# Pentavalent symmetric graphs admitting transitive non-abelian characteristically simple groups

Jia-Li Du, Yan-Quan Feng\*

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China

#### Abstract

Let  $\Gamma$  be a graph and let G be a group of automorphisms of  $\Gamma$ . The graph  $\Gamma$  is called G-normal if G is normal in the automorphism group of  $\Gamma$ . Let T be a finite non-abelian simple group and let  $G = T^l$  with  $l \geq 1$ . In this paper we prove that if every connected pentavalent symmetric T-vertex-transitive graph is T-normal, then every connected pentavalent symmetric G-vertex-transitive graph is G-normal. This result, among others, implies that every connected pentavalent symmetric G-vertex-transitive graph is G-normal except T is one of 57 simple groups. Furthermore, every connected pentavalent symmetric G-regular graph is G-normal except T is one of 20 simple groups, and every connected pentavalent G-symmetric graph is G-normal except T is one of 17 simple groups.

**Keywords:** Vertex-transitive graph, symmetric graph, Cayley graph, regular permutation group, simple group.

2010 Mathematics Subject Classification: 05C25, 20B25.

## 1 Introduction

Throughout this paper, all groups and graphs are finite, and all graphs are simple and undirected. Denote by  $\mathbb{Z}_n$ ,  $D_n$ ,  $A_n$  and  $S_n$  the cyclic group of order n, the dihedral group of order 2n, the alternating group and the symmetric group of degree n, respectively. Let G be a permutation group on a set  $\Omega$  and let  $\alpha \in \Omega$ . Denote by  $G_{\alpha}$  the stabilizer of  $\alpha$  in G, that is, the subgroup of G fixing the point  $\alpha$ . We say that G is semiregular on  $\Omega$  if  $G_{\alpha} = 1$  for every  $\alpha \in \Omega$ , and regular if it is semiregular and transitive. For a graph  $\Gamma$ , we denote its vertex set and automorphism group by  $V(\Gamma)$  and  $\operatorname{Aut}(\Gamma)$ , respectively. The graph  $\Gamma$  is said to be G-vertex-transitive or G-regular for  $G \leq \operatorname{Aut}(\Gamma)$  if G acts transitively or regularly on  $V(\Gamma)$  respectively, and G-symmetric if G acts transitively on the arc set of  $\Gamma$  (an arc is an ordered pair of adjacent vertices). In particular,  $\Gamma$  is vertex-transitive or symmetric if it is  $\operatorname{Aut}(\Gamma)$ -vertex-transitive or  $\operatorname{Aut}(\Gamma)$ -symmetric, respectively. A graph  $\Gamma$  is said to be G-normal for  $G \leq \operatorname{Aut}(\Gamma)$  if G is normal in  $\operatorname{Aut}(\Gamma)$ .

<sup>\*</sup>Corresponding author. E-mails: JiaLiDu@bjtu.edu.cn, yqfeng@bjtu.edu.cn

For a non-abelian simple group T, T-vertex-transitive graphs have received wide attentions, specially for the two extreme cases: T-symmetric graphs and T-regular graphs. It was shown in [2] that a connected pentavalent symmetric T-vertex-transitive graph  $\Gamma$  is either T-normal or  $\operatorname{Aut}(\Gamma)$  contains a non-abelian simple normal subgroup L such that  $T \leq L$  and (T, L) is one of 58 possible pairs of non-abelian simple groups.

A T-regular graph is also called a  $Cayley\ graph$  over T, and the Cayley graph is called normal if it is T-normal. Investigation of Cayley graphs over a non-abelian simple group is currently a hot topic in algebraic graph theory. One of the most remarkable achievements is the complete classification of connected trivalent symmetric non-normal Cayley graphs over non-abelian simple groups. This work was began in 1996 by Li [12], and he proved that a connected trivalent symmetric Cayley graph  $\Gamma$  over a non-abelian simple group T is either normal or  $T = A_5$ ,  $A_7$ , PSL(2, 11),  $M_{11}$ ,  $A_{11}$ ,  $A_{15}$ ,  $M_{23}$ ,  $A_{23}$  or  $A_{47}$ . In 2005, Xu  $et\ al\ [16]$  proved that either  $\Gamma$  is normal or  $T = A_{47}$ , and two years later, Xu  $et\ al\ [17]$  further showed that if  $T = A_{47}$  and  $\Gamma$  is not normal, then  $\Gamma$  must be 5-arc-transitive and up to isomorphism there are exactly two such graphs. Du  $et\ al\ [2]$  showed that a connected pentavalent symmetric Cayley graph  $\Gamma$  over T is either normal, or  $Aut(\Gamma)$  contains a non-abelian simple normal subgroup L such that  $T \leq L$  and (T, L) is one of 13 possible pairs of non-abelian simple groups.

For T-symmetric graphs, Fang and Praeger [4, 5] classified such graphs when T is a Suzuki or Ree simple group acting transitively on the set of 2-arcs of the graphs. For a connected cubic T-symmetric graph  $\Gamma$ , it was proved by Li [12] that either  $\Gamma$  is T-normal or  $(T, \operatorname{Aut}(\Gamma)) = (A_7, A_8)$ ,  $(A_7, S_8)$ ,  $(A_7, 2.A_8)$ ,  $(A_{15}, A_{16})$  or  $(\operatorname{GL}(4, 2), \operatorname{AGL}(4, 2))$ . Fang et al [3] proved that none of the above five pairs can happen, that is, T is always normal in  $\operatorname{Aut}(\Gamma)$ . Du et al [2] showed that a connected pentavalent T-symmetric graph  $\Gamma$  is either T-normal or  $\operatorname{Aut}(\Gamma)$  contains a non-abelian simple normal subgroup L such that  $T \leq L$  and (T, L) is one of 17 possible pairs of non-abelian simple groups.

Let G be the characteristically simple group  $T^l$  with  $l \geq 1$ . In this paper, we extend the above results on connected pentavalent T-vertex graphs to G-vertex graphs.

**Theorem 1.1** Let T be a non-abelian simple group and let  $G = T^l$  with  $l \ge 1$ . Assume that every connected pentavalent symmetric T-vertex-transitive graph is T-normal. Then every connected pentavalent symmetric G-vertex-transitive graph is G-normal.

In 2011, Hua *et al* [10] proved that if every connected cubic symmetric T-vertex-transitive graph is T-normal, then every connected cubic symmetric G-vertex-transitive graph is G-normal. By Theorem 1.1 and [2, Theorem 1.1], we have the following corollaries.

Corollary 1.2 Let T be a non-abelian simple group and let  $G = T^l$  with  $l \ge 1$ . Then every connected pentavalent symmetric G-vertex-transitive graph is G-normal except for  $T = \mathrm{PSL}(2,8), \ \Omega_8^-(2)$  or  $A_{n-1}$  with  $n \ge 6$  and  $n \mid 2^9 \cdot 3^2 \cdot 5$ .

Corollary 1.3 Let T be a non-abelian simple group and let  $G = T^l$  with  $l \ge 1$ . Then every connected pentavalent G-symmetric graph is G-normal except for  $T = A_{n-1}$  with  $n = 2 \cdot 3, \ 2^2 \cdot 3, \ 2^4, \ 2^3 \cdot 3, \ 2^5, \ 2^2 \cdot 3^2, \ 2^4 \cdot 3, \ 2^3 \cdot 3^2, \ 2^5 \cdot 3, \ 2^4 \cdot 3^2, \ 2^6 \cdot 3, \ 2^5 \cdot 3^2, \ 2^7 \cdot 3, \ 2^6 \cdot 3^2, \ 2^7 \cdot 3^2, \ 2^8 \cdot 3^2 \text{ or } 2^9 \cdot 3^2.$ 

Corollary 1.4 Let T be a non-abelian simple group and let  $G = T^l$  with  $l \ge 1$ . Then every connected pentavalent symmetric G-regular graph is G-normal except for T = PSL(2, 8),  $\Omega_8^-(2)$  or  $A_{n-1}$  with  $n = 2 \cdot 3, 2^3, 3^2, 2 \cdot 5, 2^2 \cdot 3, 2^2 \cdot 5, 2^3 \cdot 3, 2^3 \cdot 5, 2 \cdot 3 \cdot 5, 2^4 \cdot 5, 2^3 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5, 2^6 \cdot 3 \cdot 5, 2^5 \cdot 3^2 \cdot 5, 2^7 \cdot 3 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^7 \cdot 3^2 \cdot 5$  or  $2^9 \cdot 3^2 \cdot 5$ .

## 2 Preliminaries

In this section, we describe some preliminary results which will be used later. The first one is the vertex stabilizers of connected pentavalent symmetric graphs. By [7, Theorem 1.1], we have the following proposition.

**Proposition 2.1** Let Γ be a connected pentavalent G-symmetric graph with  $v \in V(\Gamma)$ . Then  $G_v \cong \mathbb{Z}_5$ ,  $D_5$ ,  $D_{10}$ ,  $F_{20} \times \mathbb{Z}_2$ ,  $F_{20} \times \mathbb{Z}_4$ ,  $A_5$ ,  $S_5$ ,  $A_4 \times A_5$ ,  $S_4 \times S_5$ ,  $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ , ASL(2,4), AGL(2,4), AΓL(2,4) or  $\mathbb{Z}_2^6 \rtimes \Gamma L(2,4)$ , where  $F_{20}$  is the Frobenius group of order 20,  $A_4 \rtimes \mathbb{Z}_2 = S_4$  and  $A_5 \rtimes \mathbb{Z}_2 = S_5$ . In particular,  $|G_v| = 5$ ,  $2 \cdot 5$ ,  $2^2 \cdot 5$ ,  $2^3 \cdot 5$ ,  $2^4 \cdot 5$ ,  $2^2 \cdot 3 \cdot 5$ ,  $2^3 \cdot 3 \cdot 5$ ,  $2^4 \cdot 3^2 \cdot 5$ ,  $2^6 \cdot 3^2 \cdot 5$ ,  $2^5 \cdot 3^2 \cdot 5$ ,  $2^6 \cdot 3 \cdot 5$ ,  $2^6 \cdot 3^2 \cdot 5$ ,  $2^7 \cdot 3 \cdot 5$ ,  $2^7 \cdot 3^2 \cdot 5$  or  $2^9 \cdot 3^2 \cdot 5 \cdot 5$ , respectively.

Connected pentavalent symmetric graphs admitting vertex-transitive non-abelian simple groups were classified in [2].

**Proposition 2.2** [2, Theorem 1.1] Let T be a non-abelian simple group and  $\Gamma$  a connected pentavalent symmetric T-vertex-transitive graph. Then either  $T \subseteq \operatorname{Aut}(\Gamma)$ , or  $T = \Omega_8^-(2)$ ,  $\operatorname{PSL}(2,8)$  or  $A_{n-1}$  with  $n \ge 6$  and  $n \mid 2^9 \cdot 3^2 \cdot 5$ .

The following is straightforward (also see the short proof of [2, Lemma 3.2]).

**Proposition 2.3** Let  $\Gamma$  be a connected pentavalent symmetric G-vertex-transitive graph with  $v \in V(\Gamma)$  and let  $A = \operatorname{Aut}(\Gamma)$ . If  $H \leq A$  and  $GH \leq A$ , then  $|H|/|H \cap G| = |(GH)_v|/|G_v| \mid 2^9 \cdot 3^2 \cdot 5$ , and if  $\Gamma$  is further G-symmetric then  $|H|/|H \cap G| \mid 2^9 \cdot 3^2$ .

The following proposition follows the classification of three-factor simple groups.

**Proposition 2.4** [11, Theorem I ] Let G be a non-abelian simple  $\{2,3,5\}$ -group. Then  $G = A_5$ ,  $A_6$  or PSU(4,2).

By Guralnick [8, Theorem 1], we have the following proposition.

**Proposition 2.5** Let G be a non-abelian simple group with a subgroup H such that  $|G:H| = p^a$  with p a prime and  $a \ge 1$ . Then

- (1)  $G = A_n \text{ and } H = A_{n-1} \text{ with } n = p^a;$
- (2) G = PSL(2, 11) and  $H = A_5$  with |G: H| = 11;

- (3)  $G = M_{23}$  and  $H = M_{22}$  with |G: H| = 23, or  $G = M_{11}$  and  $H = M_{10}$  with |G: H| = 11;
- (4)  $G = PSU(4, 2) \cong PSp(4, 3)$  and H is the parabolic subgroup of index 27;
- (5) G = PSL(n,q) and H is the stabilizer of a line or hyperplane with  $|G:H| = (q^n 1)/(q 1) = p^a$ .

By [13, Theorem 1] and Proposition 2.5, we have the following proposition.

**Proposition 2.6** Let G be a non-abelian simple group and H a maximal subgroup of G such that  $|G:H|=2^a\cdot 3^b\geq 6$  with  $0\leq a\leq 9$  and  $0\leq b\leq 2$ . Then G, H and |G:H| are listed in Table 1.

G	Н	G:H	G	H	G:H
$M_{11}$	PSL(2,11)	$2^2 \cdot 3$	$M_{12}$	$M_{11}$	$2^2 \cdot 3$
$M_{24}$	$M_{23}$	$2^3 \cdot 3$	PSU(3,3)	PSL(2,7)	$2^2 \cdot 3^2$
$A_9$	$S_7$	$2^2 \cdot 3^2$	PSU(4,2)	$S_6$	$2^2 \cdot 3^2$
PSp(6,2)	$S_8$	$2^2 \cdot 3^2$	$M_{12}$	PSL(2,11)	$2^4 \cdot 3^2$
PSL(2,8)	$D_7$	$2^2 \cdot 3^2$	PSL(3,3)	$\mathbb{Z}_{13} \rtimes \mathbb{Z}_3$	$2^4 \cdot 3^2$
PSL(2,9)	$A_5$	$2 \cdot 3$	PSL(2, p)	$\mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$	$p+1=2^a\cdot 3^b$
PSL(2,8)	$\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$	$3^{2}$	$A_n$	$A_{n-1}$	$n = 2^a \cdot 3^b$

Table 1: Non-abelian simple group pairs of index  $2^a \cdot 3^b$ 

Let G be a group. The *inner automorphism group* Inn(G) of G is the group of automorphisms of G induced by conjugate action of elements in G, which is a normal subgroup in the automorphism group Aut(G) of G. The quotient group Aut(G)/Inn(G) is called the *outer automorphism group* of G. By the classification of finite simple groups, we have the following proposition, which is the famous Schreier conjecture.

**Proposition 2.7** [6, Theorem 1.64] Every finite simple group has a solvable outer automorphism group.

Baddeley and Praeger [1] considered almost simple groups containing a direct product of at least two isomorphic non-abelian simple groups.

**Proposition 2.8** [1, Theorem 1.4] Let H be an almost simple group, that is,  $S \leq H \leq \operatorname{Aut}(S)$  for a non-abelian simple group S, and suppose that H = AB, where A is a proper subgroup of H not containing S, and  $B \cong T^r$  for a non-abelian simple group T and integer  $k \geq 2$ . Then  $S = A_n$  and  $A \cap S = A_{n-1}$ , where  $n = |H : A| = |S : A \cap S| \geq 10$ .

Let  $\Gamma$  be a graph and  $N \leq \operatorname{Aut}(\Gamma)$ . The quotient graph  $\Gamma_N$  of  $\Gamma$  relative to N is defined as the graph with vertices the orbits of N on  $V(\Gamma)$  and with two orbits adjacent if there is an edge in  $\Gamma$  between these two orbits.

**Proposition 2.9** [14, Theorem 9] Let  $\Gamma$  be a connected G-symmetric graph of prime valency, and let  $N \subseteq G$  have at least three orbits on  $V(\Gamma)$ . Then N is the kernel of G on  $V(\Gamma_N)$ , and semiregular on  $V(\Gamma)$ . Furthermore,  $\Gamma_N$  is G/N-symmetric with  $G/N \le \operatorname{Aut}(\Gamma_N)$ .

### 3 Proof of Theorem 1.1

Let T be a non-abelian simple group and let  $G = T^l$  with  $l \ge 1$ . Let  $\Gamma$  be a connected pentavalent symmetric G-vertex-transitive graph with  $v \in V(\Gamma)$  and let  $A = \operatorname{Aut}(\Gamma)$ . We make the following assumption throughout this section.

**Assumption:** Every connected pentavalent symmetric T-vertex-transitive graph is T-normal.

Since the complete graph  $K_6$  of order 6 has automorphism group  $S_6$  and is  $A_5$ -symmetric, we have  $T \neq A_5$  by Assumption. Since G is vertex-transitive and has no subgroup of index 2,  $\Gamma$  is not bipartite.

To prove Theorem 1.1, we apply induction on l. It suffices to show that G contains a minimal normal subgroup of A, and this is done in Lemmas 3.1 and 3.2 when rad(A) = 1, where rad(A) is the largest solvable normal subgroup of A. For  $rad(A) \neq 1$ , we need the fact  $rad(A)G = rad(A) \times G$ , which is proved in Lemmas 3.3 and 3.4.

**Lemma 3.1** Let  $\Gamma$  be X-symmetric with  $G \leq X$ , and let X have a minimal normal subgroup that is a direct product of T. Then G contains a minimal normal subgroup of X.

**Proof:** By Assumption, if l=1 then  $G \subseteq A$  and the lemma is true. Assume  $l \ge 2$ .

Let N be a minimal normal subgroup of X such that  $N = T^m$  for a positive integer m. Since  $N \cap G \subseteq G$ , we have  $N \cap G = T^n$  with  $n \le m$  and  $n \le l$ . Set D = NG. By Proposition 2.3,  $|N|/|N \cap G| = |T|^{m-n} = |D|/|G| = |D_v|/|G_v| \mid 2^9 \cdot 3^2 \cdot 5$ . Since T is non-abelian simple, we have either m - n = 1 or m = n.

Suppose m-n=1. Then  $|T| \mid 2^9 \cdot 3^2 \cdot 5$ , and by Proposition 2.4,  $T=A_6$  as  $T \neq A_5$ . Thus,  $|G| = |A_6|^l$ ,  $|D| = |A_6|^{l+1}$  and  $|D_v|/|G_v| = |A_6|$ , implying  $5 \mid |D_v|$  and  $5 \nmid |G_v|$ . It follows that  $\Gamma$  is D-symmetric, but not G-symmetric.

Let  $\bar{D} = D/G_D$ ,  $\bar{G} = G/G_D$  and  $\bar{N} = NG_D/G_D$ , where  $G_D$  is the largest normal subgroup of D contained in G. Then  $\bar{D} = \bar{G}\bar{N}$ . Since  $\Gamma$  is D-symmetric,  $D_v$  is primitive on the set of 5 neighbours of v in  $\Gamma$ , and since  $G_D \subseteq D$ , we have either  $(G_D)_v = 1$  or  $5 \mid |(G_D)_v|$ . Thus,  $(G_D)_v = 1$  because  $\Gamma$  is not G-symmetric. If  $G = G_D$ , then  $G_v = 1$  and  $|D_v| = |T| = 2^3 \cdot 3^2 \cdot 5$ , contrary to Proposition 2.1. Thus,  $G/G_D = T^{l'}$  with  $l' \ge 1$ .

If  $G_D$  is transitive on  $V(\Gamma)$ , then  $|G_v| = |G_v|/|(G_D)_v| = |G|/|G_D| = |T|^{l'}$ . Since  $T = A_6$ , we have  $5 \mid |G_v|$ , a contradiction. If  $G_D$  has two orbits on  $V(\Gamma)$ , then  $\Gamma$  is bipartite, which is impossible. Thus,  $G_D$  has at least three orbits on  $V(\Gamma)$ . By Proposition 2.9,  $\Gamma_{G_D}$  is a connected pentavalent  $G/G_D$ -vertex-transitive and  $D/G_D$ -symmetric graph. If l' = 1, by Proposition 2.2,  $G/G_D \leq D/G_D$ , that is,  $G \leq D$ , which is also impossible as otherwise  $G_v = 1$  and  $|D_v| = |T| = 2^3 \cdot 3^2 \cdot 5$ . Thus,  $\bar{G} = G/G_D = T^{l'}$  with  $l' \geq 2$  and  $|\bar{D}| = |T|^{l'+1}$  as  $|\bar{D}:\bar{G}| = |D:G| = |T|$ .

Let  $\bar{M}$  be a maximal subgroup of  $\bar{D}$  containing  $\bar{G}$ . Then  $\bar{G} \leq \bar{M}$  and  $|T|^{l'} \leq |\bar{M}| < |T|^{l'+1}$ . Consider the right multiplication of  $\bar{D}$  on  $[\bar{D}:\bar{M}]$ , the set of right cosets of  $\bar{M}$  in  $\bar{D}$ . Then  $\bar{D}/\bar{M}_{\bar{D}}$  is a primitive permutation group on  $[\bar{D}:\bar{M}]$  and  $\bar{D}/\bar{M}_{\bar{D}} \leq S_{|T|}$ , where  $\bar{M}_{\bar{D}}$  is the largest normal subgroup of  $\bar{D}$  contained in  $\bar{M}$ . Since  $|\bar{D}:\bar{M}| \mid |T| = |A_6|$ , the primitive group  $\bar{D}/\bar{M}_{\bar{D}}$  has degree dividing 360.

Let  $\bar{L}$  be a minimal normal subgroup of  $\bar{D}$ . Then  $\bar{L} \cap \bar{N} = \bar{L}$  or 1. For the former,  $\bar{L} \leq \bar{N}$ , and for the latter,  $\bar{L} = \bar{L}/\bar{L} \cap \bar{N} \cong \bar{N}\bar{L}/\bar{N} \leq \bar{D}/\bar{N} \cong \bar{G}/\bar{G} \cap \bar{N}$ . In both cases,  $\bar{L}$  is a direct product of T, and hence  $\operatorname{soc}(\bar{D}) = T^k$  for a positive integer k, where  $\operatorname{soc}(\bar{D})$  is the product of all minimal normal subgroups of  $\bar{D}$ , that is, the socle of  $\bar{D}$ .

Assume  $\bar{M}_{\bar{D}} \neq 1$ . Then  $\bar{D}$  has a minimal normal subgroup contained in  $\bar{M}_{\bar{D}}$ , say  $\bar{K}$ . Then  $\bar{K} \unlhd \operatorname{soc}(\bar{D}) = T^k$ . If  $\bar{K} \nleq \bar{G}$ , then  $|T|^{l'+1} = |\bar{D}| > |\bar{M}| \geq |\bar{G}\bar{K}| = |\bar{G}|(|\bar{K}|/|\bar{K} \cap \bar{G}|) \geq |\bar{G}||T| = |T|^{l'+1}$ , a contradiction. Thus,  $\bar{K} \leq \bar{G}$ , and we may assume  $\bar{K} = K/G_D$ , where  $G_D \leq K \leq G$ . Since  $\bar{K} \unlhd \bar{D}$ , we have  $K \unlhd D$ , and hence  $K \leq G_D$ . It follows that  $K = G_D$  and  $\bar{K} = 1$ , a contradiction.

Thus,  $\bar{M}_{\bar{D}} = 1$  and so  $\bar{D}$  is a primitive group on  $[\bar{D}:\bar{M}]$  of degree dividing |T| = 360. Since  $\mathrm{soc}(\bar{D}) = T^k = A_6^k$ , we have  $k \leq 2$  by [15, Section 7], and the primitivity implies that  $|\bar{D}| \leq |\mathrm{Aut}(\mathrm{soc}(\bar{D}))| \leq 32|T|^2$ , which is impossible because  $|\bar{D}| = |T|^{l'+1} \geq |T|^3$ .

It follows that m-n=0 and hence  $N \leq G$ , as required.

**Remark:** It is easy to check that Lemma 3.1 is also true if Assumption is replaced by the conditions that  $l \ge 2$  and  $T \ne A_5$ .

**Lemma 3.2** Let  $G \leq X$  such that rad(X) = 1 and  $\Gamma$  is X-symmetric. Then X has a minimal normal subgroup that is a direct product of T.

**Proof:** Suppose to the contrary that every minimal normal subgroup of X is not a direct product of T. Then  $G = T^l \not \subseteq X$ , and by Assumption,  $l \ge 2$ .

Let L be the socle of X, the product of all minimal normal subgroups of X. Since  $\mathrm{rad}(X)=1$ , we have  $L=S_1\times\cdots\times S_t$  with t a positive integer, and  $S_i\ncong T$  is a non-abelian simple group for each  $1\le i\le t$ . Furthermore,  $L\cap G=T^r$  with  $0\le r\le l$  as  $L\cap G\le G$ . By Proposition 2.3,  $|L|/|L\cap G|=|(GL)_v|/|G_v|$   $|2^9\cdot 3^2\cdot 5$ .

For convenience, write  $L \cap G = T^r = T_1 \times \cdots \times T_r$ . Let  $P_i$  be the projection from  $L \cap G$  to  $S_i$  for a given  $1 \leq i \leq t$ , that is,  $P_i(x) = x_i$  for any  $x = x_1 \cdots x_t \in L \cap G$  with  $x_i \in S_i$   $(1 \leq i \leq t)$ . Then  $P_i$  is a homomorphism from  $L \cap G$  to  $S_i$  and hence  $(L \cap G)/K_i \cong I_i$ , where  $K_i$  is the kernel of  $P_i$  and  $I_i \leq S_i$  is the image of  $L \cap G$  under  $P_i$ . Since  $K_i \subseteq L \cap G$ , we have  $(L \cap G)/K_i \cong T^{r_i} = I_i$ . Clearly,  $K_1 \leq S_2 \times \cdots \times S_t$  and  $K_1 \cap \cdots \cap K_t = 1$ . It follows  $T^r = L \cap G = (L \cap G)/(K_1 \cap \cdots \cap K_t) \lesssim (L \cap G)/(K_1 \times \cdots \times (L \cap G)/K_t \cong T^{r_1} \times \cdots \times T^{r_t} = I_1 \times \cdots \times I_t$ , and hence  $(|S_1:I_1| \cdots |S_t:I_t|) \mid |L:(L \cap G)| \mid 2^9 \cdot 3^2 \cdot 5$ .

Assume  $t \geq 5$ . Then at least four  $|S_i: I_i|$  are divisors of  $2^9 \cdot 3^2$ , say  $|S_i: I_i| = 2^{a_i} \cdot 3^{b_i}$  with  $2 \leq i \leq 5$ . Since  $T \neq A_5$ , Proposition 2.6 implies  $2^{a_i} \cdot 3^{b_i} \geq 2^3$  and  $|S_2: I_2| \cdots |S_t: I_t| \geq 2^{12}$ . If  $5 \nmid |S_1: I_1|$ , then  $|S_1: I_1| \geq 2^3$  and  $|S_1: I_1| \cdots |S_t: I_t| \geq 2^{15}$ . Since  $(|S_1: I_1| \cdots |S_t: I_t|) \mid 2^9 \cdot 3^2 \cdot 5$ , we have  $2^{15} \leq 2^9 \cdot 3^2 \cdot 5$ , a contradiction. Thus,  $5 \mid |S_1: I_1|$ . If  $|S_1: I_1| \neq 5$  then  $|S_1: I_1| \geq 2 \cdot 5$  and hence  $2^{13} \cdot 5 \leq 2^9 \cdot 3^2 \cdot 5$ , a contradiction. It follows  $|S_1: I_1| = 5$ , which is also impossible by Proposition 2.5. This yields  $t \leq 4$ .

Let  $C = C_X(L)$  be the centralizer of L in X. Then  $C \cap L = 1$  and hence C = 1 because L contains every minimal normal subgroup of X. Thus,  $X = X/C \leq \operatorname{Aut}(L)$  and  $G/G \cap L \cong GL/L \leq X/L \leq \operatorname{Out}(L)$ . Since  $t \leq 4$ , Proposition 2.7 implies that  $\operatorname{Out}(L)$  is solvable, and hence  $G/G \cap L = 1$  and  $G \leq L$ . Since  $G \nleq X$ , we have G < L, that is, G is a proper subgroup of L. It follows that  $L_v \neq 1$  and since  $L \subseteq X$ , we have  $5 \mid |L_v|$  and so L is symmetric.

If t=1, the Frattini argument implies that  $L=GL_v$ , and by Proposition 2.8,  $L_v=A_n$  with  $n\geq 9$ , which is impossible. Thus,  $2\leq t\leq 4$ . Since  $(|S_1:I_1|\cdots|S_t:I_t|)\mid 2^9\cdot 3^2\cdot 5$ , there is at least one  $|S_i:I_i|$  that is a divisor of  $2^9\cdot 3^2$ , say  $|S_1:I_1|\mid 2^9\cdot 3^2$ . Since  $I_1=T^{r_1}$ , Proposition 2.6 implies  $I_1=T$ , and since  $I_1\cong G/K_1$ , we have  $K_1=T^{l-1}$ . Set  $\bar{S}_1=\bar{S}_2\times\cdots\times S_t$ . Then  $L=S_1\times\bar{S}_1$  and  $K_1=G\cap\bar{S}_1$  as  $K_1\leq\bar{S}_1$  and  $G=T^l\neq K_1$ .

If  $\bar{S}_1$  is transitive on  $V(\Gamma)$ , then  $|L_v|/|(\bar{S}_1)_v| = |L|/|\bar{S}_1| = |S_1| | 2^9 \cdot 3^2 \cdot 5$  and so  $S_1 = A_5$  or  $A_6$ , forcing  $I_1 = T = A_5$ , a contradiction. If  $\bar{S}_1$  has two orbits, then  $\Gamma$  is bipartite, which is also impossible. Thus,  $\bar{S}_1$  has at least three orbits on  $V(\Gamma)$  and Proposition 2.9 implies that  $\Gamma_{\bar{S}_1}$  is a connected pentavalent  $G\bar{S}_1/\bar{S}_1$ -vertex-transitive and  $L/\bar{S}_1$ -symmetric graph. Since  $G\bar{S}_1/\bar{S}_1 \cong G/G \cap \bar{S}_1 = T$ , by Assumption,  $T \cong G\bar{S}_1/\bar{S}_1 \trianglelefteq L/\bar{S}_1 \cong S_1$ , which is impossible because  $S_1$  is a non-abelian simple group. This completes the proof.  $\square$ 

#### Remark: Let

$$\begin{array}{lll} \Delta = & \{ \mathrm{PSL}(2,8), \ \Omega_8^-(2), \ A_{n-1} \mid n \geq 6, \ n \mid 2^9 \cdot 3^2 \cdot 5 \} \\ \Delta_1 = & \{ \mathrm{PSL}(2,8), \ \Omega_8^-(2), \ A_{n-1} \mid n = 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3, 2 \cdot 5, 2^2 \cdot 5, 2^3 \cdot 3, \\ & 2^3 \cdot 5, 2 \cdot 3 \cdot 5, 2^5 \cdot 3^2 \cdot 5, 2^7 \cdot 3 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^7 \cdot 3^2 \cdot 5, 2^9 \cdot 3^2 \cdot 5 \}. \end{array}$$

Then Lemma 3.2 is true if Assumption is replaced by the conditions that  $l \geq 2$  and  $T = A_{n-1} \in \Delta - \Delta_1$ . To prove it, we only need to change the last paragraph in the proof of Lemma 3.2 as following.

Since  $|S_1:I_1| | 2^9 \cdot 3^2$ , Proposition 2.6 implies that  $I_1 = T$ , and since  $T \in \Delta - \Delta_1$ , we have  $(S_1, I_1) = (A_n, A_{n-1})$  with  $n = 2^i, 2^j \cdot 3$  or  $2^k \cdot 3^2$ , where  $4 \le i, j \le 9$  and  $1 \le k \le 9$ . Recall that  $(|S_1:I_1| \cdots |S_t:I_t|) | 2^9 \cdot 3^2 \cdot 5$  and  $2 \le t \le 4$ . Then for each  $2 \le i \le t$ , we have  $|S_i:I_i| | 2^5 \cdot 3^2 \cdot 5$  or  $2^8 \cdot 5$  and hence  $S_i$  is a primitive permutation group of degree dividing 1440 or 1280. By [15, Section 7],  $(S_i:I_i) = (A_n, A_{n-1})$ , which implies that t = 2 and  $(S_2, S_1, T) = (A_n, A_n, A_{n-1})$  with  $n = 2^4$  or  $2^4 \cdot 3$ . Thus,  $G = T^2 < L = S_1 \times S_2$ ,  $G \cap S_2 = T$  (note that  $\overline{S_1} = S_2$ ),  $|L_v|/|G_v| = |L|/|G| = n^2$  and  $2^8 \mid |L_v|$ . By Proposition 2.1,  $|L_v| = 2^9 \cdot 3^2 \cdot 5$  and so  $|G_v| = 2 \cdot 3^2 \cdot 5$  or  $2 \cdot 5$ . The former is impossible by Proposition 2.1. For the latter,  $|(GS_2)_v|/|G_v| = |GS_2|/|G| = |S_2|/|S_2 \cap G| = |S_2|/|T| = 2^4 \cdot 3$  and  $|(GS_2)_v| = 2^5 \cdot 3 \cdot 5$ , which is also impossible by Proposition 2.1.

**Lemma 3.3** Assume  $5 \mid |\operatorname{rad}(A)|$ . Then  $\operatorname{rad}(A)G = \operatorname{rad}(A) \times G$ .

**Proof:** Since  $\operatorname{rad}(A)$  is solvable,  $\operatorname{rad}(A) \cap G = 1$ . If l = 1 then by Assumption,  $G \subseteq A$ , and hence  $\operatorname{rad}(A)G = \operatorname{rad}(A) \times G$ . Thus, we may assume  $l \geq 2$ .

Set  $B = \operatorname{rad}(A)G$ . Then  $|B| = |\operatorname{rad}(A)||G|$ , and by Proposition 2.3,  $|\operatorname{rad}(A)| = |B|/|G| = |B_v|/|G_v| \mid 2^9 \cdot 3^2 \cdot 5$ . Since  $5 \mid |\operatorname{rad}(A)|$ , we have  $5 \mid |B_v|$  and  $5 \nmid |G_v|$ , that is,  $\Gamma$  is B-symmetric, but not G-symmetric. In particular,  $G_v$  is a  $\{2,3\}$ -group and  $\{2,3\}$ -group and  $\{3,4\}$ -group and  $\{4,4\}$ -g

Since rad(A) is a solvable  $\{2,3,5\}$ -group, rad(A) has a Hall  $\{2,3\}$ -subgroup, say H. Set  $\Omega = \{H^r \mid r \in \text{rad}(A)\}$ . By [9], all Hall  $\{2,3\}$ -subgroups of rad(A) are conjugate and so the conjugate action of B on  $\Omega$  is transitive. Let K be the kernel of the action of G on  $\Omega$ . Since  $5^2 \nmid \text{rad}(A)$ , we have  $|\Omega| = 1$  or 5, and hence  $G/K = T^r \leq S_5$ . Since  $T \neq A_5$ , we have G = K and so G fixes H. It follows  $H \subseteq GH \subseteq B$  and  $G \cap H = 1$ .

Set Y = GH and  $\Delta = \{Yb \mid b \in B\}$ . Then  $|\Delta| = |B:Y| = 5$ . Let  $Y_B$  be the kernel of the right multiplication action of B on  $\Delta$ . Then  $Y_B$  is the largest normal subgroup of B

contained in Y. It follows  $B/Y_B \leq S_5$ . Suppose  $G \nleq Y_B$ . Then  $G/G \cap Y_B \cong GY_B/Y_B \leq B/Y_B \leq S_5$ , and so  $G/G \cap Y_B = T = A_5$ , a contradiction. Thus,  $G \leq Y_B$ .

Since  $|Y_v|/|G_v| = |Y|/|G| = |H|$  and  $G_v$  is a  $\{2,3\}$ -group,  $Y_v$  is a  $\{2,3\}$ -group. Since  $\Gamma$  is B-symmetric,  $B_v$  is primitive on the set of the 5 neighbors of v in  $\Gamma$ , and since  $Y_B \leq B$ , we have either  $(Y_B)_v = 1$  or  $5 \mid |(Y_B)_v|$ . The latter cannot happen as  $Y_v$  is a  $\{2,3\}$ -group. Thus,  $(Y_B)_v = 1$ , and the Frattini argument implies  $Y_B = G(Y_B)_v = G$ . Thus,  $G \leq B$  and  $B = \operatorname{rad}(A) \times G$ .

**Lemma 3.4** Let  $G \leq X \leq A$  and let  $\Gamma$  be G-symmetric. Then  $\operatorname{rad}(X)G = \operatorname{rad}(X) \times G$ .

**Proof:** The lemma is true for  $|\operatorname{rad}(X)| = 1$ . In what follows we assume that  $|\operatorname{rad}(X)| \neq 1$ . Recall that  $G = T^l$  with  $l \geq 1$ . Since  $\Gamma$  is G-symmetric,  $5 \mid |G_v|$  and hence  $5 \mid |T|$ . Set  $B = \operatorname{rad}(X)G$ . Then  $G \cap \operatorname{rad}(X) = 1$  as  $\operatorname{rad}(X)$  is solvable, and so  $|B| = |\operatorname{rad}(X)||G|$ . By Proposition 2.3,  $|\operatorname{rad}(X)| = |B_v|/|G_v| \mid 2^9 \cdot 3^2$ .

Let N be a minimal normal subgroup of X contained in rad(X). Then  $N \cong \mathbb{Z}_2^s$  or  $N = \mathbb{Z}_3^t$  for some  $1 \leq s \leq 9$  or  $1 \leq t \leq 2$ , and hence  $\operatorname{Aut}(N) \leq \operatorname{GL}(s,2)$  or  $\operatorname{GL}(t,3)$ .

First we claim  $D := GN = G \times N$ . Consider the conjugate action of G on N and let K be the kernel of G in this action. Then  $K = T^r \subseteq G$  for some  $r \subseteq I$ , and  $G/K \subseteq \operatorname{Aut}(N)$ . It is easy to see that  $D = G \times N$  if and only if G = K.

Suppose  $G \neq K$ . Then r < l and  $G/K = T^{l-r}$  is insolvable. In particular,  $\operatorname{Aut}(N)$  is insolvable. Recall that  $\operatorname{Aut}(N) \leq \operatorname{GL}(s,2)$  or  $\operatorname{GL}(t,3)$ . It follows that  $N \cong \mathbb{Z}_2^s$  with  $3 \leq s \leq 9$  as both  $\operatorname{GL}(2,2)$  and  $\operatorname{GL}(2,3)$  are solvable.

Note that  $G \cap N = 1$ . By Proposition 2.3,  $|N| = |D|/|G| = |D_v|/|G_v|$ , and so  $|D_v|/|G_v|$  is a 2-power. If  $G_v \cong \mathbb{Z}_5$  then Proposition 2.1 implies that  $D_v \cong D_5$ ,  $D_{10}$ ,  $F_{20}$ ,  $F_{20}$ ,  $\mathbb{Z}_2$  or  $F_{20} \times \mathbb{Z}_4$ . It follows that  $|N| \mid 2^5$ , and this is always true by checking all other possible cases for  $G_v$  in Proposition 2.1. This means that  $N \cong \mathbb{Z}_2^s$  with  $3 \leq s \leq 5$  and hence  $G/K \leq \operatorname{Aut}(N) \leq \operatorname{PSL}(5,2)$ . Since  $5^2 \nmid |\operatorname{PSL}(5,2)|$ , we have l-r=1 and G/K=T.

If K has at least three orbits, Proposition 2.9 implies that  $\Gamma_K$  is a connected pentavalent G/K-symmetric graph, and by Assumption,  $G/K \subseteq D/K$ . It follows that  $G \subseteq D$  and hence G = K, a contradiction. Thus, K has one or two orbits. If K has two orbits, then  $\Gamma$  is bipartite, a contradiction. This yields that K is transitive. Then  $|G_v|/|K_v| = |G|/|K| = |T|$ , and since  $K \subseteq G$  and  $\Gamma$  is G-symmetric, we have either  $K_v = 1$  or  $5 \mid |K_v|$ . If  $5 \mid |K_v|$ , then  $5 \nmid |T|$  because  $|G_v|/|K_v| = |T|$ , a contradiction. It follows  $K_v = 1$ . Let L = KN. Then  $L \subseteq D$  and so  $L_v = 1$  or  $5 \mid |L_v|$ . On the other hand,  $|L_v| = |L_v|/|K_v| = |L|/|K| = |N| = 2^s$  ( $3 \le s \le 5$ ), a contradiction. Therefore,  $D = G \times N$ , as claimed.

Now we finish the proof by induction on  $|\operatorname{rad}(X)|$ . Since  $5 \nmid |N|$ , we have  $N_v = 1$ . Assume that N has one or two orbits on  $V(\Gamma)$ . Then  $5 \nmid |V(\Gamma)|$ , and so  $5^2 \nmid |G|$ , which implies that G is a simple group. By Assumption,  $G \subseteq X$  and hence  $\operatorname{rad}(X)G = G \times \operatorname{rad}(X)$ . Assume that N has at least three orbits. By Proposition 2.9,  $\Gamma_N$  is a connected pentavalent GN/N-symmetric graph. Note that  $GN/N \cong T^l$  and  $GN/N \leq X/N$ . Since  $\operatorname{rad}(X/N) = \operatorname{rad}(X)/N$ , we have  $|\operatorname{rad}(X/N)| < |\operatorname{rad}(X)|$  and the inductive hypothesis implies that  $\operatorname{rad}(X)G/N = \operatorname{rad}(X/N) \cdot GN/N = \operatorname{rad}(X)/N \times GN/N$ . Thus,  $GN \subseteq \operatorname{rad}(X)G$ . Since  $GN = G \times N$ , G is characteristic in GN and hence G is normal in  $\operatorname{rad}(X)G$ . It follows  $\operatorname{rad}(X)G = \operatorname{rad}(X) \times G$ .

Now, we are ready to prove Theorem 1.1.

The proof of Theorem 1.1: Recall that  $G = T^l$  and  $\Gamma$  is a connected pentavalent symmetric G-vertex-transitive graph with  $A = \operatorname{Aut}(\Gamma)$ . We apply induction on l. If l = 1, Theorem 1.1 is true by Assumption. Assume  $l \geq 2$ . Let R be the radical of A, and set B = RG. Then  $G \cap R = 1$  and  $|R| = |B|/|G| = |B_v|/|G_v|$ .

If R is transitive on  $V(\Gamma)$ , Proposition 2.3 implies  $|G| = |GR|/|R| = |(GR)_v|/|R_v| \mid 2^9 \cdot 3^2 \cdot 5$ , which is impossible because  $l \geq 2$ . Furthermore, R cannot have two orbits as  $\Gamma$  is not bipartite. Thus, R has at least three orbits. By Proposition 2.9,  $\Gamma_R$  is a connected pentavalent  $B/R \cong G = T^l$ -vertex-transitive and A/R-symmetric graph. Since R is the largest solvable normal subgroup of A, we have  $\operatorname{rad}(A/R) = 1$ . By Lemmas 3.2 and 3.1, A/R has a minimal normal subgroup  $M/R = T^r$  contained in B/R with  $1 \leq r \leq l$ . It follows that  $R \leq M \leq B$  and  $M \leq A$ . In particular,  $R \neq M$ .

Assume  $B = R \times G$ . Then  $M = M \cap RG = R(M \cap G) = R \times (M \cap G)$  and  $M \cap G$  is characteristic in M. It follows  $M \cap G \subseteq A$  and  $|M \cap G| = |T|^r$ . If  $M \cap G$  is transitive, then  $|G_v|/|(M \cap G)_v| = |G|/|M \cap G| = |T|^{l-r} \mid 2^9 \cdot 3^2 \cdot 5$ . It follows l-r=0 or l-r=1. The former implies  $G = M \cap G \subseteq A$  and we are done. For the latter,  $T = A_6$  as  $T \ne A_5$  and hence  $5 \mid |G_v|$  and  $5 \nmid |(M \cap G)_v|$ , that is,  $\Gamma$  is G-symmetric, but not  $(M \cap G)$ -symmetric. Since  $M \cap G \subseteq G$ , we have  $(M \cap G)_v = 1$  and  $|G_v| = |T| = 2^3 \cdot 3^2 \cdot 5$ , which is impossible by Proposition 2.1. If  $M \cap G$  has two orbits on  $V(\Gamma)$ , then  $\Gamma$  is bipartite, which is also impossible. Thus,  $M \cap G$  has at least three orbits on  $V(\Gamma)$  and Proposition 2.9 implies that  $\Gamma_{M \cap G}$  is a connected pentavalent  $G/(M \cap G)$ -vertex-transitive and  $A/(M \cap G)$ -symmetric graph. By inductive hypothesis,  $G/(M \cap G) \subseteq A/(M \cap G)$  and so  $G \subseteq A$ .

By the above paragraph, to finish the proof we only need to show  $B = R \times G$ , and to do this, we now claim  $B \subseteq A$ . If M is transitive, then  $|B_v|/|M_v| = |B|/|M| = |B/R|/|M/R| = |T|^{l-r} \mid 2^9 \cdot 3^2 \cdot 5$  and so l-r=0 or l-r=1. For the former,  $B=M \subseteq A$ , as claimed. For the latter,  $T=A_6$  as  $T \neq A_5$ , and so  $|B_v|/|M_v| = |A_6|$ . It follows  $M_v=1$  and  $|B_v| = |T| = 2^3 \cdot 3^2 \cdot 5$ , which is impossible by Proposition 2.1. If M has two orbits, then  $\Gamma$  is bipartite, which is also impossible. Thus, M has at least three orbits, and by Proposition 2.9,  $\Gamma_M$  is a connected pentavalent B/M-vertex-transitive and A/M-symmetric graph. Since  $B/M \cong (B/R)/(M/R) = T^{l-r}$  with l-r < l, the inductive hypothesis implies that  $B/M \subseteq A/M$  and so  $B \subseteq A$ , as claimed.

Since  $G \leq B \leq A$  and A is symmetric, we have  $B_v = 1$  or  $5 \mid |B_v|$ . If  $B_v = 1$ , the Frattini argument implies  $B = GB_v = G$ , and hence R = 1 and  $B = R \times G$ . If  $5 \mid |B_v|$ , then  $5 \mid |R|$  or  $5 \mid |G_v|$  as  $|B_v| = |G_v||R|$ . By Lemmas 3.3 and 3.4,  $B = R \times G$ .

The proof of Corollary 1.2: This follows from Theorem 1.1 and [2, Theorem 1.1].  $\square$ 

The proof of Corollary 1.3: It is easy to check that Lemmas 3.1-3.4 are true if Assumption is replaced by the following assumption:

**Assumption 1:** Every connected pentavalent T-symmetric graph is T-normal.

Then a similar proof to the proof of Theorem 1.1 implies that under Assumption 1, every connected pentavalent G-symmetric graph is G-normal, and by [2, Corollary 1.2], we have Corollary 1.3.

The proof of Corollary 1.4: By assumption of Corollary 1.4,  $\Gamma$  is a connected pentavalent symmetric G-regular graph with  $G = T^l$ , and hence it is G-vertex-transitive. Recall that

$$\Delta = \{ PSL(2,8), \ \Omega_8^-(2), \ A_{n-1} \mid n \ge 6, \ n \mid 2^9 \cdot 3^2 \cdot 5 \}, 
\Delta_1 = \{ PSL(2,8), \ \Omega_8^-(2), \ A_{n-1} \mid n = 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3, 2 \cdot 5, 2^2 \cdot 5, 2^3 \cdot 3, 2^3 \cdot 5, 2 \cdot 5, 2^5 \cdot 3^2 \cdot 5, 2^7 \cdot 3 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^7 \cdot 3^2 \cdot 5, 2^9 \cdot 3^2 \cdot 5 \}.$$

By Corollary 1.2,  $\Gamma$  is G-normal except for  $T \in \Delta$ , and to prove Corollary 1.4, it suffices to show that  $\Gamma$  is G-normal for  $T \in \Delta - \Delta_1$ . In what follows we always assume  $T \in \Delta - \Delta_1$ .

Clearly,  $T \neq A_5$ . If l = 1, Corollary 1.4 is true by [2, Corollary 1.3]. Assume  $l \geq 2$ . Then Lemmas 3.1 and 3.2 are true by the Remarks of these two lemmas. Then a similar argument to the proof of Theorem 1.1 (the first three paragraphs) implies that  $\Gamma$  is G-normal if  $RG = R \times G$ , where R is the radical of  $A = \operatorname{Aut}(\Gamma)$ . Set B = RG. To finish the proof, we only need to show  $B = R \times G$ .

Since R is solvable,  $G \cap R = 1$  and |B| = |G||R|, and by Proposition 2.3,  $|R| = |R|/|R \cap G| \mid 2^9 \cdot 3^2 \cdot 5$ . We may write  $|R| = 2^m \cdot 3^h \cdot 5^k$ , where  $0 \le m \le 9$ ,  $0 \le h \le 2$  and  $0 \le k \le 1$ . Since R is solvable, there exists a series of principle subgroups of B:

$$B > R = R_s > \cdots R_1 > R_0 = 1$$

such that  $R_i \subseteq B$  and  $R_{i+1}/R_i$  is an elementary abelian r-group with  $0 \le i \le s-1$ , where r=2, 3 or 5. Clearly, G has a natural action on  $B_{i+1}/B_i$  by conjugation.

Suppose to the contrary that  $B \neq R \times G$ . Then there exists some  $0 \leq j \leq s-1$  such that  $GR_j = G \times R_j$ , but  $GR_{j+1} \neq G \times R_{j+1}$ . If G acts trivially on  $R_{j+1}/R_j$  by conjugation, then  $[GR_j/R_j, R_{j+1}/R_j] = 1$ . Furthermore, since  $GR_j/R_j \cong G = T^l$ , we have  $(GR_j/R_j) \cap (R_{j+1}/R_j) = 1$ , and since  $|GR_{j+1}/R_j| = |GR_{j+1}/R_{j+1}||R_{j+1}/R_j| = |G||R_{j+1}/R_j| = |GR_j/R_j||R_{j+1}/R_j|$ , we have  $GR_{j+1}/R_j = GR_j/R_j \times R_{j+1}/R_j$ . In particular,  $GR_j \leq GR_{j+1}$  and so  $G \leq GR_{j+1}$  because  $GR_j = G \times R_j$  implies that G is characteristic in  $GR_j$ . It follows that  $GR_{j+1} = G \times R_{j+1}$ , a contradiction. Thus, G acts non-trivially on  $R_{j+1}/R_j$ . Let K be the kernel of this action. Then  $G/K = T^{l'}$  with  $I' \geq 1$ .

Since  $|R| = 2^m \cdot 3^h \cdot 5^k$  with  $0 \le m \le 9$ ,  $0 \le h \le 2$  and  $0 \le k \le 1$ ,  $R_{j+1}/R_j$  is an elementary abelian group of order  $r^\ell$ , where  $\ell \le m \le 9$  for r = 2,  $\ell \le 2$  for r = 3 and  $\ell \le 1$  for  $\ell = 5$ . Then  $\ell = 6$  for  $\ell$ 

Acknowledgements: This work was supported by the National Natural Science Foundation of China (11571035) and by the 111 Project of China (B16002).

# References

[1] R. W. Baddeley and C. E. Praeger, On primitive overgroups of quasiprimitive permutation groups, J. Alg. 263 (2003) 294-344.

- [2] J.-L. Du, Y.-Q. Feng, J.-X. Zhou, Pentavalent symmetric graphs admitting vertex-transitive non-abelian simple groups, Europ. J. Combin. 63 (2017) 132-145.
- [3] X.G. Fang, L.J. Jia, J. Wang, On the automorphism groups of symmetric graphs admitting an almost simple group, Europ. J. Combin. 29 (2008) 1467-1472.
- [4] X.G. Fang, C.E. Praeger, Finite two-arc transitive graphs admitting a Suzuki simple group, J. Algebraic Combin. 27 (1999) 3727-3754.
- [5] X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group, Comm. Algebra 27 (1999) 3755-3769.
- [6] D. Gorenstein, Finite simple groups (Plenum Press, New York, 1982).
- [7] S.-T. Guo, Y.-Q. Feng, A note on pentavalent s-transitive graphs, Discrete Math. 312 (2012) 2214-2216.
- [8] R.M. Guralnick, Subgroups of prime power index in a simple group, J Algebra. 81 (1983) 304-311.
- [9] P. Hall, A note on soluble groups, J. London Math. Soc. 3 (1928) 98-105.
- [10] X.-H. Hua, Y.-Q. Feng, Cubic graphs admitting transitive non-abelian characterically simple groups, Proceedings of the Edinburgh Mathematical Soc. 54 (2011) 113-123.
- [11] B. Huppert, W. Lempken, Simple groups of order divisible by at most four primes, Proc. of the F. Scorina Gemel State University 16 (2000) 64-75.
- [12] C.H. Li, Isomorphisms of finite Cayley graphs, Ph.D. Thesis, The University of Western Australia, 1996.
- [13] X.H. Li, M.Y. Xu, The primitive permutation groups of degree  $2^a \cdot 3^b$ , Arch Math, 86 (2006) 385-391.
- [14] P. Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency, J. Graph Theory 8 (1984) 55-68.
- [15] C. M. Roney-Dougal, The primitive permutation groups of degree less than 2500, J. Alg. 292 (2005) 154-183.
- [16] S.J. Xu, X.G. Fang, J. Wang, M.Y. Xu, On cubic s-arc transitive Cayley graphs of finite simple groups, Europ. J. Combin. 26 (2005) 133-143.
- [17] S.J. Xu, X.G. Fang, J. Wang, M.Y. Xu, 5-arc transitive cubic Cayley graphs on finite simple groups, Europ. J. Combin. 28 (2007) 1023-1036.