# Stability in the Erdős-Gallai Theorem on cycles and paths, II* 

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#### Abstract

The Erdős-Gallai Theorem states that for $k \geq 3$, any $n$-vertex graph with no cycle of length at least $k$ has at most $\frac{1}{2}(k-1)(n-1)$ edges. A stronger version of the Erdős-Gallai Theorem was given by Kopylov: If $G$ is a 2 -connected $n$-vertex graph with no cycle of length at least $k$, then $e(G) \leq \max \left\{h(n, k, 2), h\left(n, k,\left\lfloor\frac{k-1}{2}\right\rfloor\right)\right\}$, where $h(n, k, a):=\binom{k-a}{2}+a(n-k+a)$. Furthermore, Kopylov presented the two possible extremal graphs, one with $h(n, k, 2)$ edges and one with $h\left(n, k,\left\lfloor\frac{k-1}{2}\right\rfloor\right)$ edges.

In this paper, we complete a stability theorem which strengthens Kopylov's result. In particular, we show that for $k \geq 3$ odd and all $n \geq k$, every $n$-vertex 2 -connected graph $G$ with no cycle of length at least $k$ is a subgraph of one of the two extremal graphs or $e(G) \leq \max \left\{h(n, k, 3), h\left(n, k, \frac{k-3}{2}\right)\right\}$. The upper bound for $e(G)$ here is tight.


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## 1 Introduction

One of the basic Turán-type problems is to determine the maximum number of edges in an $n$-vertex graph with no $k$-vertex path. Erdős and Gallai [3] in 1959 proved the following fundamental result on this problem.

Theorem 1.1 (Erdős and Gallai [3). Fix $n, k \geq 2$. If $G$ is an $n$-vertex graph that does not contain a path with $k$ vertices, then $e(G) \leq \frac{1}{2}(k-2) n$.

When $n$ is divisible by $k-1$, the bound is best possible. Indeed, the $n$-vertex graph whose every component is the complete graph $K_{k-1}$ has $\frac{1}{2}(k-2) n$ edges and no $k$-vertex paths. Also, if $H$ is an $n$-vertex graph without a $k$-vertex path $P_{k}$, then by adding to $H$ a new vertex $v$ adjacent to all vertices of $H$ we obtain an $(n+1)$-vertex graph $H^{\prime}$ with $e(H)+n$ edges that contains no cycle of length $k+1$ or longer. Then Theorem 1.1 follows from another theorem of Erdős and Gallai:

[^0]

Figure 1: $H_{14,11,3}$.

Theorem 1.2 (Erdős and Gallai [3). Fix $n, k \geq 3$. If $G$ is an $n$-vertex graph that does not contain a cycle of length at least $k$, then $e(G) \leq \frac{1}{2}(k-1)(n-1)$.

The bound of this theorem is best possible for $n-1$ divisible by $k-2$. Indeed, any connected $n$-vertex graph in which every block is a $K_{k-1}$ has $\frac{1}{2}(k-1)(n-1)$ edges and no cycles of length at least $k$. In the 1970's, some refinements and new proofs of Theorems 1.1 and 1.2 were obtained by Faudree and Schelp [4, 5], Lewin [9, Woodall [10, and Kopylov [8] - see [7] for more details. The strongest version was proved by Kopylov [8]. His result is stated in terms of the following graphs. Let $n \geq k$ and $1 \leq a<\frac{1}{2} k$. The $n$-vertex graph $H_{n, k, a}$ is as follows. The vertex set of $H_{n, k, a}$ is the union of three disjoint sets $A, B$, and $C$ such that $|A|=a,|B|=n-k+a$ and $|C|=k-2 a$, and the edge set of $H_{n, k, a}$ consists of all edges between $A$ and $B$ together with all edges in $A \cup C$ (Fig. 1 shows $H_{14,11,3}$ ). Let

$$
h(n, k, a):=e\left(H_{n, k, a}\right)=\binom{k-a}{2}+a(n-k+a) .
$$

For a graph $G$, let $c(G)$ denote the length of a longest cycle in $G$. Observe that $c\left(H_{n, k, a}\right)<k$ : Since $|A \cup C|=k-a$, any cycle $D$ of at length at least $k$ has at least $a$ vertices in $B$. But as $B$ is independent and $2 a<k, D$ also has to contain at least $k+1$ neighbors of the vertices in $B$, while only $a$ vertices in $A$ have neighbors in $A$. Kopylov [8] showed that the extremal 2-connected $n$-vertex graphs with no cycles of length at least $k$ are $G=H_{n, k, 2}$ and $G=H_{n, k, t}$ : the first has more edges for small $n$, and the second - for large $n$.

Theorem 1.3 (Kopylov [8]). Let $n \geq k \geq 5$ and $t=\left\lfloor\frac{1}{2}(k-1)\right\rfloor$. If $G$ is an n-vertex 2 -connected graph with $c(G)<k$, then

$$
\begin{equation*}
e(G) \leq \max \{h(n, k, 2), h(n, k, t)\} \tag{1}
\end{equation*}
$$

with equality only if $G=H_{n, k, 2}$ or $G=H_{n, k, t}$.

## 2 Main results

### 2.1 A previous result

Recently, three of the present authors proved in [6] a stability version of Theorems 1.2 and 1.3 for $n$-vertex 2 -connected graphs with $n \geq 3 k / 2$, but the problem remained open for $n<3 k / 2$ when $k \geq 9$. The main result of [6] was the following:

Theorem 2.1 (Füredi, Kostochka, Verstraëte [6]). Let $t \geq 2$ and $n \geq 3 t$ and $k \in\{2 t+1,2 t+2\}$. Let $G$ be a 2-connected n-vertex graph $c(G)<k$. Then $e(G) \leq h(n, k, t-1)$ unless
(a) $k=2 t+1, k \neq 7$, and $G \subseteq H_{n, k, t}$ or
(b) $k=2 t+2$ or $k=7$, and $G-A$ is a star forest for some $A \subseteq V(G)$ of size at most $t$.

Note that

$$
h(n, k, t)-h(n, k, t-1)= \begin{cases}n-t-3 & \text { if } k=2 t+1 \\ n-t-5 & \text { if } k=2 t+2\end{cases}
$$

The paper [6] also describes the 2-connected $n$-vertex graphs with $c(G)<k \leq 8$ for all $n \geq k$.

### 2.2 The essence of the main result

Together with [6], this paper gives a full description of the 2 -connected $n$-vertex graphs with $c(G)<k$ and 'many' edges for all $k$ and $n$. Our main result is:

Theorem 2.2. Let $t \geq 4$ and $k \in\{2 t+1,2 t+2\}$, so that $k \geq 9$. If $G$ is a 2 -connected graph on $n \geq k$ vertices and $c(G)<k$, then either $e(G) \leq \max \{h(n, k, t-1), h(n, k, 3)\}$ or
(a) $k=2 t+1$ and $G \subseteq H_{n, k, t}$ or
(b) $k=2 t+2$ and $G-A$ is a star forest for some $A \subseteq V(G)$ of size at most $t$.
(c) $G \subseteq H_{n, k, 2}$.


Figure 2: $H_{n, k, t}(k=2 t+1), H_{n, k, t}(k=2 t+2), H_{n, k, 2}$; ovals denote complete subgraphs of order $t, t$, and $k-2$ respectively.

Note that the case $n<k$ is trivial and the case $k \leq 8$ was fully resolved in [6].

### 2.3 A more detailed form of the main result

In order to prove Theorem 2.2, we need a more detailed description of graphs satisfying (b) in the theorem that do not contain 'long' cycles.


Figure 3: Classes $\mathcal{G}_{2}(n, k), \mathcal{G}_{3}(n, k)$ and $\mathcal{G}_{4}(n, 10)$.

Let $\mathcal{G}_{1}(n, k)=\left\{H_{n, k, t}, H_{n, k, 2}\right\}$. Each $G \in \mathcal{G}_{2}(n, k)$ is defined by a partition $V(G)=A \cup B \cup C$ and two vertices $a_{1} \in A, b_{1} \in B$ such that $|A|=t, G[A]=K_{t}, G[B]$ is the empty graph, $G(A, B)$ is a complete bipartite graph, and $N(c)=\left\{a_{1}, b_{1}\right\}$ for every $c \in C$. Every member of $G \in \mathcal{G}_{3}(n, k)$ is defined by a partition $V(G)=A \cup B \cup J$ such that $|A|=t, G[A]=K_{t}, G(A, B)$ is a complete bipartite graph, and

- $G[J]$ has more than one component,
- all components of $G[J]$ are stars with at least two vertices each,
- there is a 2-element subset $A^{\prime}$ of $A$ such that $N(J) \cap(A \cup B)=A^{\prime}$,
- for every component $S$ of $G[J]$ with at least 3 vertices, all leaves of $S$ have degree 2 in $G$ and are adjacent to the same vertex $a(S)$ in $A^{\prime}$.

The class $\mathcal{G}_{4}(n, k)$ is empty unless $k=10$. Each graph $H \in \mathcal{G}_{4}(n, 10)$ has a 3 -vertex set $A$ such that $H[A]=K_{3}$ and $H-A$ is a star forest such that if a component $S$ of $H-A$ has more than two vertices then all its leaves have degree 2 in $H$ and are adjacent to the same vertex $a(S)$ in $A$. These classes are illustrated below:
We can refine Theorem 2.2 in terms of the classes $\mathcal{G}_{i}(n, k)$ as follows:
Theorem 2.3. (Main Theorem) Let $k \geq 9, n \geq k$ and $t=\left\lfloor\frac{1}{2}(k-1)\right\rfloor$. Let $G$ be an n-vertex 2 -connected graph with no cycle of length at least $k$. Then $e(G) \leq \max \{h(n, k, t-1), h(n, k, 3)\}$ or $G$ is a subgraph of a graph in $\mathcal{G}(n, k)$, where
(1) if $k$ is odd, then $\mathcal{G}(n, k)=\mathcal{G}_{1}(n, k)=\left\{H_{n, k, t}, H_{n, k, 2}\right\}$;
(2) if $k$ is even and $k \neq 10$, then $\mathcal{G}(n, k)=\mathcal{G}_{1}(n, k) \cup \mathcal{G}_{2}(n, k) \cup \mathcal{G}_{3}(n, k)$;
(3) if $k=10$, then $\mathcal{G}(n, k)=\mathcal{G}_{1}(n, 10) \cup \mathcal{G}_{2}(n, 10) \cup \mathcal{G}_{3}(n, 10) \cup \mathcal{G}_{4}(n, 10)$.

Since every graph in $\mathcal{G}_{2}(n, k) \cup \mathcal{G}_{3}(n, k)$ and many graphs in $\mathcal{G}_{4}(n, k)$ have a separating set of size 2 (see Fig. 3), the theorem implies the following simpler statement for 3 -connected graphs:

Corollary 2.4. Let $k \in\{2 t+1,2 t+2\}$ where $k \geq 9$. If $G$ is a 3 -connected graph on $n \geq k$ vertices and $c(G)<k$, then either $e(G) \leq \max \{h(n, k, t-1), h(n, k, 3)\}$ or $G \subseteq H_{n, k, t}$ or $k=10$ and $G$ is a subgraph of some graph $H \in \mathcal{G}_{4}(n, 10)$ such that each component of $H-A$ has at most 2 vertices.

## 3 The proof idea

### 3.1 Small dense subgraphs

First we define some more graph classes. For a graph $F$ and a nonnegative integer $s$, we denote by $\mathcal{K}^{-s}(F)$ the family of graphs obtained from $F$ by deleting at most $s$ edges.

Let $F_{0}=F_{0}(t)$ denote the complete bipartite graph $K_{t, t+1}$ with partite sets $A$ and $B$ where $|A|=t$ and $|B|=t+1$. Let $\mathcal{F}_{0}=\mathcal{K}^{-t+3}\left(F_{0}\right)$, i.e., the family of subgraphs of $K_{t, t+1}$ with at least $t(t+1)-t+3$ edges.

Let $F_{1}=F_{1}(t)$ denote the complete bipartite graph $K_{t, t+2}$ with partite sets $A$ and $B$ where $|A|=t$ and $|B|=t+2$. Let $\mathcal{F}_{1}=\mathcal{K}^{-t+4}\left(F_{1}\right)$, i.e., the family of subgraphs of $K_{t, t+2}$ with at least $t(t+2)-t+4$ edges.

Let $\mathcal{F}_{2}$ denote the family of graphs obtained from a graph in $\mathcal{K}^{-t+4}\left(F_{1}\right)$ by subdividing an edge $a_{1} b_{1}$ with a new vertex $c_{1}$, where $a_{1} \in A$ and $b_{1} \in B$. Note that any member $H \in \mathcal{F}_{2}$ has at least $|A||B|-(t-3)$ edges between $A$ and $B$ and the pair $a_{1} b_{1}$ is not an edge.

Let $F_{3}=F_{3}\left(t, t^{\prime}\right)$ denote the complete bipartite graph $K_{t, t^{\prime}}$ with partite sets $A$ and $B$ where $|A|=t$ and $|B|=t^{\prime}$. Take a graph from $\mathcal{K}^{-t+4}\left(F_{3}\right)$, select two non-empty subsets $A_{1}, A_{2} \subseteq A$ with $\left|A_{1} \cup A_{2}\right| \geq 3$ such that $A_{1} \cap A_{2}=\emptyset$ if $\min \left\{\left|A_{1}\right|,\left|A_{2}\right|\right\}=1$, add two vertices $c_{1}$ and $c_{2}$, join them to each other and add the edges from $c_{i}$ to the elements of $A_{i},(i=1,2)$. The class of obtained graphs is denoted by $\mathcal{F}\left(A, B, A_{1}, A_{2}\right)$. The family $\mathcal{F}_{3}$ consists of these graphs when $|A|=|B|=t$, $\left|A_{1}\right|=\left|A_{2}\right|=2$ and $A_{1} \cap A_{2}=\emptyset$. In particular, $\mathcal{F}_{3}(4)$ consists of exactly one graph, call it $F_{3}(4)$.

Graph $F_{4}$ has vertex set $A \cup B$, where $A=\left\{a_{1}, a_{2}, a_{3}\right\}$ and $B:=\left\{b_{1}, b_{2}, \ldots, b_{6}\right\}$ are disjoint. Its edges are the edges of the complete bipartite graph $K(A, B)$ and three extra edges $b_{1} b_{2}, b_{3} b_{4}$, and $b_{5} b_{6}$ (see Fig. 4 below). Define $F_{4}^{\prime}$ as the (only) member of $\mathcal{F}\left(A, B, A_{1}, A_{2}\right)$ such that $|A|=|B|=t=4$, $A_{1}=A_{2}$, and $\left|A_{i}\right|=3$. Let $\mathcal{F}_{4}:=\left\{F_{4}, F_{4}^{\prime}\right\}$, which is defined only for $t=4$.


Figure 4: Graphs $F_{3}(4), F_{4}$, and $F_{4}^{\prime}$.
Define $\mathcal{F}(k):= \begin{cases}\mathcal{F}_{0}, & \text { if } k \text { is odd, } \\ \mathcal{F}_{1} \cup \cdots \cup \mathcal{F}_{4}, & \text { if } k \text { is even. }\end{cases}$

### 3.2 Proof idea

For our proof, it will be easier to use the stronger induction assumption that the graphs in question contain certain dense graphs from $\mathcal{F}(k)$. We will prove the following slightly stronger version of Theorem 2.3 which also implies Theorem 2.2 .

Theorem 2.3 Let $t \geq 4, k \in\{2 t+1,2 t+2\}$, and $n \geq k$. Let $G$ be an $n$-vertex 2 -connected graph with no cycle of length at least $k$. Then $e(G) \leq \max \{h(n, k, t-1), h(n, k, 3)\}$ or
(a) $G \subseteq H_{n, k, 2}$, or
(b) $G$ is contained in a graph in $\mathcal{G}(n, k)-\left\{H_{n, k, 2}\right\}$, and $G$ contains a subgraph $H \in \mathcal{F}(k)$.

The method of the proof is a variation of that of [6]. Also, when $n$ is close to $k$, we use Kopylov's disintegration method. We take an $n$-vertex graph $G$ satisfying the hypothesis of Theorem 2.3, and iteratively contract edges in a certain way so that each intermediate graph still satisfies the hypothesis. We consider the final graph of this process $G_{m}$ on $m$ vertices and show that $G_{m}$ satisfies Theorem 2.3. Two results from [6] will be instrumental. The first is:

Lemma 3.1 (Main lemma on contraction [6]). Let $k \geq 9$ and suppose $F$ and $F^{\prime}$ are 2-connected graphs such that $F=F^{\prime} / x y$ and $c\left(F^{\prime}\right)<k$. If $F$ contains a subgraph $H \in \mathcal{F}(k)$, then $F^{\prime}$ also contains a subgraph $H^{\prime} \in \mathcal{F}(k)$.

This lemma shows that if $G_{m}$ contains a subgraph $H \in \mathcal{F}(k)$, then the original graph $G$ also contains a subgraph in $\mathcal{F}(k)$. The second result (proved in Subsection 4.5 of [6]) is:

Lemma 3.2 ([6]). Let $k \geq 9$, and let $G$ be a 2 -connected graph with $c(G)<k$ and $e(G)>h(n, k, t-$ 1). If $G$ contains a subgraph $H \in \mathcal{F}(k)$, then $G$ is a subgraph of a graph in $\mathcal{G}(n, k)-\left\{H_{n, k, 2}\right\}$.

We will split the proof into the cases of small $n$ and large $n$. The following observations can be obtained by simple calculations (for $t \geq 4$ ):

| $k$ | $h(n, k, 3) \geq h(n, k, t-1)$ | $h(n, k, 2) \geq h(n, k, t-1)$ |
| :---: | :---: | :---: |
| $2 t+1$ | If and only if $n \leq k+(t-5) / 2$ | If and only if $n \leq k+t / 2-1$ |
| $2 t+2$ | If and only if $n \leq k+(t-3) / 2$ | If and only if $n \leq k+t / 2$ |

In the case of large $n$ we will contract an edge such that the new graph still has more than $h(n-1, k, t-1)$ edges. In order to apply induction, we also need the number of edges to be greater than $h(n-1, k, 3)$. To guarantee this, we pick the cutoffs for the two cases $n \leq k+(t-1) / 2$ and $n>k+(t-1) / 2$ (therefore $n-1>k+(t-3) / 2)$.

## 4 Tools

### 4.1 Classical theorems

Theorem 4.1 (Erdős [2]). Let $d \geq 1$ and $n>2 d$ be integers, and

$$
\ell_{n, d}=\max \left\{\binom{n-d}{2}+d^{2},\binom{\left\lceil\frac{n+1}{2}\right\rceil}{ 2}+\left\lfloor\frac{n-1}{2}\right\rfloor^{2}\right\} .
$$

Then every $n$-vertex graph $G$ with $\delta(G) \geq d$ and $e(G)>\ell_{n, d}$ is hamiltonian.
Theorem 4.2 (Chvátal [1). Let $n \geq 3$ and $G$ be an n-vertex graph with vertex degrees $d_{1} \leq d_{2} \leq$ $\ldots \leq d_{n}$. If $G$ is not hamiltonian, then there is some $i<n / 2$ such that $d_{i} \leq i$ and $d_{n-i}<n-i$.
Theorem 4.3 (Kopylov [8]). If $G$ is 2 -connected and $P$ is an $x, y$-path of $\ell$ vertices, then $c(G) \geq$ $\min \{\ell, d(x, P)+d(y, P)\}$.

### 4.2 Claims on contractions

A helpful tool will be the following lemma from [6] on contraction.
Lemma 4.4 ([6]). Let $n \geq 4$ and let $G$ be an $n$-vertex 2 -connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that $G / v w$ is 2 -connected.

For an edge $x y$ in a graph $H$, let $T_{H}(x y)$ denote the number of triangles containing $x y$. Let $T(H)=\min \left\{T_{H}(x y): x y \in E(H)\right\}$. When we contract an edge $u v$ in a graph $H$, the degree of every $x \in V(H)-u-v$ either does not change or decreases by 1 . Also the degree of $u * v$ in $H / u v$ is at least $\max \left\{d_{H}(u), d_{H}(v)\right\}-1$. Thus

$$
\begin{equation*}
d_{H / u v}(w) \geq d_{H}(w)-1 \text { for any } w \in V(H) \text { and } u v \in E(H) . \text { Also } d_{H / u v}(u * v) \geq d_{H}(u)-1 \tag{2}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
T(H / u v) \geq T(H)-1 \text { for every graph } H \text { and } u v \in E(H) . \tag{3}
\end{equation*}
$$

We will use the following analog of Lemma 3.3 in [6].
Lemma 4.5. Let $h$ be a positive integer. Suppose a 2 -connected graph $G$ is obtained from a 2connected graph $G^{\prime}$ by contracting edge xy into $x * y$ chosen using the following rules:
(i) one of $x, y$, say $x$ is a vertex of the minimum degree in $G^{\prime}$;
(ii) $T_{G^{\prime}}(x y)$ is the minimum among the edges $x u$ incident with $x$ such that $G^{\prime} / x u$ is 2-connected. (Such edges exist by Lemma 4.4). If $G$ has at least $h$ vertices of degree at most $h$, then either $G^{\prime}=K_{h+2}$ or
(a) $G^{\prime}$ also has a vertex of degree at most $h$, and
(b) $G^{\prime}$ has at least $h+1$ vertices of degree at most $h+1$.

Proof. Since $G$ is 2 -connected, $h \geq 2$. Let $V_{\leq s}(H)$ denote the set of vertices of degree at most $s$ in $H$. Then by (2), each $v \in V_{\leq h}(G)-x * y$ is also in $V_{\leq h+1}\left(G^{\prime}\right)$. Moreover, then by (i),

$$
\begin{equation*}
x \in V_{\leq h+1}\left(G^{\prime}\right) . \tag{4}
\end{equation*}
$$

Thus if $x * y \notin V_{\leq h}(G)$, then (b) follows. But if $x * y \in V_{\leq h}(G)$, then by $(2)$, also $y \in V_{\leq h+1}\left(G^{\prime}\right)$. So, again (b) holds.

If $V_{\leq h-1}(G) \neq \emptyset$, then (a) holds by (2). So, if (a) does not hold, then
each $v \in V_{\leq h}(G)-x * y$ has degree $h+1$ in $G^{\prime}$ and is adjacent to both $x$ and $y$ in $G^{\prime}$.

Case 1: $\left|V_{\leq h}(G)-x * y\right| \geq h$. Then by (4), $d_{G^{\prime}}(x)=h+1$. This in turn yields $N_{G^{\prime}}(x)=V_{\leq h}(G)+y$. Since $G^{\prime}$ is 2 -connected, each $v \in V_{\leq h}(G)-x * y$ is not a cut vertex. Furthermore, $\{x, v\}$ is not a cut set. If it was, because $y$ is a common neighbor of all neighbors of $x$, all neighbors of $x$ must be in the same component as $y$ in $G^{\prime}-x-v$. It follows that

$$
\begin{equation*}
\text { for every } v \in V_{\leq h}(G)-x * y, G^{\prime} / v x \text { is 2-connected. } \tag{6}
\end{equation*}
$$

If $u v \notin E(G)$ for some $u, v \in V_{\leq h}(G)$, then by (6) and (i), we would contract the edge $x u$ rather
than $x y$. Thus $G^{\prime}\left[V_{\leq h}(G) \cup\{x, y\}\right]=K_{h+2}$ and so either $G^{\prime}=K_{h+2}$ or $y$ is a cut vertex in $G^{\prime}$, as claimed.

Case 2: $\left|V_{\leq h}(G)-x * y\right|=h-1$. Then $x * y \in V_{\leq h}(G)$. This means $d_{G^{\prime}}(x)=d_{G^{\prime}}(y)=h+1$ and $N_{G^{\prime}}[x]=N_{G^{\prime}}[y]$. So by (5), there is $z \in V(G)$ such that $N_{G^{\prime}}[x]=N_{G^{\prime}}[y]=V_{\leq h}(G) \cup\{x, y, z\}$. Again (6) holds (for the same reason that $\left.N_{G^{\prime}}[x] \subseteq N_{G^{\prime}}[y]\right)$. Thus similarly $v u \in E\left(G^{\prime}\right)$ for every $v \in V_{\leq h}(G)-x * y$ and every $u \in V_{\leq h}(G)+z$. Hence $G^{\prime}\left[V_{\leq h}(G) \cup\{x, y, z\}\right]=K_{h+2}$ and either $G^{\prime}=K_{h+2}$ or $z$ is a cut vertex in $G^{\prime}$, as claimed.

### 4.3 A property of graphs in $\mathcal{F}(k)$

A useful feature of graphs in $\mathcal{F}(k)$ is the following.
Lemma 4.6. Let $k \geq 9$ and $n \geq k$. Let $F$ be an n-vertex graph contained in $H_{n, k, t}$ with $e(F)>$ $h(n, k, t-1)$. Then $F$ contains a graph in $\mathcal{F}(k)$.

Proof. Assume the sets $A, B, C$ to be as in the definition of $H_{n, k, t}$. We will use induction on $n$.
Case 1: $k=2 t+1$. If $n=k$, then $F \in \mathcal{K}^{-t+3}\left(H_{k, k, t}\right)$ because $h(k, k, t)-h(k, k, t-1)-1=t-3$. Thus, since $H_{k, k, t} \supseteq F_{0}(t), F$ contains a subgraph in $\mathcal{F}_{0}$. Suppose now the lemma holds for all $k \leq n^{\prime}<n$. If $\delta(F) \geq t$, then each $v \in V(F)-A$ is adjacent to every $u \in A$. Hence $F$ contains $K_{t, n-t}$. If $\delta(F)<t$, then since $A$ is dominating and $n>2 t$, there is $v \in V(F)-A$ with $d_{F}(v) \leq t-1$. Then $F-v \subseteq H_{n-1, k, t}$, and we are done by induction.

Case 2: $k=2 t+2$. Let $C=\left\{c_{1}, c_{2}\right\}$. If $n=k$ then as in Case 1,

$$
e\left(H_{k, k, t}\right)-e(F) \leq h(k, k, t)-h(k, k, t-1)-1=t-4,
$$

i.e., $F \in \mathcal{K}^{-t+4}\left(H_{k, k, t}\right)$. Since $F_{1}(t) \subseteq H_{k, k, t}, F$ contains a subgraph in $\mathcal{F}_{1}$. Suppose now the lemma holds for all $k \leq n^{\prime}<n$. If $\delta(F)<t$, then there is $v \in V(F)-A$ with $d_{F}(v) \leq t-1$. Then $F-v \subseteq H_{n-1, k, t}$, and we are done by induction.

Finally, suppose $\delta(F) \geq t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of $c_{1}, c_{2}$ has at least $t-1$ neighbors in $A$. Since $\left|B \cup\left\{c_{1}\right\}\right| \geq n-t-1 \geq t+2, F$ contains a member of $\mathcal{K}^{-1}\left(F_{1}(t)\right)$. Thus $F$ contains a member of $\mathcal{F}_{1}$ unless $t=4, n=2 t+3$ and $c_{1}$ has a nonneighbor $x \in A$. But then $c_{1} c_{2} \in E(F)$, and so $F$ contains either $F_{3}(4)$ or $F_{4}^{\prime}$.

## 5 Proof of Theorem 2.3

### 5.1 Contraction procedure

If $n>k$, we iteratively construct a sequence of graphs $G_{n}, G_{n-1}, \ldots G_{m}$ where $\left|V\left(G_{j}\right)\right|=j$. In [6], the following Basic Procedure (BP) was used:

At the beginning of each round, for some $j: k \leq j \leq n$, we have a $j$-vertex 2-connected graph $G_{j}$ with $e\left(G_{j}\right)>h(j, k, t-1)$.
(BP1) If $j=k$, then we stop.
(BP2) If there is an edge $u v$ with $T_{G_{j}}(u v) \leq t-2$ such that $G_{j} / u v$ is 2-connected, choose one such edge so that
(i) $T_{G_{j}}(u v)$ is minimum, and subject to this
(ii) $u v$ is incident to a vertex of minimum possible degree.

Then obtain $G_{j-1}$ by contracting $u v$.
(BP3) If (BP2) does not hold, $j \geq k+t-1$ and there is $x y \in E\left(G_{j}\right)$ such that $G_{j}-x-y$ has at least 3 components and one of the components, say $H_{1}$ is a $K_{t-1}$, then let $G_{j-t+1}=G_{j}-V\left(H_{1}\right)$.
(BP4) If neither (BP2) nor (BP3) occurs, then we stop.

Remark 5.1. By definition, (BP3) applies only when $j \geq k+t-1$. As observed in [6], if $j \leq 3 t-2$, then a $j$-vertex graph $G_{j}$ with a 2 -vertex set $\{x, y\}$ separating the graph into at least 3 components cannot have $T_{G_{j}}(u v) \geq t-1$ for every edge $u v$. It also was calculated there that if $3 t-1 \leq j \leq 3 t$, then any $j$-vertex graph $G^{\prime}$ with such 2 -vertex set $\{x, y\}$ and $T_{G^{\prime}}(u v) \geq t-1$ for every edge $u v$ has at most $h(j, k, t-1)$ edges and so cannot be $G_{j}$.

In this paper, we also use a quite similar Modified Basic Procedure (MBP): start with a 2connected, $n$-vertex graph $G=G_{n}$ with $e(G)>h(n, k, t-1)$ and $c(G)<k$. Then
(MBP0) if $\delta\left(G_{j}\right) \geq t$, then apply the rules ( BP 1 )-( BP 4$)$ of ( BP ) given above;
(MBP1) if $\delta\left(G_{j}\right) \leq t-1$ and $j=k$, then stop;
(MBP2) otherwise, pick a vertex $v$ of smallest degree, contract an edge $v u$ with the minimum $T_{G_{j}}(v u)$ among the edges $v u$ such that $G_{j} / v u$ is 2-connected, and set $G_{j-1}=G_{j} / u v$.

### 5.2 Proof of Theorem 2.3 for the case $n \leq k+(t-1) / 2$

Apply to $G$ the Modified Basic Procedure (MBP) starting from $G_{n}=G$. By Remark 1, (BP3) was never applied, since $k+(t-1) / 2<k+t-1$. Therefore at every step, we only contracted an edge. Denote by $G_{m}$ the terminating graph of (MBP). Then $G_{j}$ is 2-connected and $c\left(G_{j}\right) \leq c(G)<k$ for each $m \leq j \leq n$. By construction, after each contraction, we lose at most $t-1$ edges. It follows that $e\left(G_{m}\right)>h(m, k, t-1)$.
If $m>k$, then the same argument as in [6] gives us the following structural result:
Lemma 5.1 (6]). Let $m>k \geq 9$ and $n \geq k$.

- If $k \neq 10$, then $G_{m} \subseteq H_{m, k, t}$.
- If $k=10$, then $G_{m} \subseteq H_{m, k, t}$ or $G_{m} \supseteq F_{4}$.

If $k=10$ and $G_{m} \supseteq F_{4}$, then $G_{m}$ contains a subgraph in $\mathcal{F}(k)$. Otherwise, by Lemma 4.6, again $G_{m}$ has a subgraph in $\mathcal{F}(k)$. Next, undo the contractions that were used in (MBP). At every step for $m \leq j \leq n$, by Lemma 3.1, $G_{j}$ contains some subgraph $H^{\prime} \in \mathcal{F}(k)$. In particular, $G=G_{n}$ contains such a subgraph. Thus by Lemma 3.2, we get our result. So, below we assume

$$
\begin{equation*}
m=k \tag{7}
\end{equation*}
$$

Since $c\left(G_{k}\right)<k, G_{k}$ does not have a hamiltonian cycle. Denote the vertex degrees of $G_{k} d_{1} \leq$ $d_{2} \leq \ldots \leq d_{k}$. By Theorem 4.2, there exists some $2 \leq i \leq t$ such that $d_{i} \leq i$ and $d_{k-i}<k-i$. Let $r=r\left(G_{k}\right)$ be the smallest such $i$.

Because $G_{k}$ has $r$ vertices of degree at most $r$, similarly to [2],

$$
e\left(G_{k}\right) \leq r^{2}+\binom{k-r}{2}
$$

For $k=2 t+1, r^{2}+\binom{k-r}{2}>h(n, k, t-1)$ only when $r=t$ or $r<(t+4) / 3$, and for $k=2 t+2$, when $r=t$ or $r<(t+6) / 3$. If $r=t$, then repeating the argument in [6 yields:

Lemma 5.2 (6]). If $r\left(G_{k}\right)=t$ then $G_{k} \subseteq H_{k, k, t}$.
Then by Lemma 4.6, Lemma 3.1, and Lemma 3.2, $G \subseteq H_{n, k, t}$ and contains some subgraph in $\mathcal{F}(k)$. So we may assume that

$$
\begin{equation*}
\text { if } k=2 t+1 \text { then } r<(t+4) / 3 \text {, and if } k=2 t+2 \text { then } r<(t+6) / 3 \tag{8}
\end{equation*}
$$

Our next goal is to show that $G$ contains a large "core", i.e., a subgraph with large minimum degree. For this, we recall the notion of disintegration used by Kopylov [8].
Definition: For a natural number $\alpha$ and a graph $G$, the $\alpha$-disintegration of a graph $G$ is the process of iteratively removing from $G$ the vertices with degree at most $\alpha$ until the resulting graph has minimum degree at least $\alpha+1$. This resulting subgraph $H=H(G, \alpha)$ will be called the $\alpha$-core of $G$. It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

Claim 5.3. The $t$-core $H(G, t)$ of $G$ is not empty.
Proof of Claim 5.3. We may assume that for all $m \leq j<n$, graph $G_{j}$ was obtained from $G_{j+1}$ by contracting edge $x_{j} y_{j}$, where $d_{G_{j+1}}\left(x_{j}\right) \leq d_{G_{j+1}}\left(y_{j}\right)$. By Rule (MBP2), $d_{G_{j+1}}\left(x_{j}\right)=\delta\left(G_{j+1}\right)$, provided that $\delta\left(G_{j+1}\right) \leq t-1$.
By definition, $\left|V_{\leq r}\left(G_{k}\right)\right| \geq r$. So by Lemma 4.5 (applied several times), for each $k+1 \leq j \leq k+t-r$, because each $G_{j}$ is not a complete graph (otherwise it would have a hamiltonian cycle),

$$
\begin{equation*}
\delta\left(G_{j}\right) \leq j-k+r-1 \text { and }\left|V_{\leq j-k+r}\left(G_{j}\right)\right| \geq j-k+r . \tag{9}
\end{equation*}
$$

To show that

$$
\begin{equation*}
\delta\left(G_{j}\right) \leq t-1 \text { for all } k \leq j \leq n \tag{10}
\end{equation*}
$$

by (9) and (8), it is enough to observe that

$$
\delta\left(G_{j}\right) \leq j-k+r-1 \leq(n-k)+r-1 \leq \frac{t-1}{2}+\frac{t+6}{3}-1=\frac{5 t+3}{6}<t .
$$

We will apply a version of $t$-disintegration in which we first manually remove a sequence of vertices and count the number of edges they cover. By (10) and (MBP2), $d_{G_{n}}\left(x_{n-1}\right)=\delta\left(G_{n}\right) \leq n-k+r-1$. Let $v_{n}:=x_{n-1}$. Then $G-v_{n}$ is a subgraph of $G_{n-1}$. If $x_{n-2} \neq x_{n-1} * y_{n-1}$ in $G_{n-1}$, then let $v_{n-1}:=x_{n-2}$, otherwise let $v_{n-1}:=y_{n-1}$. In both cases, $d_{G-v_{n}}\left(v_{n-1}\right) \leq n-k+r-2$. We continue
in this way until $j=k$ : each time we delete from $G-v_{n}-\ldots-v_{j+1}$ the unique survived vertex $v_{j}$ that was in the preimage of $x_{j-1}$ when we obtained $G_{j-1}$ from $G_{j}$. Graph $G-v_{n}-\ldots-v_{k+1}$ has $r \geq 2$ vertices of degree at most $r$. We additionally delete 2 such vertices $v_{k}$ and $v_{k-1}$. Altogether, we have lost at most $(r+n-k-1)+(r+n-k-2)+\ldots+r+2 r$ edges in the deletions.
Finally, apply $t$-disintegration to the remaining graph on $k-2 \in\{2 t-1,2 t\}$ vertices. Suppose that the resulting graph is empty.

Case 1: $n=k$. Then

$$
e(G) \leq r+r+t(2 t-1-t)+\binom{t}{2}
$$

where $r+r$ edges are from $v_{k}$ and $v_{k-1}$, and after deleting $v_{k}$ and $v_{k-1}$, every vertex deleted removes at most $t$ edges, until we reach the final $t$ vertices which altogether span at most $\binom{t}{2}$ edges.

For $k=2 t+1$,
$h(k, k, t-1)-e(G) \geq\binom{ 2 t+1-(t-1)}{2}+(t-1)^{2}-\left[r+r+t(2 t-1-t)+\binom{t}{2}\right]=t+2-2 r$,
which is nonnegative for $r<(t+3) / 3$. Therefore $e(G) \leq h(k, k, t-1)$, a contradiction.
Similarly, if $k=2 t+2$,

$$
\begin{gathered}
e(G) \leq r+r+t(2 t-t)+\binom{t}{2} \text {, and } \\
h(k, k, t-1)-e(G) \geq\binom{ 2 t+2-(t-1)}{2}+(t-1)^{2}-\left[r+r+t(2 t-t)+\binom{t}{2}\right]=t+4-2 r,
\end{gathered}
$$

which is nonnegative when $r<(t+6) / 3$.
Case 2: $k<n \leq k+(t-1) / 2$. Then for $k=2 t+1$,

$$
\begin{aligned}
e(G) & \leq[(r+n-k-1)+(r+n-k-2)+\ldots+r]+2 r+t(2 t-1-t)+\binom{t}{2} \\
& \leq[(t-1)+(t-1)+\ldots+(t-1)]+h(k, k, t-1) \\
& =(t-1)(n-k)+h(k, k, t-1) \\
& =h(n, k, t-1),
\end{aligned}
$$

where the last inequality holds because $r+n-k-1 \leq t-1$.
Similarly, for $k=2 t+2$,

$$
\begin{aligned}
e(G) & \leq[(r+n-k-1)+(r+n-k-2)+\ldots+r]+2 r+t(2 t-t)+\binom{t}{2} \\
& \leq(n-k)(t-1)+h(k, k, t-1) \\
& =h(n, k, t-1) .
\end{aligned}
$$

This contradiction completes the proof of Claim 5.3.
For the rest of the proof of Theorem [2.3], we will follow the method of Kopylov in [8] to show that $G \subseteq H_{n, k, 2}$. Let $G^{*}$ be the $k$-closure of $G$. That is, add edges to $G$ until adding any additional
edges creates a cycle of length at least $k$. In particular, for any non-edge $x y$ of $G^{*}$, there is an $(x, y)$-path in $G^{*}$ with at least $k-1$ edges.

Because $G$ has a nonempty $t$-core, and $G^{*}$ contains $G$ as a subgraph, $G^{*}$ also has a nonempty $t$-core (which contains the $t$-core of $G$ ). Let $H=H\left(G^{*}, t\right)$ denote the $t$-core of $G^{*}$. We will show that

$$
\begin{equation*}
H \text { is a complete graph. } \tag{11}
\end{equation*}
$$

Indeed, suppose there exists a nonedge in $H$. Choose a longest path $P$ of $G^{*}$ whose terminal vertices $x \in V(H)$ and $y \in V(H)$ are nonadjacent. By the maximality of $P$, every neighbor of $x$ in $H$ is in $P$, similar for $y$. Hence $d_{P}(x)+d_{P}(y)=d_{H}(x)+d_{H}(y) \geq 2(t+1) \geq k$, and also $|P|=k-1$ (edges). By Kopylov' Theorem 4.3, $G^{*}$ must have a cycle of length at least $k$, a contradiction.
Therefore $H$ is a complete subgraph of $G^{*}$. Let $\ell=|V(H)|$. Because every vertex in $H$ has degree at least $t+1, \ell \geq t+2$. Furthermore, if $\ell \geq k-1$, then $G^{*}$ has a clique $K$ of size at least $k-1$. Because $G^{*}$ is 2-connected, we can extend a $(k-1)$-cycle of $K$ to include at least one vertex in $G^{*}-H^{\prime}$, giving us a cycle of length at least $k$. It follows that

$$
\begin{equation*}
t+2 \leq \ell \leq k-2, \tag{12}
\end{equation*}
$$

and therefore $k-\ell \leq t$. Apply a weaker $(k-\ell)$-disintegration to $G^{*}$, and denote by $H^{\prime}$ the resulting graph. By construction, $H \subseteq H^{\prime}$.

Case 1: There exists $v \in V\left(H^{\prime}\right)-V(H)$. Since $v \notin V(H)$, there exists a nonedge between a vertex in $H$ and a vertex in $H^{\prime}-H$. Pick a longest path $P$ with terminal vertices $x \in V\left(H^{\prime}\right)$ and $y \in V(H)$. Then $d_{P}(x)+d_{P}(y) \geq(k-\ell+1)+(\ell-1)=k$, and therefore $c\left(G^{*}\right) \geq k$.
Case 2: $H=H^{\prime}$. Then

$$
e\left(G^{*}\right) \leq\binom{\ell}{2}+(n-\ell)(k-\ell)=h(n, k, k-\ell)
$$

If $3 \leq(k-\ell) \leq t-1$, then $e(G) \leq \max \{h(n, k, 3), h(n, k, t-1)\}$, so by (12), $k-\ell=2$, and $H$ is the complete graph with $k-2$ vertices. Let $D=V\left(G^{*}\right)-V(H)$. If there is an edge $x y$ in $G^{*}[D]$, then because $G^{*}$ is 2 -connected, there exist two vertex-disjoint paths, $P_{1}$ and $P_{2}$, from $\{x, y\}$ to $H$ such that $P_{1}$ and $P_{2}$ only intersect $\{x, y\} \cup H$ at the beginning and end of the paths. Let $a$ and $b$ be the terminal vertices of $P_{1}$ and $P_{2}$ respectively that lie in $H$. Let $P$ be any $(a, b)$-hamiltonian path of $H$. Then $P_{1} \cup P \cup P_{2}+x y$ is a cycle of length at least $k$ in $G^{*}$, a contradiction.
Therefore $D$ is an independent set, and since $G^{*}$ is 2-connected, each vertex of $D$ has degree 2 . Suppose there exists $u, v \in D$ where $N(u) \neq N(v)$. Let $N(u)=\{a, b\}, N(v)=\{c, d\}$ where it is possible that $b=c$. Then we can find a cycle $C$ of $H$ that covers $V(H)$ which contains edges $a b$ and $c d$. Then $C-a b-c d+u a+u b+v c+v d$ is a cycle of length $k$ in $G^{*}$. Thus for every $v \in D$, $N(v)=\{a, b\}$ for some $a, b \in H$. I.e., $G^{*}=H_{n, k, 2}$, and thus $G \subseteq H_{n, k, 2}$.

### 5.3 Proof of Theorem 2.3 for all $n$

We use induction on $n$ with the base case $n \leq k+(t-1) / 2$. Suppose $n \geq k+t / 2$ and for all $k \leq n^{\prime}<n$, Theorem 2.3 holds. Let $G$ be a 2 -connected graph $G$ with $n$ vertices such that

$$
\begin{equation*}
e(G)>\max \{h(n, k, t-1), h(n, k, 3)\} \text { and } c(G)<k . \tag{13}
\end{equation*}
$$

Apply one step of (BP). If (BP4) was applied (so neither (BP2) nor (BP3) applies to $G$ ), then $G_{m}=G$ (with $G_{m}$ defined as in the previous case). By Lemmas 5.1, 4.6, and 3.2, the theorem holds.

Therefore we may assume that either (BP2) or (BP3) was applied. Let $G^{-}$be the resulting graph. Then $c\left(G^{-}\right)<k$, and $G^{-}$is 2-connected.

Claim 5.4.

$$
\begin{equation*}
e\left(G^{-}\right)>\max \left\{h\left(\left|V\left(G^{-}\right)\right|, k, t-1\right), h\left(\left|V\left(G^{-}\right)\right|, k, 3\right)\right\} . \tag{14}
\end{equation*}
$$

Proof. If (BP2) was applied, i.e., $G^{-}=G / u v$ for some edge $u v$, then

$$
e\left(G^{-}\right) \geq e(G)-(t-1)>h(n-1, k, t-1) \geq h(n-1, k, 3),
$$

so (14) holds. Therefore we may assume that (BP3) was applied to obtain $G^{-}$. Then $n \geq k+t-1$ and $e(G)-e\left(G^{-}\right)=\binom{t+1}{2}-1$. So by 13),

$$
\begin{equation*}
e\left(G^{-}\right)>h(n, k, t-1)-\binom{t+1}{2}+1 . \tag{15}
\end{equation*}
$$

The right hand side of (15) equals $h(n-(t-1), k, t-1)+t^{2} / 2-5 t / 2+2$ which is at least $h(n-(t-1), k, t-1)$ for $t \geq 4$, proving the first part of (14).
We now show that also $e\left(G^{-}\right)>h(n-(t-1), k, 3)$. Indeed, for $k=2 t+1$,

$$
\begin{gathered}
e\left(G^{-}\right)-h(n-(t-1), k, 3)>\binom{t+2}{2}+(t-1)(n-t-2)-\binom{t+1}{2}+1 \\
-\left[\binom{2 t-2}{2}+3(n-(t-1)-(2 t-2))\right] \geq 0 \text { when } n \geq 3 t .
\end{gathered}
$$

Similarly, for $k=2 t+2$,

$$
\begin{gathered}
e\left(G^{-}\right)-h(n-(t-1), k, 3)>\binom{t+3}{2}+(t-1)(n-t-3)-\binom{t+1}{2}+1 \\
-\left[\binom{2 t-1}{2}+3(n-(t-1)-(2 t-1))\right]>0 \text { when } n \geq 3 t+1 .
\end{gathered}
$$

Thus if $n \geq 3 t+1$, then (14) is proved. But if $n \in\{3 t-1,3 t\}$ then by Remark 5.1, no graph to which (BP3) applied may have more than $h(n, k, t-1)$ edges.

By (14), we may apply induction to $G^{-}$. So $G^{-}$satisfies either (a) $G^{-} \subseteq H_{\left|V\left(G^{-}\right)\right|, n, 2}$, or (b) $G^{-}$ is contained in a graph in $\mathcal{G}(n, k)-H_{\left|V\left(G^{-}\right)\right|, k, 2}$ and contains a subgraph $H \in \mathcal{F}(k)$. Suppose first
that $G^{-}$satisfies (b). If (BP3) was applied to obtain $G^{-}$from $G$, then because $G^{-}$contains a subgraph $H \in \mathcal{F}(k)$ and $G^{-} \subseteq G, G$ also contains $H$. If (BP2) was applied, then by Lemma 3.1, $G$ contains a subgraph $H^{\prime} \in \mathcal{F}(k)$. In either case, Lemma 3.2 implies that $G$ is a subgraph of a graph in $\mathcal{G}(n, k)-H_{n, k, 2}$.
So we may assume that (a) holds, that is, $G^{-}$is a subgraph of $H_{\left|V\left(G^{-}\right)\right| n, 2}$. Because $\delta\left(G^{-}\right) \leq 2$, $\delta(G) \leq 3$, and so $G$ has edges in at most $2 \leq t-2$ triangles. Therefore (BP2) was applied to obtain $G^{-}$, where $G / u v=G^{-}$. Let $D$ be an independent set of vertices of $G^{-}$of size $(n-1)-(k-2)$ with $N(D)=\{a, b\}$ for some $a, b \in V\left(G^{-}\right)$. Since $T_{G^{-}}(x a), T_{G^{-}}(x b) \leq 1$ for every $x \in D$, we have that $T_{G}(u v) \leq 2$ with equality only if $T(G)=2$ where $T(G)=\min _{x y \in E(G)} T_{G}(x y)$.
We want to show that $T_{G}(u v) \leq 1$. If not, suppose first that $u * v \in D \subseteq V\left(G^{-}\right)$. Then there exists $x \in D-u * v$, and $x$ and $u * v$ are not adjacent in $G^{-}$. Therefore $x$ was not in a triangle with $u$ and $v$ in $G$, and hence $T_{G}(x a)=T_{G^{-}}(x a) \leq 1$, so the edge $x a$ should have been contracted instead. Otherwise if $u * v \notin D$, at least one of $\{a, b\}$, say $a$, is not $u * v$. If $T(G)=2$, then for every $x \in D \subseteq V(G), T_{G}(x a)=2$, therefore each such edge $x a$ was in a triangle with $u v$ in $G$. Then $T_{G}(u v) \geq|D|=(n-1)-(k-2) \geq k+t / 2-1-k+2 \geq 3$, a contradiction.

Thus $T_{G}(u v) \leq 1$ and $e(G) \leq 2+e\left(G^{-}\right) \leq 2+h(n-1, k, 2)=h(n, k, 2)$. But for $n \geq k+t / 2$, we have $h(n, k, t-1) \geq h(n, k, 2)$, a contradiction.

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