Hall sets, Lazard sets and comma-free codes

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Abstract

We investigate the relationship between two constructions of maximal comma-free codes described respectively by Eastman and by Scholtz and the notions of Hall sets and Lazard sets introduced in connection with factorizations of free monoids and bases of free Lie algebras.

Contents

1	Introduction	1
2	Hall sets and Lazard sets2.1Hall sets2.2Lazard sets	2 3 4
3	Comma-free codes	6
4	Scholtz algorithm	6
5	Dips and superdips	7
6	Eastman algorithm	10
7	A new Lazard set	12
8	The Melançon algorithm	15

1 Introduction

The notion of comma_free code has been introduced by Golomb, Gordon and $\frac{ColombGordonWelch1958}{[4]}$ after the mention of their possible role in molecular genetics [2]. These codes are defined by a property of nonoverlap which makes the decoding very simple. The definition implies that the words of a comma-free code are primitive and that it cannot contain two distinct conjugate words. It was conjectured in [4] that for every odd integer *n* there exists a comma-free code which is a system of representatives of the conjugacy classes of words of length *n*. This conjecture was proved by Eastman 1965 ferent construction. In [1] and in [7], the construction of Scholtz was described using concepts related with factorizations of free monoids and bases of free Lie algebras, namely Hall sets and Lazard sets. These concepts introduced initially by Schützenberger form a remarkable interaction of notions from classical algebra, such as free Lie algebras and notions from information theory such as comma-free codes. They were studied extensively by Viennot [9] who introduced the terms of Lazard and Hall sets. Recently Knuth [5] has incorporated the problem of constructing comma-free

Recently Knuth [5] has incorporated the problem of constructing comma-free codes as an example involving techniques important for backtrack programming. He gives in particular a simplified description of Eastman's construction in Exercise 32.

The aim of this article is to show that Eastman construction is also related to Hall and Lazard sets. We prove that there exist a Lazard set of of words such that for any odd integer n, its words of length n form the comma-free codes obtained by Eastman's construction (Theorem 25).

Eastman's construction has an advantage over Schotz construction: it gives directly for a comma-free code X constructed by his method a polynomial time algorithm to find the conjugate of a given primitive word which belongs to X. This allows to perform the decoding in polynomial time. We will show here that this is also the case for Scholtz construction using an algorithm due to Melançon to find the conjugate of a primitive word which belongs to a given Hall set. The paper is organized as follows. In Section $\frac{2}{2}$ we recall the definition and

The paper is organized as follows. In Section 2 we recall the definition and basic properties of Hall sets of trees and words and of Lazard sets of words. In Section 3, we recall some definitions and properties of codes. We define

In Section \overline{B} , we recall some definitions and properties of codes. We define circular codes and the subfamily of comma-free codes.

In Section \overline{b} , $\overline{we give}$ an account of the notions of dips and superdips introduced by Knuth [5] to describe Eastmn's algorithm.

In Section **6**, we describe Eastman's algorithm in terms of the notions introduced in the preceding section and prove that it gives a maximal comma-free code of length n for each odd integer n (Proposition 16). In Section 7 we show the existence of a Lazard set Z such that for every odd

In Section $\overline{I'}$ we show the existence of a Lazard set Z such that for every odd integer n the set $Z \cap A^n$ is the result of Eastman's construction.

In the last section, we describe the algorithm of Melançon (see [7]) which allows to find the conjugate of a primitive word which belongs to a Lazard set Z. It can be applied to find the conjugate of a primitive word which belongs to a comma-free code of the form $Z \cap A^n$. In particular it gives such an algorithm for the code obtained by Scholtz construction.

sectionHall

2 Hall sets and Lazard sets

We begin by recalling the notions of Hall and Lazard sets. These sets were <u>Reutenauer1993</u> introduced in connexion with the description of bases of free Lie algebras (see [7] for the historical background).

2.1 Hall sets

The free magma M(A) on the alphabet A is the set of all terms containing the letters and closed under the binary operation $x, y \mapsto (x, y)$. It can be identified with the set of complete binary trees with leaves labeled by A.

A totally ordered subset H of M(A) containing A is called a *Hall set* (of trees) if for any $x, y \in H$, one has $(x, y) \in H$ if and only if x < y and either $y \in A$ or y = (z, t) with $z \leq x$. Moreover, in this case x < (x, y). An element of H is called a *Hall tree*.

We warn the reader that we follow here the notation of Viennot in [9]. The notation used in [7], following Lothaire [6], is symmetrical and a Hall set H in [7] is such that for any $x, y \in H$, one has $(x, y) \in H$ if and only x < y and either $x \in A$ or x = (v, w) with $y \leq w$. Moreover, in this case (x, y) < y. To recover the above definition, one has to reverse the order and to take the mirror image.

The foliage of an element z of M(A) is the word $f(z) \in A^*$ defined by f(a) = a if $a \in A$ and f(x, y) = f(x)f(y). Thus the foliage of z is obtained by erasing the parentheses when z is viewed as a term and by following the frontier of the tree if z is viewed as a binary tree.

A Hall set of words is the foliage of a Hall set tree. Its elements are called Hall words

Fix a Hall set. By [7, Corollary 4.5], each Hall word is the foliage of a unique Hall tree.

Two words x, y are *conjugate* if they have the form x = uv and y = vu. Since a conjugate of a word is just cyclic shift, conjugacy is an equivalence on words. A word x is *primitive* if it is not a power of another word. The conjugacy class of a primitive word is made of primitive words and a primitive word has |x|distinct conjugates (see 6) for a more detailed presentation of these properties).

A Lyndon word is a primitive word which is minimal in its conjucacy class.

Example 1 The set of Lyndon words on A is a Hall set. Indeed, for each Lyndon word x which is not a letter, its *standard factorization* is the pair (y, z) such that x = yz where y is the longest proper prefix of x which is a Lyndon word. We then associate to any Lyndon word x a tree $\pi(x)$ by $\pi(a) = a$ if $a \in A$ and $\pi(x) = (\pi(y), \pi(z))$ if (y, z) is the standard factorization of x. One may then verify the condition defining a Hall set (see [9] p. 15 or [1, Exercise 8.1.4]). The set of Lyndon words of length 4 on $A = \{a, b\}$ is $\{aaab, aabb, abbb\}$. The corresponding trees are represented in Figure 11.

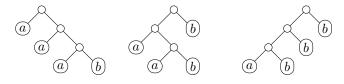


Figure 1: The 3 Lyndon trees of degree 4

figureLyndon4

2.2 Lazard sets

We denote $A^{\leq n} = \varepsilon \cup A \cup \ldots \cup A^n$ the set of words of length at most n.

A totally ordered set Z of words on the alphabet A is called a *Lazard set* if the following holds. For any integer $n \ge 1$, denote the set $Z \cap A^{\le n} = \{z_1, z_2, \ldots, z_k\}$ with

$$z_1 < z_2 < \ldots < z_k.$$
 (1) equationz_i

For $1 \le i \le k$, let Z_i be the sequence of sets defined by $Z_1 = A$ and for $1 \le i \le k$,

$$Z_{i+1} = z_i^* (Z_i \setminus z_i). \tag{2}$$
 equationZ

Then $z_i \in Z_i$ for $1 \leq i \leq k$ and

$$Z_{k+1} \cap A^{\leq n} = \emptyset. \tag{3} \quad \text{equation} \mathbb{Z}_{k+1}$$

_i

The same remark on the choice of right or left made for Hall sets holds for Lazard sets. We follow here the choice of Viennot [9] and of [1]. The choice made by the second author in [7] is symmetrical. The sets Z_i are defined by $Z_{i+1} = (Z_i \setminus z_i)z_i^*$.

By a result due to Viennot [9], Hall sets of words and Lazard sets coincide (see [7, Therem 4.18]).

The Hall set of trees corresponding to a given Lazard set Z is obtained via the mapping $\pi: Z \to M(A)$ defined as follows. First $\pi(a) = a$ for any letter $a \in A$. Next, let $n \geq 1$, let $Z \cap A^{\leq n} = \{z_1, z_2, \ldots, z_k\}$ with $z_1 \leq z_2 \leq z_k$ and let Z_1, \ldots, Z_n be the sequence of prefix codes defined by (Z). Let $z \in Z \cap A^{\leq n}$. If $z \in A$, we set $\pi(z) = z$. Otherwise, let *i* be the least index such that $z \in Z_{i+1}$. One has then $z = z_i y$ with $y \in Z$ and we set $\pi(z) = (\pi(z_i), \pi(y))$.

 corresponding sequence of prefix codes is

=	$\{a,b\},$
=	$\{aaaab, aaab, aab, ab, b\}$
=	$\{aaab, aab, ab, b\},$
=	$\{aaabb,aab,ab,b\},$
=	$\{aab, ab, b\},\$
=	$\{aabab, aabb, ab, b\},$
=	$\{aabb, ab, b\},$
=	$\{aabbb, ab, b\},$
=	$\{ab,b\},$
=	$\{ababb, abb, b\},$
=	$\{abb,b\},$
=	$\{abbb, b\},$
=	$\{abbbb, b\},$
=	$\{b\},$

The following result is analogous with Proposition 4.1 in $\begin{bmatrix} Reutenauer1993 \\ \hline I \end{bmatrix}$ which is stated for Hall sets (and requires a total order on M(A)).

propositionLazard

Proposition 3 Assume that A^* is totally ordered with an order such that for any $u, v \in A^*$, if u < v, then u < uv. Then there is a unique Lazard set Z ordered by the restriction to Z of this order.

Proof. We show for each $n \geq 1$ the existence and uniqueness of a sequence (z_i, Z_i) of pairs of a word, z_i and a set Z_i such that $z_i \in Z_i$ and satisfying for $1 \leq i \leq k$ conditions (I), (2) and (B).

Let $n \ge 1$. Starting with $Z_1 = A$, we define for $i \ge 1$ a pair (z_i, Z_i) of a word z_i and a maximal prefix code Z_i as follows. We choose $z_i \in Z_i$ as the minimal element of Z_i of length at most n and set $Z_{i+1} = z_i^*(Z_i \setminus z_i)$. We stop when $Z_i \cap A^{\le n} = \emptyset$.

Let k be the first index such that $Z_{k+1} \cap A^{\leq n} = \emptyset$. Then, $z_1 < z_2 < \ldots < z_k$. Indeed, let $1 \leq i \leq k-1$ and set $z_{i+1} = z_i^p z$ with $z \in Z_i \setminus z_i$. Let us show by induction on $0 \leq q \leq p$ that $z_i < z_i^q z$. It is true for q = 0 since z_i is minimal in Z_i . Next, for $q \geq 1$, set $u = z_i$ and $v = z_i^{q-1} z$. Then u < v by induction hypothesis and thus u < uv by the hypothesis made on the order. Thus $z_i < z_i^q z$.

The sequence defined as above for each n is clearly an initial part of the sequence defined for n + 1. Thus there is a unique set Z such that $Z \cap A^{\leq n} = \{z_1, \ldots, z_k\}$.

sectionCodes

3

Comma-free codes

In this section, we introduce some basic notions concerning codes (see $\begin{bmatrix} BerstelPerrinReutenauer2010 \\ |1| & for a \end{bmatrix}$ more detailed presentation). Let A be an alphabet. A set $X \subset A^+$ is a code if every word in A^* has at most one factorization in words of X. Formally, for any x_1, \ldots, x_n and y_1, \ldots, y_m in X one has

$$x_1 \cdots x_n = y_1 \cdots y_m$$

only if n = m and $x_i = y_i$ for $i = 1, \ldots, n$.

A prefix code is a set $X \subset A^+$ such that no word of X is a prefix of another word of X.

A circular code is a set $X \subset A^+$ such that any word written on a circle has at most one factorization in words of X. Formally, for any x_1, \ldots, x_n and $y_1, \ldots y_m$ in X and $p \in A^*$, $s \in A^+$, one has

$$sx_2\cdots x_np = y_1\cdots y_m, \ x_1 = ps$$

only if n = m, $p = \varepsilon$, and $x_i = y_i$ for $i = 1, \ldots, n$.

It can be shown that a code X is circular if and only if the submonoid X^* satisfies the following condition. For any $u, v \in A^*$, one has

$$uv, vu \in X^* \Rightarrow u, v \in X^*.$$

Let n > 1. A set $X \subset A^n$ is a *comma-free code* if the following property holds. For any $x, y, z \in X$, if x is a factor of yz, then x = y or x = z.

One may verify that $X \subset A^n$ is comma-free if and only if for any $x \in X^+$ and $u, v \in A^*$,

$$uxv \in X^* \Rightarrow u, v \in X^*.$$
 (4) eqComm

It is clear that a comma-free code is circular. In particular a comma-free code of length n contains only primitive words and at most one element of each conjugacy class of primitive words of length n.

Denote by $\ell_n(k)$ the number of primitive conjugacy classes of primitive words of length n on an alphabet with k elements.

theoremEastman

sectionScholtz

Theorem 4 (Eastman) For any alphabet A with k letters and for any odd integer $n \geq 1$, there is a comma-free code $X \subset A^n$ with $\ell_n(k)$ elements.

4 Scholtz algorithm

We reproduce here the construction of Scholtz giving a proof of Theorem 4 (see T for more details).

We say that a sequence $(x_i, X_i)_{i>1}$ of pais of a word x_i and a set X_i containing x_i is a Hall sequence if it is obtained as follows. Let $X_1 = A$ and let x_1 be an arbitrary letter. If x_i and X_i are already defined, then X_{i+1} is defined by

$$X_{i+1} = x_i^*(X_i \setminus x_i), \tag{5} \quad \text{eqHallSequence}$$

na

and x_{i+1} is an element of X_{i+1} such that $|x_{i+1}| \ge |x_i|$.

To describe Scholtz construction, we build a particular Hall sequence. For each $i \geq 1$, assuming $(x_1, X_1), \ldots, (x_i, X_i)$ already build, we choose x_{i+1} in X_i as a word of minimal odd length in X_i and we define X_{i+1} by (b). The following is given in [1] as a proof of Theorem 4. Set k = Card(A).

Theorem 5 For every odd integer n, the set of words of length n in the union of the X_i is a comma-free code with $\ell_n(k)$ elements.

We now relate Scholtz construction with Hall sets.

Let $(X_n, x_n)_{n \ge 1}$ be a Hall sequence and let $U = \bigcup_{n \ge 1} X_n$. We use for $u \in U$, as in [1, Section 7.3], the notation

$$\nu(u) = \min\{i \ge 1 \mid u \in X_i\} - 1, \delta(u) = \sup\{i \ge 1 \mid u \in X_i\}.$$

We define a parenthesis map $\pi: U \to M(A)$ by $\pi(a) = a$ for $a \in A$ and then inductively $\pi(x) = (\pi(x_n), \pi(y))$ if $\nu(x) = n$ and $x = x_n y$.

Proposition 6 The set $\pi(U)$ is a Hall set for the order defined by $\pi(u) < \pi(v)$ if |u| is odd and |v| even or if |u|, |v| have the same parity and u < v for the radix order.

Actually, this follows from the general result that Hall sets coincide with Lazard sets, as seen above.

This can be verified directly as follows. For any $x, y \in U$, one has

$$(\pi(x), \pi(y)) \in \pi(U) \Leftrightarrow \nu(y) \le \delta(x) < \delta(y)$$

and if $\pi(y) = (\pi(z), \pi(t))$, one has $\nu(y) = \delta(z)$. Thus $\pi(U)$ is a Hall set.

Dips and superdips

We now come to the descrition of Eastman's construction, as presented in $\frac{\text{Knuth2015}}{[5,]}$ Exercise 32].

Let A be a totally ordered alphabet with at least two elements. Consider the set

$$D(A) = \{a_1 a_2 \cdots a_n \mid a_i \in A, n \ge 2, a_1 \ge a_2 \ge \dots \ge a_{n-1} < a_n\}.$$

The elements of D(A) are called *dips*. Let S(A) be the words formed of a dip of odd length followed by a (possibly empty) sequence of dips of even length. The elements of S(A) are called *superdips*.

Example 7 For $A = \{a, b, c\}$ and a < b < c, we have $D(A) = c^*b^*a^*(ab+ac) + ac^*b^*a^*(ab+ac) + b^*a^*(ab+ac) + b^*a^$ $c^*b^*bc.$

sectionDips

 $\mathbf{5}$

Let $X \subset A^*$ be a maximal prefix code. A word $x \in X^*$ is synchronizing if for any $u \in A^*$, one has $ux \in X^*$.

Proposition 8 The set D(A) is a maximal prefix code such that each word of D(A) of length at least 3 is synchronizing.

Proof. It is clear that D(A) is a prefix code. To show that it is maximal, assume that w has no prefix in D(A). Then $w = a_1 a_2 \cdots a_k$ with $a_1 \ge \ldots \ge a_k$. If a_k is not the largest letter, then $wb \in D(A)$ for a letter $b > a_k$. Otherwise, wab is in D(A) for $a_k > a < b$ (a, b exist since A has at least two elements).

Let $u \in A^*$ and let $x \in D(A)$ be of length at least 3. We show that $ux \in D(A)^*$, which will imply that x is a synchronizing word. Set x = pbac with $p \in A^*$ and $a, b, c \in A$ and $b \ge a < c$. Set ux = yq with $y \in D(A)^*$ and q a proper prefix of D(A). Since a < c, the word ac is not an internal factor of D(A) nor a proper prefix of D(A). This implies that q = c or q = 1. The first case is not possible because y would end with ba which is not a suffix of D(A). Thus $ux \in D(A)^*$.

Note that a word of D(A) of length 2 may be not synchronizing. Indeed, for a < b < c, we have $bc \in D(A)$ although $aabc \notin D(A)^*$ and thus bc is not synchronizing.

Proposition 9 The set S(A) is a code on the alphabet A.

Proof. The set S(A) is a suffix code on the alphabet D(A) and D(A) is a prefix code on the alphabet A.

Note that S(A) is actually a maximal code on the alphabet A. Indeed, D(A) is a thin maximal prefix code on the alphabet A and S(A) is a thin maximal suffix code on the alphabet D(A). Thus S(A) is a thin and complete code on the alphabet A by [1, Proposition 2.6.13].

propositionConjugate

Proposition 10 Any primitive word w of odd length $m \ge 3$ has a conjugate in $D(A)^*$ and thus in $S(A)^*$. More precisely, the conjugate of w starting after the shortest prefix of w^2 which ends with a dip of length at least 3 is in $D(A)^*$.

Proof. We first show that w has a conjugate in $D(A)^*$. Let p be the shortest prefix of w^2 which ends with a dip of length at least 3.

To show the existence of p, we consider a factorization w = uav with a the largest letter among the letters occurring in w. The word vua has at least some ascent (that is, a factor bc with b < c), since a is its largest letter, and since vua is of length at least 3 and not a power of a. Hence vua has a prefix q which is a dip of length at least 2. Thus w^2 has the prefix p = uaq which ends with a dip of length at least 3.

Assume first that p is a prefix of w. Set w = ps. Since p is synchronizing, we have $sp \in D(A)^*$, whence the conclusion. Assume next that w is a prefix of

p. Set p = wr and w = rs. Since p is synchronizing, we have $p, wp \in D(A)^*$. Since wp = wwr = wrsr = psr, we have $sr \in D(A)^*$.

We may now assume that $w \in D(A)^*$. Since w has odd length, at least one of the dips forming w has odd length. The conjugate starting before this dip is in $S(A)^*$.

exampleAbracadabra

Example 11 We illustrate Proposition 10 by finding the conjugate of the word

w = abracadabra which is in $S(A)^*$.

The shortest prefix of w^2 which ends with a dip of length at least 3 is p = abrac. The conjugate of w starting after p is ad ab raab rac which factorizes in dips as indicated. Its conjugate rac ad ab raab is in S(A)

propositionOverlap

Proposition 12 No word of S(A) overlaps nontrivially the product of two words in S(A), in the sense that for $x, y, z \in S(A)$, if $x = x_1x_2$ with y, x_1 (resp. z, x_2) comparable for the suffix (resp. prefix) order, then x_1 or x_2 is empty.

Proof.

Let $a_1a_2 \cdots a_k$ be the last dip of y and let $b_1b_2 \cdots b_\ell$ be the first dip of z. Note that since $z \in S(A)$, ℓ is odd. We assume that x_1, x_2 are nonempty.

Assume first that x_1 is a letter. Then $x_1 = a_k$. If $a_k < b_1$, the first dip of x is $a_k b_1$ and has even length, which is impossible. Otherwise, the first dip of x is $a_k b_1 b_2 \cdots b_\ell$ and thus has also even length, a contradiction.

Assume next that x has length 2. Then $x_1 = a_{k-1}a_k$. But then the first dip of x has length 2, which is again impossible.

Thus x_1 has length at least 3. We distinguish two cases.

Case 1. Assume that x_1 is shorter than y. Set $y = ux_1$ (see Figure $\begin{bmatrix} figureSuperdip \\ 2 & on the \\ 2 & on the \\ \end{bmatrix}$ left). Since x_1 ends with $a_{k-1}a_k$ which is not an internal factor of D(A), the first dip of x is a prefix of x_1 . Set $x_1 = ts$ where t is the first dip of x. Since t has odd length, its length is at least 3. Since a dip of length at least 3 is synchronizing, and since $y = ux_1 = uts$, we have $ut, s \in D(A)^*$ and thus also $x_1 \in D(A)^*$ since $t, s \in D(A)^*$. This implies that $x_2 \in D(A)^*$ a contradiction, since the first dip of x_2 is equal to the first dip of z which has odd length and since x is a superdip.

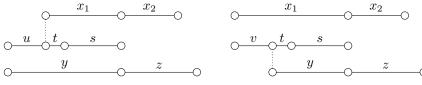


Figure 2: The two cases

figureSuperdip

Case 2. Assume now that x_1 is longer than y. Set $x_1 = vy$ (see Figure $\begin{bmatrix} figureSuperdip \\ 2 & 0n \end{bmatrix}$ the right). Let t be the first dip of y. Since t is synchronizing as in Case 1, we have $t, vt \in D(A)^*$. Set y = ts. Since $y, t \in D(A)^*$, we have $s \in D(A)^*$. Thus $x_2 \in D(A)^*$ a contradiction, as in Case 1.

The following corollary shows that the code S(A) satisfies Condition (4).

corollaryComma

Corollary 13 For any $x \in S(A)^+$ and $u, v \in A^*$ such that $uxv \in S(A)^*$, one has $u, v \in S(A)^*$.

Proof. Set $x = x_1 \cdots x_k$ and $uxv = y_1 \cdots y_\ell$ with $y_i \in S(A)$. We assume that u, v are nonempty. We may assume that u is a prefix of y_1 since otherwise we may simplify by y_1 on the left. Next, ux_1 cannot be a proper prefix of y_1 because we would have $u, v \in D(A)^*$ and all dips of x_1 would be even, a contradiction. Thus x overlaps y_1y_2 which implies that $u = y_1$.

6 Eastman algorithm

sectionEastman

Consider now the radix order on A^* , defined by u < v if |u| < |v| or if |u| = |v|and u precedes v in lexicographic order.

Let $(S_n(A))_{n\geq 0}$ be the sequence of maximal codes on the alphabet A obtained as follows. We start with $S_0(A) = A$. For $n \geq 1$, let $D_n(A) = D(S_{n-1}(A))$ and $S_n(A) = S(S_{n-1}(A))$ where D, S are defined in the previous exercise with $S_{n-1}(A)$ instead of A as an alphabet, using the order on $S_{n-1}(A)$ induced by the radix order on A^* .

Note that each $S_n(A)$ is formed of words of odd length. Indeed, this is true for n = 0 and assuming that it is true for n - 1, we obtain the property for $D_n(A)$ using the following lemma whose proof is left to the reader. A graded alphabet is a set A with a map deg : $A \to \mathbb{N}$ associating to each letter its degree. The degree of a word is the sum of the degrees of its letters.

Lemma 14 Let A be a graded alphabet such that deg(a) is odd for every $a \in A$. Then every word of odd length has odd degree.

The elements of $D_n(A)$ are called *n*-dips and those of $S_n(A)$ are called *n*superdips. Recognizing if a word w is in $D_n(A)^*$ or $S_n(A)^*$ can be done operating for increasing values of i = 1, ..., n. Assume that $w \in S_{i-1}(A)^*$. Then one may use a left to right scan of w to write w = xp with $x \in D_i(A)^*$ and pa proper prefix of $D_i(A)^*$. Set $x = x_1 \cdots x_k$ with $x_j \in D_i(A)$. Selecting the blocks beginning with an odd *i*-dip, we obtain $x = qy_1 \cdots y_\ell$ with $y_j \in S_i(A)$. Then $w \in S_i(A)^*$ if and only if $p = q = \varepsilon$.

Consider the following algorithm to compute a conjugate of a primitive word $w \in A^*$ of odd length $m \ge 3$ which is in $S_n(A)$. For successive values of $n \ge 1$, we perform the following steps.

- 1. Let p be the shortest prefix of w^2 which ends with an n-dip of length at least 3 on the alphabet $S_{n-1}(A)$ (such a prefix exists because the length of w on the alphabet $S_{n-1}(A)$ is at least 3). Replace w by its conjugate starting after p. Now $w \in D_n(A)^*$ (see below).
- 2. Let q be the first dip of odd length of w (it exists because w has odd length). Replace w by its conjugate starting before q. Now $w \in S_n(A)^*$.

We stop when w is in $S_n(A)$. It follows from Proposition $\frac{\text{propositionConjugate}}{10$, since $D_n(A) = D(S_{n-1}(A))$, that the conjugate of w chosen as in step 1 of the algorithm is in $D_n(A)^*$.

Example 15 We perform the above algoritm on the word *abracadabra* (already considered in Example $\overline{\text{III}}$).

Write periodically the word *abracadabra* and factorize it in dips. We obtain

 $abracadabra \ abracadabra \cdots = ab \ rac \ ad \ ab \ raab \ rac \ ad \ ab \ raab \cdots$

We thus replace *abracadabra* by its conjugate

racadabraab = rac ad ab raab.

Since it is a superdip (a dip of length 3 followed by dips of lengths 2, 2, 4), the algorithm stops.

We conclude from the above that one has the following result.

propositionEastman

Proposition 16 For every odd integer m, the set of words of length m which belongs to some $S_n(A)$ is a comma-free code which meets all conjugacy classes of primitive words of length m.

Proof. Set $U = \bigcup_{n\geq 0} S_n(A)$. The set $U \cap A^m$ meets every conjugacy class of of primitive words of length m because the algorithm above gives this conjugate. To verify that it is comma-free, we prove by induction on |x| + |y| + |z| that a word x in U does not overlap nontrivially a product yz of words $y, z \in U$ in the sense that if $x = x_1x_2$ with x_1 a proper suffix of y and x_2 a proper prefix of z, then x_1 or x_2 is empty.

The property is true if one of x, y, z is a letter.

Otherwise, we have $x, y, z \in S_1(A)^*$. Set $y = ux_1$ and $z = x_2 v$. By Corollary 13, we have $u, v \in S_1(A)^*$ and thus also $x_1, x_2 \in S_1(A)^*$. Since $S_n(A) = S_{n-1}(S_1(A))$ for all $n \ge 1$, we may apply the induction hypothesis to the words x', y', z' obtained from x, y, z by considering $S_1(A)$ as a new alphabet, obtaining the conclusion.

The following example shows that the comma-free codes obtained by Eastman algorithm are not the same as those obtained by Scholtz algorithm.

exampleDiff

sectionNewLazard

Example 17 Let $A = \{a, b, c\}$, the word $a^{12}ca^{10}ca^2ca^2ba^2c$ is in $S_2(A)$ since all words $a^{12}c, a^{10}c, a^2c, a^2b$ are odd length dips and $a^{12}c \ge a^{10}c \ge a^2c \ge a^2b < a^2c$. Thus it belongs to the comma-free code of length 33 obtained by Eastman algorithm. But it is not in the code obtained by Scholtz algorithm which contains the conjugate $a^{10}ca^2ca^2ba^2ca^{12}c$.

7 A new Lazard set

The aim of this section is to show that the comma-free code obtained by Eastman algorithm is a Lazard set and thus can be obtained by an elimination method, as in Scholtz algorithm, although using a different order (which does not respect length and thus in not defined by a Hall sequence).

Let $D_n(A)$ be the set of *n*-dips as defined in the previous section and let $P_n(A)$ denote the set of proper prefixes of $D_n(A)$. One has

$$A^* \supset D_1(A)^+ P_1(A) \supset \cdots \supset D_n(A)^+ P_n(A) \supset \cdots$$

For $w \in A^*$, the *index* of w, denoted index(w), is the largest integer n such that $w \in D_n(A)^+ P_n(A)$.

Example 18 The index of *aabaa* is 1.

Note that if $x \in S_n(A)$, then index(x) = n. Indeed, $x \in D_n(A)^+$ and x cannot have a prefix in $D_{n+1}(A)$ since the words of $D_{n+1}(A)$ have length at least 2 on the alphabet $S_n(A)$.

We define the following order on A^* . For $x, y \in A^*$, we define $x \prec y$ if

- (i) x has odd length and y has even length, or
- (ii) the lengths of x and y the same parity and
 - (a) index(x) < index(y) or
 - (b) index(x) = index(y) and x < y for the radix order.

With the objective of applying Proposition B, we prove the following property of this order.

Proposition 19 For $x, y \in A^*$, if $x \prec y$, then $x \prec xy$.

Proof. Assume first that y has even length. Then the lengths of x, xy have the same parity. Assume that index(x) = n. Set x = zp with $z \in D_n(A)^+$ and $p \in P_n(A)$. Since $D_n(A)$ is a maximal prefix code, we have $py \in D_n(A)^*P_n(A)$. Thus $xy \in D_n(A)^+P_n(A)$ showing that $index(xy) \ge n$. If index(xy) > n, then $x \prec xy$. Otherwise, x is shorter than xy and thus $x \prec xy$.

Next, if y has odd length, then x is also of odd length and xy has even length, which implies $x \prec xy$.

Let Z be the Lazard set corresponding to the order \prec , (which exists and is proposition.Lazard unique by Proposition $\frac{3}{3}$.

Let $N \geq 1$ be an integer and set $Z \cap A^{\leq N} = \{z_1, z_2, \dots, z_M\}$ with $z_1 \prec z_2 \prec$ $\ldots \prec z_M$. Let Z_1, \ldots, Z_{M+1} be the sequence of maximal prefix codes defined, starting with $Z_1 = A$, by $Z_{n+1} = z_n^*(Z_n \setminus z_n)$.

The standard factorization of a word z of $Z \cap A^{\leq N}$ which is not a letter is the pair (z_n, y) such that $z = z_n y$ with $1 \le n < M$ and $y \in z_n^*(Z_n \setminus z_n)$. In this way the Hall tree corresponding to z is $\pi(z) = (\pi(z_n), \pi(y)).$

We need another definition. For some $k \ge 0$, an element $y \in Z$ is consistently in $D_{k+1}(A)$ if $y = y_1 \cdots y_s$ with $y_i \in S_k(A) \cap Z, y_1 \ge \ldots \ge y_{s-1} < y_s$, and $\pi(y) = (\pi(y_1), \pi(y_2 \cdots y_s))$. Thus, when y is consistently in $D_{k+1}(A)$, its standard factorization is given by its (k + 1)-dip structure.

Note that any element of $D_1(A)$ is in Z (for large enough N) and is consistently in $D_1(A)$. Indeed, if $y = a_1 \cdots a_k$ with $a_i \in A$ and $a_1 \geq \ldots \geq a_{k-1} < a_k$, then $\pi(y) = (a_1 \dots (a_{k-1}, a_k)).$

Note also that there can be elements of $Z \cap D_{k+1}(A)$ which are not consistently in $D_{k+1}(A)$, as shown in the example below.

exampleNotConsistent

Example 20 The word z = aadaabaacab is in $D_2(A)$ since $aad, aab, aacab \in$ $S_1(A)$ and aad > aab < aacab. We have also $z \in Z$ with the decomposition $\pi(y) = ((\pi(aad), (\pi(aab), \pi(aac))), \pi(ab)).$ But the two decompositions do not coincide (see Figure $\frac{1}{3}$ where the tree is only partially developed).

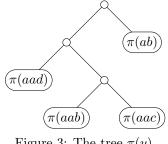


Figure 3: The tree $\pi(y)$.

figureNonConsistent

We will use two lemmas concerning these notions. The first one describes a situation where the structure of the Lazard set coincides with that of the dip and superdip structure.

lemma2

Lemma 21 Let $k \ge 0$ and $n \ge 1$ be such that $x \in S_k(A)$ and $y \in Z_{n+1}$ with |y|odd. If y is consistently in $D_{k+1}(A)$, then $z_n y$ is in consistently in $D_{k+1}(A)$.

Proof. Set $y = y_1 \cdots y_s$ with $y_i \in S_k(A) \cap Z$ and $y_1 \ge \ldots \ge y_{s-1} < y_s$. By the hypothesis, $y_1 = z_m$ with m < n and thus $y_1 \prec z_n$. Thus $z_n \ge y_1 \ge \ldots \ge$ $y_{s-1} < y_s$, which shows that xy is in $D_{k+1}(A)$. It is consistently in $D_{k+1}(A)$ since $\pi(z_n y) = (\pi(x), \pi(y)).$

The next lemma describes a situation where the two structures are distinct.

lemma3

propositionInduction

Lemma 22 For $k \ge 0$ and $n \ge 1$, if $x \in S_k(A)$ and $y \in D_m(A)$ with |y| even and $m \le k$, then $xy \in S_k(A)$.

Proof. We prove the statement by induction on k - m. It is true if m = k by definition of $S_k(A)$. Assume m < k and set $x = u_1 \cdots u_s$ with $u_i \in D_k(A)$. Set $u_s = v_1 \cdots v_t$ with $v_i \in S_{k-1}(A)$ and $v_1 \ge \ldots \ge v_{t-1} < v_t$. We have $v_t y \in S_{k-1}(A)$ by induction hypothesis. Since $v_t < v_t y$ in the radix order, this implies $u_s y = v_1 \cdots v_{t-1}(v_t y) \in D_k(A)$. Since |y| is even, the lengths of u_s and $u_s y$ have the same parity. This implies that $xy = u_1 \cdots u_{s-1}(u_s y)$ is in $S_k(A)$.

Example 23 Set x = aadaabaac and y = ab as in Example 20. Then, as we have seen, $xy \in D_2(A)$ but the decomposition of xy in 1-dips is (aad, aab, aacab) although $\pi(xy) = ((\pi(aad), (\pi(aab), \pi(aac))), \pi(ab)).$

Let $\Delta = \bigcup_{n>1} D_n(A)$ and $\Sigma = \bigcup_{n>0} S_n(A)$.

Proposition 24 The words of odd length in Z are in Σ .

Proof. For a given integer $k \ge 0$, we define a set \mathcal{P}_k which is formed of the words $y \in Z$ such that the following holds.

- (i) If |y| is odd, then y is in $S_k(A)$ or consistently in $D_{k+1}(A)$.
- (ii) If |y| is even, then y is consistently in $D_{k+1}(A)$ or in $D_m(A)$ with $m \leq k$.

We prove by induction on $n \ge 1$ that if n = 1 or if z_{n-1} has odd length, one has $Z_n \subset \Delta \cup \Sigma$, and more precisely that, for some integer $k \ge 0$, all words of Z_n are in \mathcal{P}_k .

This is true for n = 1 with k = 0. Indeed, $Z_1 = A = S_0(A)$. Thus $Z_1 \subset \mathcal{P}_0$. Assume now that $n \ge 2$ and that z_{n-1} has odd length. Then by the induction hypothesis $Z_{n-1} \subset \mathcal{P}_{k-1}$ for some $k \ge 1$. Since z_{n-1} has odd length it is either in $S_{k-1}(A)$ or in $D_k(A)$.

Case 1 Assume first that $z_{n-1} \in S_{k-1}(A)$. We show that in this case $Z_n \subset \mathcal{P}_{k-1}$. We have to prove that for $y \in Z_n \cap \mathcal{P}_{k-1}$, one has $z_{n-1}y \in \mathcal{P}_{k-1}$.

- 1.1 Suppose that y has odd length. Then y is in $S_{k-1}(A)$ or consistently in $D_k(A)$.
 - 1.1.1 If $y \in S_{k-1}(A)$, then $z_{n-1}y$ is consistently in $D_k(A)$ and thus $z_{n-1}y \in \mathcal{P}_{k-1}$
 - 1.1.2 If $y \in D_k(A)$, then $z_{n-1}y$ is consistently in $D_k(A)$ by Lemma 21 and thus $z_{n-1}y \in \mathcal{P}_{k-1}$.
- 1.2 Suppose now that y has even length. Then y is either consistently in $D_k(A)$ or in $D_m(A)$ for m < k 1.

- 1.2.1 If y is consistently in $D_k(A)$, then $z_{n-1}y$ is consistently in $D_k(A)$ by Lemma 21.
- 1.2.2 Otherwise, it is in $D_m(A)$ for m < k-1 and thus $z_{n-1}y$ is in $S_{k-1}(A)$ by Lemma 22.

In both cases, $z_{n-1}y \in \mathcal{P}_{k-1}$.

Case 2 Assume now that z_{n-1} is in $D_k(A)$. We show that in this case $Z_n \subset \mathcal{P}_k$. Since $z_{n-1} \prec z$ for every $z \in Z_{n-1} \setminus z_{n-1}$, the words of odd length in Z_{n-1} cannot be in $S_{k-1}(A)$. Thus $Z_{n-1} \subset \mathcal{P}_k$ and we have to prove that for any $y \in Z_n \cap \mathcal{P}_k$, we have $z_{n-1}y \in \mathcal{P}_k$. The proof is the same as in Case 1 with k instead of k-1.

Using Proposition 24, we now prove the following result.

theoremMain

Theorem 25 For any odd integer m, the set of words of Σ of length m is equal to the set of words of length m in Z.

Proof. We have shown that $Z \cap A^m$ is contained in $\Sigma \cap A^m$. Since Z is a Lazard set, by [1, Proposition 8, 1.10], the family $\{z \mid z \in Z\}$ is a complete factorization of A^* . Thus, by [1, Corollary 8.1.7], it is a set of representatives of the primitive conjugacy classes. Since $\Sigma \cap A^m$ is comma-free by Proposition 16, this forces $Z \cap A^m = \Sigma \cap A^m$.

sectionMelancon

8

The Melançon algorithm

The last part, taken from [7], describes an algorithm due to Melançon to find the conjugate of a primitive word which belongs to a Lazard set. We include it here because it gives an algorithm to compute the conjugate of a word of even length which belongs to the comma-free code obtained by Scholtz algorithm.

Let Z be a Lazard set. Consider the following algorithm starting with a primitive word $w = a_1 a_2 \cdots a_m$ and operating on a sequence $s = (s_1, \ldots, s_n)$ of n elements of Z, not all equal. Initially, $s = (a_1, a_2, \ldots, a_m)$. The main step transformes s as follows. Since not all s_i are equal, there is an index i with $1 \leq i \leq n$ such that $s_i < s_{i+1}$ (taking the indices cyclically) with s_i minimal among s_1, \ldots, s_n . If i < n, change s into $(s_1, \ldots, s_{i-1}, s_i s_{i+1}, s_{i+2}, \ldots, s_n)$. If i = n, change s into $(s_n s_1, s_2, \ldots, s_{n-1})$. The algorithm stops when n = 1.

Example 26 We use the algorithm to find the conjugate of *abracadabra* which is in the code S_1 , using the order \prec .

A first sequence of iterations transforms s = (a, b, r, a, c, a, d, a, b, r, a) into s = (ab, r, ac, ad, ab, r, a). At this step, the minimum is the last one and we obtain s = (aab, r, ac, ad, ab, r). The minimum is now r and thus we obtain in two steps s = (raab, rac, ad, ab). The minimum is now rac and we obtain in two steps s = (raab, rac, ad, ab). The last one being the minimum, we finally obtain s = (racadabraab).

We claim that the result of the algorithm is the conjugate of w which is in Z.

Let $Z \cap A^{[m]} = \{z_1, z_2, \ldots, z_k\}$ and let Z_i be the sequence defined, as in the definition of Lazard sets, by $Z_1 = A$ and $Z_{i+1} = z_i^*(Z_i \setminus z_i)$. For $z \in Z$, we denote $\nu(z) = \min\{i \ge 1 \mid z \in Z_i\} - 1$ and $\delta(z) = \max\{i \ge 1 \mid z \in Z_i\}$. Note that $\delta(z_i) = i$ and that y < z if and only if $\delta(y) < \delta(z)$. Then, as for Hall sequences, for $y, z \in Z$, one has $yz \in Z$ if and only if $\nu(z) \le \delta(y) < \delta(z)$. Moreover, in this case, $\nu(yz) = \delta(y)$.

Let us show that any sequence $s = (s_1, \ldots, s_n)$ obtained during the algorithm is such that all s_i are in Z and for any s_i , either $s_i \in A$ or $\nu(s_i) \leq \delta(s_1), \ldots, \delta(s_n)$. This is true for the initial value of s. Next, if we assume that s has this property and is not constant, let i be such that $s_i < s_{i+1}$ and $s_i = \min\{s_1, \ldots, s_n\}$. Then, since $\nu(s_{i+1}) \leq \delta(s_i) < \delta(s_{i+1})$, we have $s_i s_{i+1} \in Z$ and $\nu(s_i s_{i+1}) = \delta(s_i)$. Since $\delta(s_i) = \min\{\delta(s_1), \ldots, \delta(s_n)\}$, we conlude that $\nu(s_i s_{i+1}) \leq \delta(s_1), \ldots, \delta(s_n)$. For the other s_j , we have $\nu(s_j) \leq \delta(s_i)$ by induction hypothesis and thus $\nu(s_j) < \delta(s_i s_{i+1})$. When the algorithm stops, we obtain a word in Z.

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