i=n$, change $s$ into $\left(s_{n} s_{1}, s_{2}, \ldots, s_{n-1}\right)$. The algorithm stops when $n=1$.

Example 26 We use the algorithm to find the conjugate of abracadabra which is in the code $S_{1}$, using the order $\prec$.

A first sequence of iterations transforms $s=(a, b, r, a, c, a, d, a, b, r, a)$ into $s=(a b, r, a c, a d, a b, r, a)$. At this step, the minimum is the last one and we obtain $s=(a a b, r, a c, a d, a b, r)$. The minimum is now $r$ and thus we obtain in two steps $s=(r a a b, r a c, a d, a b)$. The minimum is now rac and we obtain in two steps $s=($ raab, racadab $)$. The last one being the minimum, we finally obtain $s=($ racadabraab $)$.

We claim that the result of the algorithm is the conjugate of $w$ which is in $Z$.
Let $Z \cap A^{[m]}=\left\{z_{1}, z_{2}, \ldots, z_{k}\right\}$ and let $Z_{i}$ be the sequence defined, as in the definition of Lazard sets, by $Z_{1}=A$ and $Z_{i+1}=z_{i}^{*}\left(Z_{i} \backslash z_{i}\right)$. For $z \in Z$, we denote $\nu(z)=\min \left\{i \geq 1 \mid z \in Z_{i}\right\}-1$ and $\delta(z)=\max \left\{i \geq 1 \mid z \in Z_{i}\right\}$. Note that $\delta\left(z_{i}\right)=i$ and that $y<z$ if and only if $\delta(y)<\delta(z)$. Then, as for Hall sequences, for $y, z \in Z$, one has $y z \in Z$ if and only if $\nu(z) \leq \delta(y)<\delta(z)$. Moreover, in this case, $\nu(y z)=\delta(y)$.

Let us show that any sequence $s=\left(s_{1}, \ldots, s_{n}\right)$ obtained during the algorithm is such that all $s_{i}$ are in $Z$ and for any $s_{i}$, either $s_{i} \in A$ or $\nu\left(s_{i}\right) \leq$ $\delta\left(s_{1}\right), \ldots, \delta\left(s_{n}\right)$. This is true for the initial value of $s$. Next, if we assume that $s$ has this property and is not constant, let $i$ be such that $s_{i}<s_{i+1}$ and $s_{i}=\min \left\{s_{1}, \ldots, s_{n}\right\}$. Then, since $\nu\left(s_{i+1}\right) \leq \delta\left(s_{i}\right)<\delta\left(s_{i+1}\right)$, we have $s_{i} s_{i+1} \in Z$ and $\nu\left(s_{i} s_{i+1}\right)=\delta\left(s_{i}\right)$. Since $\delta\left(s_{i}\right)=\min \left\{\delta\left(s_{1}\right), \ldots, \delta\left(s_{n}\right)\right\}$, we conlude that $\nu\left(s_{i} s_{i+1}\right) \leq \delta\left(s_{1}\right), \ldots, \delta\left(s_{n}\right)$. For the other $s_{j}$, we have $\nu\left(s_{j}\right) \leq \delta\left(s_{i}\right)$ by induction hypothesis and thus $\nu\left(s_{j}\right)<\delta\left(s_{i} s_{i+1}\right)$. When the algorithm stops, we obtain a word in $Z$.

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