ON THE NUMBER OF RAINBOW SPANNING TREES IN EDGE-COLORED COMPLETE GRAPHS

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ABSTRACT. A spanning tree of a properly edge-colored complete graph, K_n, is rainbow provided that each of its edges receives a distinct color. In 1996, Brualdi and Hollingsworth conjectured that if K_{2m} is properly $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m rainbow spanning trees except when $m = 2$. By means of an explicit, constructive approach, in this paper we construct $\lfloor \sqrt{6m+9}/3 \rfloor$ mutually edge-disjoint rainbow spanning trees for any positive value of m. Not only are the rainbow trees produced, but also some structure of each rainbow spanning tree is determined in the process. This improves upon best constructive result to date in the literature which produces exactly three rainbow trees.

1. INTRODUCTION

A spanning tree T of a connected graph G is an acyclic connected subgraph of G for which $V(T) = V(G)$. A proper k-edge-coloring of a graph G is a mapping from $E(G)$ into a set of colors, $\{1, 2, ..., k\}$, such that adjacent edges of G receive distinct colors. The chromatic index $\chi'(G)$ of a graph G is the minimum number k such that G is k-edge-colorable. It is well known that $\chi'(K_{2m}) = 2m - 1$ and thus, if K_{2m} is properly $(2m-1)$-edge-colored, each color appears at every vertex exactly once. All edge-colorings considered in this paper are proper.

A subgraph in an edge-colored graph is said to be rainbow (sometimes called multicolored or polychromatic) if all of its edges receive distinct colors. Observe that with any $(2m-1)$-edge-coloring of K_{2m}, it is not hard to find a rainbow spanning tree by taking the spanning star, S_v, with center $v \in V(K_{2m})$. Further, K_{2m} has $m(2m-1)$ edges and it is well known that these edges can be partitioned into m spanning trees. This led Brualdi and Hollingsworth [5] to make the following conjecture in 1996.

Conjecture 1 ([5]). *If K_{2m} is $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m rainbow spanning trees except when $m = 2$.*

Based on Brualdi and Hollingsworth's concept, the following related conjectures were proposed in 2002.

Conjecture 2 ([7], Constantine). *K_{2m} can be edge-colored with $2m - 1$ colors in such a way that the edges can be partitioned into m isomorphic rainbow spanning trees except when $m = 2$.*

Conjecture 2 was proved to be true by Akbari, Alipour, Fu, and Lo in 2006 [2].

Conjecture 3 ([7], Constantine). *If K_{2m} is $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m isomorphic rainbow spanning trees except when $m = 2$.*

Conjecture 4 ([12], Kaneko, Kano, Suzuki). *Every properly colored K_n contains $\lfloor \frac{n}{2} \rfloor$ edge-disjoint isomorphic rainbow spanning trees.*

Concerning Conjecture 1, in [5], Brualdi and Hollingsworth proved that every properly $(2m-1)$-edge-colored K_{2m} has two edge-disjoint rainbow spanning trees for $m > 2$, and in 2000, Krussel, Marshall, and Verrall [13] improved this result to three spanning trees. Kaneko, Kano, and Suzuki [12] then improved the previous result slightly by showing that three edge-disjoint rainbow spanning trees exist in any proper edge-coloring of K_{2m}. Recently, Horn [10] showed that for m sufficiently large there is an $\epsilon > 0$ such that every properly $(2m-1)$-edge-colored K_{2m} has $\epsilon 2m$ edge-disjoint rainbow spanning trees. And a recently submitted paper by Pokrovskiy and Sudakov [14] shows that every properly $(n-1)$-edge-colored K_n has $\frac{n}{9}$ edge-disjoint spanning rainbow trees.

Balogh, Liu, and Montgomery [4] have also recently submitted a result showing that every properly edge-colored K_n contains at least $\frac{n}{10^{12}}$ edge-disjoint rainbow spanning trees. They then use this result to show that any properly $(2m-1)$-edge-colored K_{2m} contains linearly many edge-disjoint rainbow spanning trees.

Relaxing the restriction that the coloring be proper, Akbari and Alipour [1] were able to show that two edge-disjoint rainbow spanning trees exist in any edge-coloring of K_{2m} with each color class containing $\leq m$ edges. With the same assumption that the coloring be not necessarily proper and each color appears on at most m edges, Carraher, Hartke, and Horn [6] showed that if m is sufficiently large ($m \geq 500,000$) then K_{2m} contains at least $\left\lfloor \frac{m}{500 \log(2m)} \right\rfloor$ edge-disjoint rainbow spanning trees.

Essentially not much was done on Conjecture 3 until recently. In 2015 Fu and Lo [8] proved that three isomorphic rainbow spanning trees exist in any $(2m-1)$-edge-colored K_{2m}, $m \geq 14$ and in 2017, Pokrovskiy and Sudakov [14] proved the existence of $10^{-6}n$ edge-disjoint rainbow spanning t-spiders in any properly edge-colored K_n, $0.0007n \leq t \leq 0.2n$. Note that a t-spider is a tree obtained from a star by subdividing t of its edges once.

In this paper, we focus on Conjecture 1 by proving that in any $(2m-1)$-edge-coloring of K_{2m}, $m \geq 1$, there exist at least $\left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor$ mutually edge-disjoint rainbow spanning trees. Asymptotically, this is not as good as the bound in [6] or [10], but our result applies to all values of m and it is better until m is extremely large (over 5.7×10^7 for the bound in [6]). Instead of using the probabilistic method to prove the result, as was used in [6] and [10], we derive our bound by means of an explicit, constructive approach. So, not only do we actually produce the rainbow trees, but also some structure of each rainbow spanning tree is determined in the process. It should be noted that the best constructive result (before ours) is the one in the paper by Krussel, Marshall, and Verrall [13] which produces just three rainbow spanning trees, though the recent result by Pokrovskiy and Sudakov is stronger.

Here is our main result.

Theorem 1. *Let K_{2m} be a properly $(2m-1)$-edge-colored graph. Then there exist $\Omega_m = \left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor$ mutually edge-disjoint rainbow spanning trees, say $T_1, T_2, \ldots, T_{\Omega_m}$, with the following properties.*

  (i)   *Each tree has a designated distinct root.*
  (ii) *The root of T_1 has degree $(2m-1) - 2(\Omega_m - 1)$ and has at least $(2m-1) - 4(\Omega_m - 1)$ adjacent leaves.*
  (iii) *For $2 \leq i \leq \Omega_m$, the root of T_i has degree $(2m-1) - i - 2(\Omega_m - i)$ and has at least $(2m-1) - 2i - 4(\Omega_m - i)$ adjacent leaves.*

It is worth mentioning here that the above conjectures will play important roles in certain applications if they are true. Notice that a rainbow spanning tree is orthogonal to the 1-factorization of K_{2m} (induced by any $(2m-1)$-edge-coloring). An application of parallelisms of complete designs to population genetics data can be found in [3]. Parallelisms are also useful in partitioning consecutive positive integers into sets of equal size with equal power sums [11]. In addition, the discussions of applying colored matchings and design parallelisms to parallel computing appeared in [9].

2. Proof of Theorem 1

We will use induction on the number of trees to prove this result. We can assume $m \geq 5$ since for $1 \leq m \leq 4$, $\Omega_m = 1$ and the spanning star, S_r, in which $r \in V(K_{2m})$ and r is joined to every other vertex, is clearly a rainbow spanning tree of K_{2m}. When the value of m is clear, it will cause no confusion to simply refer to Ω_m as Ω. It is worth noting that the following inductive proof can be used as a recursive construction to create Ω rainbow edge-disjoint spanning trees, $T_1, T_2, ..., T_{\Omega}$.

For $1 \leq \psi \leq \Omega$, the rainbow edge-disjoint spanning trees, $T_1, T_2, ..., T_{\psi}$, are constructed to satisfy the proposition $f(\psi)$, defined to be the conjunction of the following three degree and structural characteristics:

Each tree has a designated distinct root. $\hfill (1)$

The root of T_1 has degree $(2m - 1) - 2(\psi - 1)$ and has at least $(2m - 1) - 4(\psi - 1)$ adjacent leaves. $\hfill (2)$

For $2 \leq i \leq \psi$, The root of T_i has degree $(2m - 1) - i - 2(\psi - i)$ and has at least $(2m - 1) - 2i - 4(\psi - i)$ adjacent leaves. $\hfill (3)$

In particular, note here that by (3), if $\psi > 1$, then the root of T_2 has degree $(2m-1)-2-2(\psi-2) = (2m - 1) - 2(\psi - 1)$ and at least $(2m - 1) - 4 - 4(\psi - 2) = (2m - 1) - 4(\psi - 1)$ adjacent leaves, sharing these characteristics with T_1 (as stated in (2)).

We begin with some necessary notation. All vertices defined in what follows are in $V(K_{2m})$, the given edge-colored complete graph.

The proof proceeds inductively, producing a list of j edge-disjoint rainbow spanning trees from a list of $j - 1$ edge-disjoint rainbow spanning trees; so for $1 \leq i \leq j \leq \Omega$, let T_i^j be the ith rainbow spanning tree of the jth induction step and let r_i be the designated root of T_i^j. Notice that r_i is independent of j.

Suppose T is any spanning tree of the complete graph K_{2m} with root r containing vertices $y, v, w,$ and v', where ry and rv are distinct pendant edges in T (so $y \neq v$ are leaves of T) and $y, v \notin \{w, v'\}$ (note that w could equal v'). Then define $T' = T[r; y, v; w, v']$ to be the new graph formed from T with edges ry and rv removed and edges yw and vv' added. Formally, $T' = T[r; y, v; w, v'] = T - ry - rv + yw + vv'$. We note here that T' is also a spanning tree of K_{2m} because y and v are leaves in T, and thus adding edges yw and vv' does not create a cycle in T'.

Our inductive strategy will be to assume we have $k - 1$ (where $1 < k \leq \Omega$) edge-disjoint rainbow spanning trees with suitable characteristics satisfying proposition $f(k - 1)$ that yield properties (1), (2), and (3) with $\psi = k - 1$. From those trees we will construct k edge-disjoint rainbow spanning trees with suitable characteristics that allow properties (1), (2), and (3) to be eventually established when $\psi = k$, thus satisfying $f(k)$.

For this construction, given any T_i^{j-1} with root r_i and distinct pendant edges $r_i y_i^j$ and $r_i v_i^j$, we define T_i^j in the following way:

$$ T_i^j = T_i^{j-1}[r_i; y_i^j, v_i^j; w_i^j, (v')_i^j] = T_i^{j-1} - r_i y_i^j - r_i v_i^j + y_i^j w_i^j + v_i^j (v')_i^j \hfill (4) $$

The choice of the vertices defined in (4) will eventually be made precise, based on the discussion which follows.

When the value of j is clear, it will cause no confusion to refer to the vertices $y_i^j, v_i^j; w_i^j, (v')_i^j$ by omitting the superscript and instead writing $T_i^j = T_i^{j-1}[r_i; y_i, v_i; w_i, v_i']$. We now make the following remarks about the definition of T_i^j above. Recall that for $1 \leq i \leq j \leq \Omega$, r_i is independent of j, and thus is the root of both T_i^{j-1} and T_i^j. The following is easily seen to be true.

If φ is any proper edge-coloring of K_{2m} and T_i^{j-1} is a rainbow spanning tree of K_{2m} with root r_i and distinct pendant edges $r_i y_i$ and $r_i v_i$, then T_i^j as defined in (4) is also   (5)
a rainbow spanning tree of K_{2m} if $\varphi(r_i y_i) = \varphi(v_i v_i')$ and $\varphi(r_i v_i) = \varphi(y_i w_i)$.

Next, for $1 \le i \le j \le \Omega$, let $L_i^j = \{x \mid x r_i$ is a pendant edge in $T_i^j\}$ (so x is a leaf adjacent to r_i in T_i^j). Define

$$L_j = \bigcap_{i=1}^{j} L_i^j. \tag{6}$$

Notice that if $x \in L_j$, then for $1 \le i \le j$, $x r_i$ is a pendant edge in T_i^j .

We now begin our inductive proof with induction parameter k. Specifically we will prove that for $1 \le k \le \Omega$ there exist k edge-disjoint rainbow spanning trees, $T_1^k, T_2^k, ..., T_k^k$ satisfying the inductive parameter $f(k)$ (stated below as $f(k-1)$ in the inductive step).

Base Step. The case $k = 1$ is seen to be true for all properly edge-colored complete graphs, K_{2m}, by letting r_1 be any vertex in $V(K_{2m})$ and defining $T_1^1 = S_{r_1}$, the spanning star with root r_1. It is also clear that S_{r_1} satisfies $f(1)$ since r_1 has degree $2m - 1$ and has $2m - 1$ adjacent leaves, as required in (2). Property (3) is vacuously true.

Induction Step. Suppose that φ is a proper edge-coloring of K_{2m} and that for some k with $1 < k \le \Omega$, K_{2m} contains $k-1$ edge-disjoint rainbow spanning trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$, satisfying $f(k-1)$:

(1) r_i is the root of tree T_i^{k-1} and $r_i \ne r_c$ for $1 \le i, c < k$, $i \ne c$,
(2) $d_{T_1^{k-1}}(r_1) = (2m - 1) - 2(k - 2)$ and r_1 is adjacent to at least $(2m - 1) - 4(k - 2)$ leaves in T_1^{k-1}, and
(3) For $2 \le i \le k - 1$, $d_{T_i^{k-1}}(r_i) = (2m - 1) - i - 2(k - 1 - i)$ and r_i is adjacent to at least $(2m - 1) - 2i - 4(k - 1 - i)$ leaves in T_i^k.

It thus remains to construct k edge-disjoint rainbow spanning trees satisfying $f(k)$.

We note here that $f(k-1)$ and the definition of L_{k-1} in (6) guarantee that a lower bound for $|L_{k-1}|$ can be obtained by starting with a set containing all $2m$ vertices, then removing the $k - 1$ roots of $T_1^{k-1}, T_2^{k-2}, ..., T_{k-1}^{k-1}$, the (at most $4(k - 2)$) vertices in $V(T_1^{k-1} \backslash \{r_1\})$ which are not leaves adjacent to r_1, and for $2 \le i < k$, the (at most $2i + 4(k - 1 - i)$) vertices in $V(T_i^{k-1} \backslash \{r_i\})$ which are not leaves adjacent to r_i. Formally,

$$
\begin{aligned}
|L_{k-1}| &\ge 2m - (k - 1) - 4(k - 2) - \sum_{i=2}^{k-1}(2i + 4(k - 1 - i)) \\
&= 2m - (k - 1) - 4(k - 2) - (3k^2 - 11k + 10) \\
&= 2m - 3k^2 + 6k - 1.
\end{aligned} \tag{7}
$$

Knowing $|L_{k-1}|$ is useful because later (see (15)) we will show that if $|L_{k-1}| > 6k - 5$, then from $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ we can construct k rainbow edge-disjoint spanning trees which satisfy proposition $f(k)$. As the reader might expect, it is from here that the bound on Ω is obtained: it actually follows that since $k \le \Omega$, $|L_{k-1}| > 6k - 5$.

First, select any two distinct vertices $r_k, w_k^k \in L_{k-1}$; since it will cause no confusion, we will write w_k for w_k^k. Set r_k equal to the root of the kth tree, T_k^k. Later, $r_k w_k$ will be an edge removed from

T_k^k. For now, the two special vertices r_k and w_k play a role in the construction of T_i^k from T_i^{k-1} for $1 \le i < k$. For convenience we explicitly state and observe the following

$$\text{Since } r_k \text{ and } w_k \text{ are distinct vertices in } L_{k-1} \text{ (defined in (6)), } r_k \text{ and} \tag{8}$$
$$w_k \text{ are leaves adjacent to } r_i \text{ for } 1 \le i < k.$$

For the sake of clarity, having selected r_k and w_k, we now discuss how to construct the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ before returning to our discussion of the construction of T_k^k (though in actuality T_k^k is formed recursively as we are constructing $T_1^k, T_2^k, ..., T_{k-1}^k$).

For $1 \le i < k$, we will find suitable vertices v_i^k, w_i^k, and $v_i^{k'}$, which for convenience we refer to as v_i, w_i, and v_i' respectively, and define T_i^k in the following way:

$$T_i^k = T_i^{k-1}[r_i; r_k, v_i; w_i, v_i'] \tag{9}$$
$$\text{where } \varphi(r_i r_k) = \varphi(v_i v_i') \text{ and } \varphi(r_i v_i) = \varphi(r_k w_i)$$

It is clear by (5) that for $1 \le i < k$, since T_i^{k-1} is a rainbow spanning tree of K_{2m}, if v_i is chosen so that $v_i r_i$ is a pendant edge in T_i^{k-1} with $v_i \ne r_k$, then T_i^k is also a rainbow spanning tree of K_{2m} (recall from (8) that $r_k \in L_{k-1}$, so by (6) $r_k r_i$ is a pendant edge in T_i^{k-1}).

$$\text{Further, since } r_k, w_k \in L_{k-1}, \text{ it is clear from (9) that (1) } r_k, v_i \notin L_k,$$
$$\text{and (2) all leaves adjacent to } r_i \text{ in } T_i^k \text{ are leaves adjacent to } r_i \text{ in } T_i^{k-1}. \tag{10}$$
$$\text{Therefore } |L_k| < |L_{k-1}|.$$

By the induction hypothesis, the trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ satisfy $f(k-1)$. We now show that the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ satisfy properties (1), (2), and (3) of $f(k)$. We will construct a T_k^k below in (12) and together, $T_1^k, ..., T_{k-1}^k, T_k^k$ will be a collection of trees satisfying $f(k)$.

First, clearly (1) is satisfied. Further, for $1 \le i < k$, when T_i^k is formed from T_i^{k-1} (see (9)), it can easily be seen that the degree of r_i is decreased by 2 and the number of leaves adjacent to r_i is decreased by at most 4.

(i.) T_1^k

By our induction hypothesis, we have that $d_{T_1^{k-1}}(r_1) = (2m-1) - 2(k-2)$ and that r_1 is adjacent to at least $(2m-1) - 4(k-2)$ leaves in T_1^{k-1}. From (9) we have that $d_{T_1^k}(r_1) = d_{T_1^{k-1}}(r_1) - 2 = (2m-1) - 2(k-2) - 2 = (2m-1) - 2(k-1)$ and that r_1 is adjacent to at least $(2m-1) - 4(k-2) - 4 = (2m-1) - 4(k-1)$ leaves in T_1^k. So (2) of $f(k)$ is satisfied.

(ii.) $T_i^k, 2 \le i < k$

By our induction hypothesis, we have that $d_{T_i^{k-1}}(r_i) = (2m-1) - i - 2(k-1-i)$ and that r_i is adjacent to at least $(2m-1) - 2i - 4(k-1-i)$ leaves in T_i^k. From (9) we have that $d_{T_i^k}(r_i) = d_{T_i^{k-1}}(r_i) - 2 = (2m-1) - i - 2(k-1-i) - 2 = (2m-1) - i - 2(k-i)$ and that r_i is adjacent to at least $(2m-1) - 2i - 4(k-1-i) - 4 = (2m-1) - 2i - 4(k-i)$ leaves in T_i^k. So (3) of $f(k)$ is satisfied when $2 \le i < k$.

Lastly, we can observe that once v_i is selected, vertices w_i and v_i' are determined by the required property from (9) that $\varphi(r_i r_k) = \varphi(v_i v_i')$ and $\varphi(r_i v_i) = \varphi(r_k w_i)$.

It remains to ensure that the trees, $T_1^k, T_2^k, ..., T_{k-1}^k$, are all edge-disjoint. This is also proved using the induction hypothesis that $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge-disjoint.

Now, while forming the rainbow edge-disjoint spanning trees, $T_1^k, T_2^k, ..., T_{k-1}^k$, we simultaneously construct the kth rainbow spanning tree, T_k^k, from a sequence of inductively defined graphs, $T_k^k(1), T_k^k(2), ..., T_k^k(k) = T_k^k$ where $T_k^k(1) = S_{r_k}$ and at the ith induction step, the formation of $T_k^k(i)$ depends on the choice of v_i used in the construction of T_i^k. For $2 \le i \le k$ define

$$T_k^k(i) = T_k^k(i-1) - r_k w_i + w_i w_i',$$
$$\text{where } \varphi(w_1 w_1') = \varphi(r_k w_k) \text{ and } \varphi(w_i w_i') = \varphi(r_k w_{i-1}) \text{ for } 2 \leq i \leq k. \tag{11}$$

Note that for $1 \leq i \leq k-1$, the choice of v_i determines $T_k^k(i)$; the formation of $T_k^k(k)$ is dictated by $T_k^k(k-1)$ since w_k' is determined by requiring that $\varphi(w_k w_k') = \varphi(r_k w_{k-1})$. It is worth explicitly stating that

$$T_k^k = T_k^k(k) = S_{r_k} - r_k w_1 - ... - r_k w_k + w_1 w_1' + ... + w_k w_k',$$
$$\text{where } \varphi(w_1 w_1') = \varphi(r_k w_k) \text{ and } \varphi(w_c w_c') = \varphi(r_k w_{c-1}) \text{ for } 2 \leq c \leq k \tag{12}$$

Observe that T_k^k is a rainbow graph since each edge removed from S_{r_k} is replaced by a corresponding edge of the same color. Also, one can easily see that T_k^k has $2m-1$ edges; $d_{T_k^k}(r_k) = (2m-1)-k$ since $r_k \notin \{w_1', w_2', ..., w_k'\}$; and r_k has at least $(2m-1) - 2k$ adjacent leaves. Therefore, condition (3) of $f(k)$ is satisfied. So it remains to show that T_k^k is acyclic and contains no edges in the trees T_i^k for $1 \leq i \leq k-1$.

For future reference, it is worth gathering two observations just made into one:

For $1 \leq i < k$, once v_i is chosen, T_i^k and $T_k^k(i)$ are completely determined by the constructions described in (9) and (11) respectively. $\tag{13}$

Due to the fact highlighted above in (13), our strategy will be to select a suitable v_i and construct T_i^k from T_i^{k-1}, while simultaneously constructing $T_k^k(i)$ from $T_k^k(i-1)$. In doing so, we restrict the choices for each v_i in order to achieve the following three properties:

(C1) The edges in $T_a^k, 1 \leq a < i$ do not appear in T_i^k,
(C2) The edges in T_k^k do not appear in T_i^k, $1 \leq i < k$, and
(C3) T_k^k is acyclic

To that end, we let

$$L_{k-1}^* = L_{k-1} \backslash \{r_k, w_k\} \tag{14}$$

and let v_i be any vertex for which the following properties are satisfied (so by (13), this choice completes the formation of T_i^k and $T_k^k(i)$ for $1 \leq i < k$):

(R1) $v_i \in L_{k-1}^*$,
(R2) For $1 \leq c < k$, $c \neq i$, $\varphi(v_i r_c) \neq \varphi(r_i r_k)$,
(R3) For $1 \leq a < i$, $\varphi(v_i r_i) \neq \varphi(r_a v_a)$,
(R4) For $i < b < k$, $\varphi(v_i r_i) \neq \varphi(r_k r_b)$,
(R5) $\varphi(v_i r_i) \neq \varphi(r_k w_k)$,
(R6) For $1 \leq a < i$, $\varphi(v_i r_i) \neq \varphi(r_k w_a')$,
(R7) For $2 \leq i < k$, $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$,
     where α is the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$,
(R8) For $i = 1$ and for $1 \leq c < k$, $\varphi(v_1 r_1) \neq \varphi(r_k \alpha)$,
     for each vertex α incident with the edge of color $\varphi(r_k w_k)$ in T_c^{k-1},
(R9) For $2 \leq i < k$, $1 \leq a < i$, and for $i \leq b < k$, $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$,
     for each vertex α incident with the edge of color $\varphi(r_k w_{i-1})$ in T_a^k and in T_b^{k-1},
(R10) For $1 \leq i < k$, $\varphi(v_i w_k) \neq \varphi(r_i r_k)$,
(R11) For $1 \leq d \leq k-2$, $\varphi(v_{k-1} r_{k-1}) \neq \varphi(w_k r_d)$.

From the observation in (7), we know that $\left| L_{k-1}^* \right| \geq 2m - 3k^2 + 6k - 3$.

As i increases, the number of vertices eliminated in each item increases or is constant, except for (R4) and (R8). However, we observe here that the number eliminated in items (R3) and (R4) is $(i-1)+(k-i-1) = (k-2)$, a constant, and the number eliminated in (R8) and (R9) is $2i$, which is maximized when $i = k-1$. Therefore, an upper bound for the number of vertices eliminated through

items (R2) - (R11) as candidates for v_i is achieved when $i = k - 1$. When $i = k - 1$, the number of vertices eliminated by (R2), (R3), ..., (R11) is $(k-2), (k-2), 0, 1, (k-2), 1, 0, 2(k-1), 1, (k-2)$ respectively, the sum of which is $6k - 7$. Now, since the induction hypothesis includes the condition $k \leq \Omega$, we can observe the following.

First, from $f(\Omega)$ and the definition of $L_{\Omega-1}$, we can follow the same steps as we did in (7) to see that $|L_{\Omega-1}| \geq 2m - 3\Omega^2 + 6\Omega - 1$ and further, that $|L_{\Omega-1}^*| \geq 2m - 3\Omega^2 + 6\Omega - 3$. Now, since by the induction hypothesis $k \leq \Omega$ and by (10) and (14), $|L_{i-1}^*| > |L_i^*|$ for $2 \leq i \leq k - 1$, we have the following by our choice of Ω:

$$|L_{k-1}^*| \geq |L_{\Omega-1}^*| \geq 2m - 3\Omega^2 + 6\Omega - 3 > 6\Omega - 7 \geq 6k - 7. \tag{15}$$

Therefore, since $|L_{k-1}^*| > 6k - 7$, such a vertex v_i meeting the restrictions in (R1) - (R11) exists. The following cases show that this choice of v_i ensures that (C1), (C2), and (C3) hold.

2.1. Case 1. (C1) Edges in $T_a^k, 1 \leq a < i$ do not appear in T_i^k.

First, by the induction hypothesis we know that the trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all rainbow edge-disjoint and spanning. Inductively, we also assume for some i with $2 \leq i < k$ the trees $T_1^k, T_2^k, ..., T_{i-1}^k$ are edge-disjoint rainbow spanning trees as well. By (9), regardless of the choice of v_i, the only edges in T_i^k ($1 \leq i < k$) that are not in T_i^{k-1} are $v_i v_i'$ and $r_k w_i$. Thus, if we can prove that the edges in $(E(T_i^{k-1}) \backslash \{r_i v_i, r_i r_k\}) \cup \{v_i v_i', r_k w_i\}$ are not in T_a^k, $1 \leq a < i$, we will have shown that the trees $T_1^k, T_2^k, ..., T_i^k$ are all edge-disjoint rainbow and spanning; so by induction, $T_1^k, T_2^k, ..., T_{k-1}^k$ are edge-disjoint rainbow spanning trees.

To that end, for the remainder of Case 1 suppose that $2 \leq i < k$, $1 \leq a < i$, and $i < b < k$ and define the following sets of edges.

(1) $E_{old}(T_a^k) = \{xy \mid xy \in E(T_a^{k-1}) \cap E(T_a^k)\}$
(2) $E_{new}(T_a^k) = E(T_a^k) \backslash E(T_a^{k-1}) = \{v_a v_a', r_k w_a\}$
(3) $E_{old}(T_i^k) = \{xy \mid xy \in E(T_i^{k-1}) \cap E(T_i^k)\}$
(4) $E_{new}(T_i^k) = E(T_i^k) \backslash E(T_i^{k-1}) = \{v_i v_i', r_k w_i\}$

Observe that by (9), $E_{old}(T_a^k) \cap E_{new}(T_a^k) = \emptyset$ and $E(T_a^k) = E_{old}(T_a^k) \cup E_{new}(T_a^k)$. Similarly, $E_{old}(T_i^k) \cap E_{new}(T_i^k) = \emptyset$ and $E(T_i^k) = E_{old}(T_i^k) \cup E_{new}(T_i^k)$.

Since the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ are formed sequentially, it is clearly necessary to prohibit edges $v_i v_i'$ and $r_k w_i$ from appearing in T_a^k. It is also very useful to prohibit edges $v_i v_i'$ and $r_k w_i$ from appearing in T_b^{k-1}.

Consequently, when v_i was selected to satisfy (R1) - (R11) it was done in such a way that ensures $E_{new}(T_i^k) \cap (E(T_a^k) \cup E(T_b^{k-1})) = \emptyset$ and $E_{old}(T_i^k) \cap (E(T_a^k) \cup E(T_b^{k-1})) = \emptyset$. To prove this, six cases are considered:

(P1) $v_i v_i', r_k w_i \notin E_{old}(T_a^k)$,
(P2) $v_i v_i', r_k w_i \notin E_{new}(T_a^k)$,
(P3) $v_i v_i', r_k w_i \notin E(T_b^{k-1})$,
(P4) $E_{old}(T_i^k) \cap E_{old}(T_a^k) = \emptyset$,
(P5) $E_{old}(T_i^k) \cap E_{new}(T_a^k) = \emptyset$,
(P6) $E_{old}(T_i^k) \cap E(T_b^{k-1}) = \emptyset$.

It is clear that if properties (P1) - (P6) are satisfied, then T_i^k is edge-disjoint from the trees, T_a^k and T_b^{k-1}. We consider edges $v_i v_i'$ and $r_k w_i$ in turn for properties (P1) - (P3), then address properties (P4) - (P6).

2.1.1. Properties (P1) and (P3) for $v_i v_i'$.

Since $E_{old}(T_a^k) \subset E(T_a^{k-1})$, we can prove $v_i v_i'$ is not an edge in $E_{old}(T_a^k)$ or T_b^{k-1} by showing that $v_i v_i' \notin E(T_c^{k-1})$ for $1 \leq c < k$, $c \neq i$.

Recall from (R1) and (14) that because $v_i \in L_{k-1}^*$, v_i is a leaf adjacent to the root r_c in T_c^{k-1}. Therefore, to show that $v_i v_i' \notin E(T_c^{k-1})$, we need only prove that $v_i' \neq r_c$. The following argument shows that (R2) guarantees this property.

Suppose to the contrary that $v_i' = r_c$. Then $v_i v_i' = v_i r_c$ and by (9), $\varphi(v_i r_c) = \varphi(v_i v_i') = \varphi(r_i r_k)$, contradicting (R2). It follows that $v_i' \neq r_c$ so $v_i v_i' \notin E(T_c^{k-1})$, as required.

2.1.2. *Property (P2) for* $v_i v_i'$.
Recall that $E_{new}(T_a^k) = \{v_a v_a', r_k w_a\}$. Thus, to prove that $v_i v_i' \notin E_{new}(T_a^k)$ for $1 \leq a < i$, we need only show that $v_i v_i' \neq v_a v_a'$ and $v_i v_i' \neq r_k w_a$. We consider each in turn.

(i.) $v_i v_i' \neq v_a v_a'$
By (9), we have that $\varphi(v_i v_i') = \varphi(r_i r_k)$ and $\varphi(v_a v_a') = \varphi(r_a r_k)$. But, by property (1) of $f(\psi)$ when $\psi = k - 1$ we know $r_i \neq r_a$ and so $\varphi(r_i r_k) \neq \varphi(r_a r_k)$. It follows that $\varphi(v_i v_i') \neq \varphi(v_a v_a')$ and, therefore, $v_i v_i' \neq v_a v_a'$.

(ii.) $v_i v_i' \neq r_k w_a$
Assume that $v_i v_i' = r_k w_a$ and recall from (14) that because $v_i \in L_{k-1}^*$, $v_i \neq r_k$. Therefore, $v_i = w_a$. By (9), $\varphi(v_i v_i') = \varphi(r_k r_i)$, so since we are assuming that $v_i v_i' = r_k w_a$, clearly $\varphi(r_k r_i) = \varphi(r_k w_a)$ and so $w_a = r_i = v_i$. But because $v_i \in L_{k-1}^*$, $v_i \neq r_i$ and this is a contradiction.

Combining the above two arguments, it is clear that $v_i v_i' \notin E_{new}(T_a^k)$, as required.

2.1.3. *Property (P1) for* $r_k w_i$.
Recall from (6) that $r_k \in L_{k-1}$, so $r_k r_a$ is a pendant edge in T_a^{k-1} with leaf r_k. Therefore, from (9) it is clear that $r_k r_a \notin E(T_a^k)$ since it is removed from T_a^{k-1} in forming T_a^k. So r_k is not incident with any edges in $E_{old}(T_a^k)$ and thus, $r_k w_i$ cannot be an edge in $E_{old}(T_a^k)$, as required.

2.1.4. *Property (P2) for* $r_k w_i$.
Recall that $E_{new}(T_a^k) = \{v_a v_a', r_k w_a\}$. To show that $r_k w_k \notin E_{new}(T_a^k)$, we prove that $r_k w_i \neq r_k w_a$ and $r_k w_i \neq v_a v_a'$ for $1 \leq a < i$. We consider each in turn.

(i.) $r_k w_i \neq r_k w_a$
To show that $r_k w_i \neq r_k w_a$, we need only show that $w_i \neq w_a$.
By (9) we have that $\varphi(r_k w_i) = \varphi(r_i v_i)$ and $\varphi(r_k w_a) = \varphi(r_a v_a)$. So if $r_k w_i = r_k w_a$, then $\varphi(v_i r_i) = \varphi(r_a v_a)$, contradicting (R3). Therefore, $r_k w_i \neq r_k w_a$, as required.

(ii.) $r_k w_i \neq v_a v_a'$
Assume that $r_k w_i = v_a v_a'$. Recall from (14) that because $v_a \in L_{k-1}^*$, $v_a \neq r_k$. Therefore, $v_a = w_i$. By (9), $\varphi(v_a v_a') = \varphi(r_a r_k)$, so since we are assuming that $r_k w_i = v_a v_a'$, then $\varphi(r_k w_i) = \varphi(r_k r_a)$ and it follows that $r_a = w_i = v_a$. But this is a contradiction because $v_a \in L_{k-1}^*$ so by (14), $v_a \neq r_a$.

Combining the above two arguments, it is clear that $r_k w_i \notin E_{new}(T_a^k)$, as required.

2.1.5. *Property (P3) for* $r_k w_i$.
Recall that by (8), because r_k was chosen to be in L_{k-1}, r_k is a leaf adjacent to the root of T_b^{k-1}, $i < b < k$. Thus, to show $r_k w_i \notin E(T_b^{k-1})$, we need only prove that $w_i \neq r_b$.
By (9), we have that $\varphi(r_k w_i) = \varphi(v_i r_i)$. So if $w_i = r_b$, then $r_k w_i = r_k r_b$ and $\varphi(v_i r_i) = \varphi(r_k r_b)$, contradicting (R4). Therefore, $r_k w_i \notin E(T_b^{k-1})$, as required.

2.1.6. *Properties (P4), (P5), and (P6).*
We consider each property, (P4), (P5), and (P6), in turn.

(i.) Property (P4)
By our induction hypothesis, the trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge disjoint. So (P4) follows because $E_{old}(T_i^k) \subset E(T_i^{k-1})$ and $E_{old}(T_a^k) \subset E(T_a^{k-1})$.

(ii.) Property (P5)

Since $a < i$, from (P3) (replacing i with a), it follows that $\{v_a v_a', r_k w_a\} \cap E(T_c^{k-1}) = \emptyset$, for $a < c < k$. In particular, since $i > a$, it follows that $E_{new}(T_a^k) \cap E(T_i^{k-1}) = \emptyset$. And lastly, since $E_{old}(T_i^k) \subset E(T_i^{k-1})$, we have that $E_{old}(T_i^k) \cap E_{new}(T_a^k) = \emptyset$.

(iii.) Property (P6)

Again, by our induction hypothesis, the trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge-disjoint. It follows that $E_{old}(T_i^k) \cap E(T_b^{k-1}) = \emptyset$ because $E_{old}(T_i^k) \subset E(T_i^{k-1})$.

Therefore, properties (P4) - (P6) hold for $E_{old}(T_i^k)$.

The above Sections $2.1.1 - 2.1.6$ ensure that properties (P1) - (P6) hold. As stated above, since these six properties hold, the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ are all edge-disjoint and further, from (9), are also rainbow and spanning.

2.2. Case 2. (C2) Edges in T_k^k do not appear in T_i^k.

Recall from (11) that T_k^k is defined by a sequence, $T_k^k(1), T_k^k(2), ..., T_k^k(k)$, and from (13) that at the ith induction step, $T_k^k(i)$ was determined by the choice of v_i. For the remainder of Case 2, suppose that $1 \le i < k$, $1 \le a < i$, and $i < b < k$.

In order to prevent edges in T_k^k from also appearing in T_i^k, we will now show that T_k^k has been constructed in such a way that $T_k^k(i)$ and T_k^k satisfy the following properties:

(P7) $E(T_k^k(i)) \cap E(T_a^k) = \emptyset$

(P8) $E(T_k^k(i)) \cap E(T_b^{k-1}) = \{r_k r_b\}$

(P9) $E(T_k^k(i)) \cap E_{old}(T_i^k) = \emptyset$

(P10) $E(T_k^k(i)) \cap E_{new}(T_i^k) = \emptyset$

(P11) $w_k w_k' \notin E(T_i^k)$

We note here that by (9), when T_b^k was constructed from T_b^{k-1}, edge $r_k r_b$ was removed, so it does not appear in T_b^k. Therefore, it is not necessary to prevent $r_k r_b$ from being an edge in $T_k^k(i)$ nor T_k^k.

Proving the above five properties will be done inductively. We show in the base step that $T_k^k(1)$ satisfies properties (P7) - (P10) with $i = 1$, and then show that for $2 \le i < k$, $T_k^k(i)$ satisfies the same four properties before finally proving property (P11).

The following preliminary result will be useful in proving properties (P7) - (P11).

2.2.1. Preliminary Result: $w_i \ne w_k$.

Recall from (8) that $w_k \in L_{k-1}$ was selected with r_k before any of the rainbow spanning trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ were revised. It will be useful to show that the vertices $w_i \in T_i^k$, $1 \le i < k$, cannot equal w_k.

From (9), we have that $\varphi(v_i r_i) = \varphi(r_k w_i)$. So if $w_i = w_k$, then $\varphi(v_i r_i) = \varphi(r_k w_k)$ contradicting (R5). Therefore, $w_i \ne w_k$.

2.2.2. Base Step: $i = 1$.

Observe that for $2 \le b < k$, $E(S_{r_k}) \cap E(T_b^{k-1}) = \{r_k r_b\}$ and $E(S_{r_k}) \cap E_{old}(T_1^k) = \emptyset$ since by (9), $r_k r_1$ is removed from T_1^{k-1} when forming T_1^k. Further, it is clear from (11) that the only edge in $T_k^k(1)$ that is not in S_{r_k} is $w_1 w_1'$.

(i.) (P7)

Since $i = 1$, there do not exist any such trees T_a^k since $1 \le a < i$ and so property (P7) is vacuously true.

(ii.) (P8) and (P9)

First, recall that $E_{old}(T_1^k) \subset E(T_1^{k-1})$. To establish properties (P8) and (P9), we show that $w_1 w_1' \notin E(T_c^{k-1})$ for $1 \le c < k$.

Suppose to the contrary that $w_1 w_1' \in E(T_c^{k-1})$. Recall from (11) that $\varphi(w_1 w_1') = \varphi(r_k w_k)$. So if $w_1 w_1' \in E(T_c^{k-1})$, then w_1 is a vertex incident to the edge of color $\varphi(r_k w_k)$ in T_c^{k-1}. But this is impossible since from (9) we have that $\varphi(v_1 r_1) = \varphi(r_k w_1)$ and from (R8) that $\varphi(v_1 r_1) \neq \varphi(r_k \alpha)$, where α is a vertex incident to the edge of color $\varphi(r_k w_k)$ in T_c^{k-1}. Therefore, $w_1 w_1' \notin E(T_c^{k-1})$ and $T_k^k(1)$ satisfies properties (P8) and (P9).

(iii.) (P10)

Recall that $E_{new}(T_i^k) = \{v_i v_i', r_k w_i\}$. To establish (P10) for $T_k^k(1)$, we need only show that $w_1 w_1' \neq v_1 v_1'$ and $w_1 w_1' \neq r_k w_1$. We consider each in turn.

(a.) $w_1 w_1' \neq v_1 v_1'$

Recall from (9) that $\varphi(v_1 v_1') = \varphi(r_k r_1)$ and from (11) that $\varphi(w_1 w_1') = \varphi(r_k w_k)$. So if $w_1 w_1' = v_1 v_1'$, then $\varphi(r_k w_k) = \varphi(r_k r_1)$ and so $w_k = r_1$. But this is not possible because by (8) $w_k \in L_{k-1}$ and so $w_k \neq r_1$. Therefore, $w_1 w_1' \neq v_1 v_1'$.

(b.) $w_1 w_1' \neq r_k w_1$

Recall from (11) that $\varphi(w_1 w_1') = \varphi(r_k w_k)$. So if $w_1 w_1' = r_k w_1$, then $\varphi(r_k w_k) = \varphi(r_k w_1)$ and so $w_k = w_1$, contradicting the result in Section 2.2.1. Thus, $w_1 w_1' \neq r_k w_1$.

Therefore, property (P10) holds for $T_k^k(1)$ and we have established our base step.

2.2.3. Property (P7) for $2 \leq i < k$.

From (11), it is clear that the only edge in $T_k^k(i)$ that differs from $T_k^k(i-1)$ is $w_i w_i'$. Therefore, since by induction we have that $T_k^k(i-1)$ satisfies (P7), in order to prove property (P7) is satisfied for $T_k^k(i)$, we need only show that $w_i w_i'$ is not an edge in T_a^k, $1 \leq a < i$.

To that end, suppose to the contrary that $w_i w_i' \in E(T_a^k)$. Recall from (11) that $\varphi(w_i w_i') = \varphi(r_k w_{i-1})$. So if $w_i w_i' \in E(T_a^k)$, then w_i is a vertex incident to the edge of color $\varphi(r_k w_{i-1})$ in T_a^k. But this is impossible since from (9) we have that $\varphi(v_i r_i) = \varphi(r_k w_i)$ and from (R9) that $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$, where α is a vertex incident to the edge of color $\varphi(r_k w_{i-1})$ in T_a^k. Therefore, $w_i w_i' \notin E(T_a^k)$ and $T_k^k(i)$ satisfies property (P7).

2.2.4. Properties (P8) and (P9) for $2 \leq i < k$.

Observe again that $E_{old}(T_i^k) \subset E(T_i^{k-1})$. As in Section 2.2.3, to prove properties (P8) and (P9) for $T_k^k(i)$, we can show that $w_i w_i' \notin E(T_d^{k-1})$, $i \leq d < k$.

For $i \leq d < k$, property (R9), which guarantees $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$, where α is a vertex incident to the edge of color $\varphi(r_k w_{i-1})$ in T_d^{k-1}, ensures $w_i w_i' \notin E(T_d^{k-1})$, thus ensuring that (P8) and (P9) hold for $T_k^k(i)$. The argument has been omitted here due to its similarity to the argument used above for (P7) in Section 2.2.3.

2.2.5. Property (P10) for $2 \leq i < k$.

To prove (P10) for $T_k^k(i)$, we need only show that $w_i w_i' \neq v_i v_i'$ and $w_i w_i' \neq r_k w_i$. We consider each in turn.

(i.) $w_i w_i' \neq v_i v_i'$

Recall from (9) that $\varphi(v_i v_i') = \varphi(r_k r_i)$ and from (11) that $\varphi(w_i w_i') = \varphi(r_k w_{i-1})$. If $w_i w_i' = v_i v_i'$, then $\varphi(r_k w_{i-1}) = \varphi(r_k r_i)$ and so $w_{i-1} = r_i$. But $r_k r_i \in E(T_i^{k-1})$ and $r_k w_{i-1} \in E(T_{i-1}^k)$; so if $w_{i-1} = r_i$, this contradicts property (P3) in the $i-1^{th}$ induction step, which in particular (i.e. when $b = i$) ensures that $r_k w_{i-1} \notin E(T_i^{k-1})$. Therefore, $w_i w_i' \neq v_i v_i'$, as required.

(ii.) $w_i w_i' \neq r_k w_i$

Recall from (11) that $\varphi(w_i w_i') = \varphi(r_k w_{i-1})$. If $w_i w_i' = r_k w_i$, then $\varphi(r_k w_{i-1}) = \varphi(r_k w_i)$ and so $w_{i-1} = w_i$. However, this is impossible by the result in Section 2.1.4 which, in particular, proved that $r_k w_i \neq r_k w_a$ for $1 \leq a < i$. Thus, $w_i w_i' \neq r_k w_i$.

Therefore, property (P10) holds for $T_k^k(i)$, as required.

2.2.6. *Property (P11) for $w_k w_k'$.*
The above sections of Case 2 ensure that the rainbow spanning trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the rainbow spanning graph, $T_k^k(k-1)$ are all edge-disjoint. Thus, it remains to show that $T_1^k, T_2^k, ..., T_{k-1}^k$ and T_k^k are all edge-disjoint. As above, recall from (11) that the only edge in T_k^k that differs from $T_k^k(k-1)$ is $w_k w_k'$. Therefore, showing property (P11) holds will prove that $T_1^k, T_2^k, ... T_{k-1}^k$ and T_k^k are edge-disjoint.

First, observe from (8) that since $w_k \in L_{k-1}$, w_k is a leaf adjacent to the root r_i in T_i^{k-1} for $1 \le i < k$. So if $w_k w_k' \in E(T_i^k)$, $w_k w_k' = w_i r_k$, $v_i v_i'$, or $w_k r_i$. We consider each in turn.

(i.) $w_k w_k' \ne w_i r_k$
From (8) we know that $w_k \ne r_k$. So if $w_k w_k' = w_i r_k$, then $w_k = w_i$, contradicting the preliminary result in Section 2.2.1. Therefore, $w_k w_k' \ne w_i r_k$, as required.

(ii.) $w_k w_k' \ne v_i v_i'$
Recall from (14) that since $v_i \in L_{k-1}^*$, $v_i \ne w_k$. So if $w_k w_k' = v_i v_i'$, then $w_k = v_i'$. From (9) we know that $\varphi(v_i v_i') = \varphi(r_i r_k)$, so if $w_k = v_i'$, then $\varphi(v_i w_k) = \varphi(r_i r_k)$, contradicting (R10). Therefore, $w_k w_k' \ne v_i v_i'$, as required.

(iii.) $w_k w_k' \ne w_k r_i$
Recall from (11) that $\varphi(w_k w_k') = \varphi(r_k w_{k-1})$ and suppose that $w_k w_k' = w_k r_i$. First observe that $i \ne k-1$ since $r_k w_{k-1} \in E(T_{k-1}^k)$ and we know from (8) and Section 2.2.1 that $w_k \ne r_k$ and $w_k \ne w_{k-1}$.

Now, for $1 \le i \le k-2$, if $w_k w_k' = w_k r_i$ then $r_i = w_k'$. But from (9) and (11) if $r_i = w_k'$ then $\varphi(w_k w_k') = \varphi(r_k w_{k-1}) = \varphi(v_{k-1} r_{k-1}) = \varphi(w_k r_i)$, contradicting (R11). Therefore, $w_k w_k' \ne w_k r_i$, as required.

It follows that $w_k w_k' \notin E(T_i^k)$, $1 \le i < k$.

The above Sections 2.2.1 - 2.2.6 ensure that the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the graph T_k^k are all edge-disjoint. Further, from (9) it is clear that $T_1^k, T_2^k, ..., T_{k-1}^k$ are all rainbow spanning trees and from (12) that T_k^k is a spanning rainbow graph (since for every leaf, w_c, $1 \le c \le k$, which is adjacent to r_k and for which $r_k w_c$ is removed from T_k^k, there exists w_c' such that the edge $w_c w_c'$ is added to T_k^k and edge $w_d w_d'$ in T_k^k such that $\varphi(w_d w_d') = \varphi(r_k w_c)$, where $d \equiv c+1 \mod k$.)

2.3. Case 3. (C3) Preventing cycles from appearing in T_k^k.

Properties (C1) and (C2) in the previous sections guarantee that the rainbow spanning trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the rainbow spanning graph T_k^k are all edge-disjoint. Thus, it remains to prove that T_k^k is acyclic and, therefore, a tree. This is proved inductively, showing that for $1 \le i \le k$, $T_k^k(i)$ is acyclic. //

For our base step, we let $T_k^k(0) = S_{r_k}$ and observe that this graph is clearly acyclic.

Recall from (11) that for $1 \le i \le k$, $T_k^k(i) = T_k^k(i-1) - r_k w_i + w_i w_i'$. Therefore, since by induction we have that $T_k^k(i-1)$ is acyclic, in order to prove $T_k^k(i)$ is acyclic, we need only show that adding $w_i w_i'$ to $T_k^k(i-1) - r_k w_i$ does not create a cycle. Let $T_k^k(i-1)^* = T_k^k(i-1) - r_k w_i$.

Now, from (11) observe that all of the edges in $T_k^k(i-1)$ are of the form $r_k x$, $r_k w_a'$, and $w_a w_a'$, where $1 \le a < i$ and $x \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{i-1} w_a, w_a'\} \cup \{r_k\})$. Thus, $w_i \in \{r_k, x, w_a, w_a'\}$. We now show that $w_i = x$ and, further, that since $w_i = x$, $T_k^k(i)$ is acyclic. //

In showing that $w_i = x$, we first consider the case where $1 \le i < k$ and then the case where $i = k$ before then showing $T_k^k(i)$ is acyclic for $1 \le i \le k$.

(i.) $w_i = x$, $1 \le i < k$
First observe that $w_i \ne r_k$ since $r_k w_i$ is an edge in T_i^k. Also, $w_i \ne w_a$ (this property is established by (R3) and was discussed in Section 2.1.4). Lastly, recall from (9) that

$\varphi(v_i r_i) = \varphi(r_k w_i)$. So if $w_i = w'_a$ then $\varphi(v_i r_i) = \varphi(r_k w'_a)$, contradicting (R6). Therefore, $w_i \neq w'_a$ and it follows that $w_i = x$.

(ii.) $w_k = x$

Begin by observing that $w_k \neq r_k$ (since by (8) w_k and r_k were chosen to be distinct vertices) and, for $1 \leq i < k$, $w_k \neq w_i$ (this property was established by (R5) and discussed in Section 2.2.1). The following argument shows $w_k \neq w'_i$.

First, observe that $w_k \neq w'_1$ since $\varphi(w_1 w'_1) = \varphi(r_k w_k)$, so if $w_k = w'_1$ then $w_1 = r_k$, which we know from (9) cannot be the case.

Now, for $2 \leq i < k$, let $\alpha \in V(K_{2m})$ be the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$ and recall from (12) that $\varphi(w_i w'_i) = \varphi(r_k w_{i-1})$. Suppose that $w_k = w'_i$. Then since $\varphi(w_k \alpha) = \varphi(r_k w_{i-1}) = \varphi(w_i w'_i) = \varphi(w_i w_k)$, α must equal w_i. But from (9), we have that $\varphi(v_i r_i) = \varphi(r_k w_i)$, so if $w_i = \alpha$ then $\varphi(v_i r_i) = \varphi(r_k \alpha)$, contradicting (R7) which ensures that $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$, where α is the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$. Therefore, $w_k \neq w'_i$, $2 \leq i < k$.

Combining the above arguments, it is clear that $w_k = x$.

(iii.) $T_k^k(i)$ is acyclic, $1 \leq i \leq k$

Observe that since $w_i = x$, $w_i \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{i-1} w_a, w'_a\} \cup \{r_k\})$ and w_i is a leaf adjacent to r_k in $T_k^k(i-1)$. Now, in order for $w_i w'_i$ to create a cycle in $T_k^k(i)$, there would have to exist a path from w_i to w'_i in $T_k^k(i-1)^*$. But, as we just observed, w_i is a leaf in $T_k^k(i-1)$ and since $T_k^k(i-1)^* = T_k^k(i-1) - r_k w_i$, w_i is an isolated vertex in $T_k^k(i-1)^*$ so it follows that no such path exists. Therefore, $T_k^k(i)$ is acyclic, as required.

The above three arguments show that T_k^k is acyclic, thus completing the proof of the theorem.

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