# A lower bound on the acyclic matching number of subcubic graphs 

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#### Abstract

The acyclic matching number of a graph $G$ is the largest size of an acyclic matching in $G$, that is, a matching $M$ in $G$ such that the subgraph of $G$ induced by the vertices incident to an edge in $M$ is a forest. We show that the acyclic matching number of a connected subcubic graph $G$ with $m$ edges is at least $m / 6$ except for two small exceptions.


Keywords: Acyclic matching; subcubic graph

## 1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation. A matching $M$ in a graph $G$ is acyclic [7] if the subgraph of $G$ induced by the set of vertices that are incident to some edge in $M$ is a forest, and the acyclic matching number $\nu_{a c}(G)$ of $G$ is the maximum size of an acyclic matching in $G$. While the ordinary matching number $\nu(G)$ of $G$ is tractable [4], it has been known for some time that the acyclic matching number is NP-hard for graphs of maximum degree 5 [7,15]. Recently, we [6] showed that just deciding the equality of $\nu(G)$ and $\nu_{a c}(G)$ is already NP-complete when restricted to bipartite graphs $G$ of maximum degree 4 . The complexity of the acyclic matching number for cubic graphs is unknown.

In the present paper we establish a tight lower bound on the acyclic matching number of subcubic graphs. Similar results were obtained for the matching number [2, 8, ,9, 14], and also for the induced matching number [11-13]. Baste and Rautenbach [1] studied acyclic edge colorings, and showed that the acyclic chromatic index $\chi_{a c}^{\prime}(G)$ of a graph $G$, that is, the minimum number of acyclic matchings in $G$ into which the edge set of $G$ can be partitioned, is at most $\Delta(G)^{2}$, where $\Delta(G)$ denotes the maximum degree of $G$. This implies $\nu_{a c}(G) \geq m(G) / \Delta(G)^{2}$, where $m(G)$ denotes the size of $G$, which, for subcubic graphs, simplifies to $\nu_{a c}(G) \geq m(G) / 9$. This latter bound also follows from a lower bound [12] on the induced matching number, which is always at most the acyclic matching number. While the bound is tight for $K_{3,3}$, excluding some small graphs allows a considerable improvement. Let $K_{4}^{+}$be the graph that arises by subdividing one edge of $K_{4}$ once.

We prove the following.
Theorem 1 If $G$ is a connected subcubic graph that is not isomorphic to $K_{4}^{+}$or $K_{3,3}$, then $\nu_{a c}(G) \geq m(G) / 6$.

Since every subcubic graph $G$ of order $n(G)$ satisfies $m(G) \leq 3 n(G) / 2$, Theorem 1 is an immediate consequence of the following stronger result. For two graphs $G$ and $H$, let $\kappa_{G}(H)$ denote the number of components of $G$ that are isomorphic to $H$.

Theorem 2 If $G$ is a subcubic graph without isolated vertices, then

$$
\nu_{a c}(G) \geq \frac{1}{4}\left(n(G)-\kappa_{G}\left(K_{2,3}\right)-\kappa_{G}\left(K_{4}^{+}\right)-2 \kappa_{G}\left(K_{3,3}\right)\right) .
$$

Note that Theorem 2 is tight; examples are $K_{4}, K_{2,2}, K_{1,3}$, or the graph obtained from $K_{1,3}$ by replacing each endvertex with an endblock isomorphic to $K_{2,3}$. The proof of Theorem 2 is postponed to the second section. The reduction arguments within that proof easily lead to a polynomial time algorithm computing acyclic matchings of the guaranteed size.

In a third section, we conclude with some open problems.

## 2 Proof of Theorem 2

The proof is by contradiction. Therefore, suppose that $G$ is a counterexample to Theorem 22 that is of minimum order $n$. A graph is special if it is isomorphic to $K_{2,3}, K_{4}^{+}$, or $K_{3,3}$. Clearly, $G$ is connected, not special, and $n$ is at least 5 . Note that $\nu_{a c}(G)<n / 4$.

We derive a contradiction using a series of claims.
Claim 1 No subgraph of $G$ is isomorphic to $K_{4}^{+}$.
Proof of Claim 11: Suppose that $G$ has a subgraph $H$ that is isomorphic to $K_{4}^{+}$. Let $v_{1}$, $v_{2}, v_{3}$, and $v_{4}$ be the vertices of degree 3 in $H$, and let $u$ the vertex of degree 2 in $H$. Let $G^{\prime}=G-\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$. Since $G$ is connected, the graph $G^{\prime}$ is connected. Since $u$ has degree 1 in $G^{\prime}$, the graph $G^{\prime}$ is not special. By the choice of $G$, the graph $G^{\prime}$ is no counterexample to Theorem 2, and, hence, it has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $v_{1} v_{2}$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.

Claim 2 No endblock of $G$ is isomorphic to $K_{2,3}$.
Proof of Claim 园: Suppose that some endblock $B$ of $G$ is isomorphic to $K_{2,3}$. Let $u$ be the unique cutvertex of $G$ in $B$. Clearly, the vertex $u$ has degree 2 in $B$. The graph $G^{\prime}=G-(V(B) \backslash\{u\})$ is connected, and, since $u$ has degree 1 in $G^{\prime}$, it is not special. Therefore, by the choice of $G$, the graph $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding an edge of $B$ that is not incident to $u$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.

Claim 3 No two vertices of degree 1 have a common neighbor.
Proof of Claim [3: Suppose that $u$ and $v$ are two vertices of degree 1, and that $w$ is their common neighbor. Let $G^{\prime}=G-\{u, v, w\}$. Since $G^{\prime}$ is connected and not isomorphic to $K_{3,3}$, the choice of $G$ implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-1\right) / 4=$ $n / 4-1$. Since $w$ does not lie on any cycle in $G$, adding the edge $u w$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.

Claim 4 No vertex of degree 1 is adjacent to a vertex that does not lie on a cycle.
Proof of Claim 4: Suppose that $u$ is a vertex of degree 1 that is adjacent to a vertex $v$ that does not lie on a cycle. By Claim 3, the graph $G^{\prime}=G-\{u, v\}$ has no isolated vertex. Since $G^{\prime}$ has at most two components, and no component of $G^{\prime}$ is isomorphic to $K_{3,3}$, the choice of $G$ implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-2\right) / 4=n / 4-1$. Since $v$ does not lie on a cycle, adding the edge $u v$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.

Claim 5 The minimum degree of $G$ is at least 2.
Proof of Claim 5: Suppose that $u$ is a vertex of degree 1. By Claim 4, the neighbor $v$ of $u$ lies on a cycle $C$ in $G$. Let $x$ and $w$ be the neighbors of $v$ on $C$.

First, suppose that $w$ has no neighbor of degree 1 .
If $G-\{u, v, w\}$ contains an isolated vertex, then this is necessarily the vertex $x$, and $N_{G}(x)=\{v, w\}$. In this case, let $G^{\prime}=G-\{u, v, w, x\}$. Clearly, the graph $G^{\prime}$ is connected and not isomorphic to $K_{3,3}$. If isomorphic to $K_{4}^{+}$or $K_{2,3}$, then it follows easily that $\nu_{a c}(G) \geq$ $3>9 / 4=n / 4$, which is a contradiction. Hence, $G^{\prime}$ is not special, which implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u v$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction. Hence, we may assume that $G^{\prime}=G-\{u, v, w\}$ has no isolated vertex.

Since there are at most three edges between $\{u, v, w\}$ and $V\left(G^{\prime}\right)$ in $G$, Claim 2 implies that at most one component of $G^{\prime}$ is isomorphic to $K_{2,3}$. By the choice of $G$, this implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-1\right) / 4=n / 4-1$. Adding the edge $u v$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction. Hence, by symmetry, we may assume that $x$ and $w$ both have a neighbor of degree 1 .

Let $y$ be a neighbor $w$ of degree 1 . If $x$ and $w$ are adjacent, then $\nu_{a c}(G)=2>6 / 4=n / 4$, which is a contradiction. Hence, $x$ and $w$ are not adjacent. In view of the cycle $C$, the graph $G^{\prime}=G-\{u, v, w, y\}$ is connected. Since $G^{\prime}$ has a vertex of degree 1 , it is not special, which implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u v$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.

For a set $X$ of vertices of $G$, let $N_{G}[X]=\bigcup_{u \in X} N_{G}[u]$.
Claim 6 No subgraph of $G$ is isomorphic to $K_{2,3}$.
Proof of Claim 6: Suppose that $G$ has a subgraph $H$ that is isomorphic to $K_{2,3}$. Claim 1 implies that $H$ is an induced subgraph of $G$. Let $u_{1}, u_{2}$, and $u_{3}$ be the vertices of degree 2 in $H$, and let $v_{1}$ and $v_{2}$ be the vertices of degree 3 in $H$.

First, suppose that $u_{1}$ has degree 2 in $G$. Since $G$ is not special, we may assume that $u_{2}$ has degree 3 in $G$. By Claim [5, the graph $G^{\prime}=\left(V(H) \backslash\left\{u_{2}\right\}\right)$ has no isolated vertex, and, since $u_{2}$ has degree 1 in $G^{\prime}$, it is not special. It follows that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u_{1} v_{1}$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction. Hence, by symmetry, we may assume that all vertices in $U=\left\{u_{1}, u_{2}, u_{3}\right\}$ have degree 3 in $G$.

Next, suppose that $u_{1}$ and $u_{2}$ have a common neighbor $u$ that is distinct from $v_{1}$ and $v_{2}$. Let $G^{\prime}=G-N_{G}[U]$. Note that there are at most 3 edges between $N_{G}[U]$ and $V\left(G^{\prime}\right)$ in $G$. By Claim 5, the graph $G^{\prime}$ has at most one isolated vertex, and, by Claim [1, at most one component of $G^{\prime}$ is isomorphic to $K_{2,3}$. Furthermore, the graph $G^{\prime}$ does not have an isolated vertex as well as a component isomorphic to $K_{2,3}$. This implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-1\right) / 4=n / 4-2$. Adding the two edges $u u_{1}$ and $u_{3} v_{1}$ to $M^{\prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction. Hence, by symmetry, no two vertices in $U$ have a common neighbor that is distinct from $v_{1}$ and $v_{2}$.

The graph $G^{\prime}$ that arises by contracting all edges of $H$ is simple and connected. If $G^{\prime}$ is special, then $G$ has order at most 11, and an acyclic matching consisting of the three edges between $N_{G}[U]$ and $V(G) \backslash N_{G}[U]$ in $G$, which is a contradiction. Hence, $G^{\prime}$ is not special, which implies that $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Let $M^{\prime \prime}$ be the acyclic matching in $G$ corresponding to $M^{\prime}$. Since $M^{\prime \prime}$ covers at most one vertex
in $U$, say $u_{1}$, adding the edge $u_{2} v_{1}$ to $M^{\prime \prime}$ yields an acyclic matching in $G$ of size at least $n / 4$, which is a contradiction.
Claim 1, Claim 6, and the choice of $G$ imply that every proper induced subgraph $G^{\prime}$ of $G$ with $i\left(G^{\prime}\right)$ isolated vertices has an acyclic matching $M^{\prime}$ such that

$$
\begin{equation*}
\left|M^{\prime}\right| \geq \frac{n\left(G^{\prime}\right)-i\left(G^{\prime}\right)}{4} \tag{1}
\end{equation*}
$$

Claim 7 No two vertices of degree 2 are adjacent.
Proof of Claim 7: Suppose that $u$ and $v$ are adjacent vertices of degree 2, and that $w$ is the neighbor of $u$ distinct from $v$. By Claim 5, the graph $G^{\prime}=G-\{u, v, w\}$ has at most one isolated vertex, and, hence, by (11), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-1\right) / 4=n / 4-1$. Adding the edge $u v$ to $M^{\prime}$ yields a contradiction.

Claim 8 No vertex of degree 2 lies on a triangle.
Proof of Claim 8: Suppose that $u_{1} u_{2} u_{3} u_{1}$ is a triangle in $G$ such that $u_{1}$ has degree 2. By Claim 7, the vertices $u_{2}$ and $u_{3}$ have degree 3. Since $n \geq 5$, the graph $G^{\prime}=G-\left\{u_{1}, u_{2}, u_{3}\right\}$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4>n / 4-1$. Adding the edge $u_{1} u_{2}$ to $M^{\prime}$ yields a contradiction.

Claim 9 No vertex of degree 2 lies on a cycle of length 4.
Proof of Claim 9: Suppose that $u_{1} u_{2} u_{3} u_{4} u_{1}$ is a cycle in $G$ such that $u_{1}$ has degree 2. By Claims 7 and 8 , the vertices $u_{2}$ and $u_{4}$ have degree 3, and are not adjacent. By Claims 6 and 8, the graph $G^{\prime}=G-\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ has no isolated vertex, and, hence, by (11), it has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u_{1} u_{2}$ to $M^{\prime}$ yields a contradiction.

Claim 10 No cycle of length 5 contains two vertices of degree 2.
Proof of Claim 10: Suppose that the cycle $u_{1} u_{2} u_{3} u_{4} u_{5} u_{1}$ contains two vertices of degree 2. By Claim 7, we may assume that $u_{1}$ and $u_{4}$ have degree 2 , and that $u_{2}, u_{3}$, and $u_{5}$ have degree 3. Let $G^{\prime}=G-\left(N_{G}\left[u_{5}\right] \cup\left\{u_{2}, u_{3}\right\}\right)$. Since there are at most 4 edges between $N_{G}\left[u_{5}\right] \cup\left\{u_{2}, u_{3}\right\}$ and $V\left(G^{\prime}\right)$ in $G$, the graph $G^{\prime}$ has at most two isolated vertices, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-2\right) / 4=n / 4-2$. Adding the edges $u_{1} u_{2}$ and $u_{4} u_{5}$ to $M^{\prime}$ yields a contradiction.

Claim $11 G$ is cubic.
Proof of Claim 11: Suppose that $u$ is a vertex of degree 2. By Claims 7, 8, and 9, the neighbors of $u$, say $v$ and $w$, have degree 3 , are not adjacent, and have no common neighbor except for $u$. Let $x$ be a neighbor of $v$ distinct from $u$. By Claims 8, 9, and 10, the graph $G^{\prime}=G-\{u, v, w, x\}$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u v$ to $M^{\prime}$ yields a contradiction.

Claim $12 G$ is triangle-free.
Proof of Claim 12: Suppose that $u_{1} u_{2} u_{3} u_{1}$ is a triangle in $G$. By Claims 1 and 11, the graph $G^{\prime}=G-N_{G}\left[u_{1}\right]$ has no isolated vertex, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-1$. Adding the edge $u_{1} u_{2}$ to $M^{\prime}$ yields a contradiction.
Let $C: u_{1} u_{2} \ldots u_{g} u_{1}$ be a shortest cycle in $G$. For $i \in[g]$, let $v_{i}$ be the neighbor of $u_{i}$ not on $C$. By Claim 12, we have $g \geq 4$.

Claim $13 g \geq 5$.
Proof of Claim 13: Suppose that $g=4$. By Claims 6 and 12, the vertices $v_{1}, v_{2}, v_{3}$, and $v_{4}$ are distinct. Let $w_{1}$ and $w_{2}$ be the neighbors of $v_{1}$ distinct from $u_{1}$.

First, suppose that $w_{1}=v_{2}$. By Claim 11, the graph $G^{\prime}=G-\left(N_{G}\left[v_{1}\right] \cup\left\{u_{2}, u_{3}, u_{4}\right\}\right)$ has at most one isolated vertex, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-1\right) / 4=n / 4-2$. Adding the edges $u_{1} v_{1}$ and $u_{2} u_{3}$ to $M^{\prime}$ yields a contradiction. Hence, we may assume, by symmetry, that $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ is independent.

Next, suppose that there is some vertex $x$ outside of $N_{G}\left[\left\{v_{1}, u_{1}, u_{3}\right\}\right]$ such that $N_{G}(x) \subseteq$ $N_{G}\left[\left\{v_{1}, u_{1}, u_{3}\right\}\right]$. By Claim 6, $x$ is not adjacent to both $u_{2}$ and $u_{4}$. Hence, by Claim 11, we may assume that $x$ is adjacent to $w_{1}$ but not to $u_{2}$. By Claim 11, the graph $G^{\prime}=$ $G-N_{G}\left[\left\{v_{1}, u_{1}, u_{3}, w_{1}\right\}\right]$ has at most two isolated vertices, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-2\right) / 4=n / 4-3$. Adding the edges $x w_{1}$, $u_{1} v_{1}$, and $u_{2} u_{3}$ to $M^{\prime}$ yields a contradiction. Hence, we may assume that the graph $G^{\prime}=$ $G-N_{G}\left[\left\{v_{1}, u_{1}, u_{3}\right\}\right]$ has no isolated vertex. By (11), $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-2$. Adding the edges $u_{1} v_{1}$ and $u_{2} u_{3}$ to $M^{\prime}$ yields a contradiction.

Claim $14 g \geq 6$.
Proof of Claim 14: Suppose that $g=5$. By Claim 13, the vertices $v_{1}, v_{2}, v_{3}, v_{4}$, and $v_{5}$ are distinct. Suppose that there is some vertex $x$ outside of $N_{G}\left[\left\{u_{1}, u_{2}, u_{4}\right\}\right]$ such that $N_{G}(x) \subseteq N_{G}\left[\left\{u_{1}, u_{2}, u_{4}\right\}\right]$. By Claims 11 and 13, we obtain $N_{G}(x)=\left\{v_{1}, v_{2}, v_{4}\right\}$. By Claim 11, the graph $G^{\prime}=G-N_{G}\left[\left\{v_{1}, u_{1}, u_{2}, u_{4}\right\}\right]$ has at most two isolated vertices, and, hence, by (1), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-2\right) / 4=n / 4-3$. Adding the edges $x v_{1}, u_{1} u_{2}$, and $u_{3} u_{4}$ to $M^{\prime}$ yields a contradiction. Hence, we may assume that the graph $G^{\prime}=G-N_{G}\left[\left\{u_{1}, u_{2}, u_{4}\right\}\right]$ has no isolated vertex. By (11), the graph $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-2$. Adding the edges $u_{1} u_{2}$ and $u_{3} u_{4}$ to $M^{\prime}$ yields a contradiction.

## Claim $15 g \geq 7$.

Proof of Claim 15: Suppose that $g=6$. Let $w_{1}$ and $w_{2}$ be the neighbors of $v_{1}$ distinct from $u_{1}$. By Claim (14, the vertices $v_{i}$ for $i \in[6] \backslash\{4\}, w_{1}$, and $w_{2}$ are distinct. Suppose that there is some vertex $x$ outside of $N_{G}\left[\left\{v_{1}, u_{3}, u_{5}, u_{6}\right\}\right]$ such that $N_{G}(x) \subseteq N_{G}\left[\left\{v_{1}, u_{3}, u_{5}, u_{6}\right\}\right]$. By Claims 11 and 14, we obtain that $x$ is adjacent to $v_{3}$, to one vertex in $\left\{v_{5}, v_{6}\right\}$, and to one vertex in $\left\{w_{1}, w_{2}\right\}$. Let $G^{\prime}=G-N_{G}\left[\left\{v_{1}, v_{3}, u_{3}, u_{5}, u_{6}\right\}\right]$. By Claim [14, no isolated vertex in $G^{\prime}$ is adjacent to $u_{2}$ or $u_{4}$. Since there are at most 10 edges between $N_{G}\left[\left\{v_{1}, v_{3}, u_{3}, u_{5}, u_{6}\right\}\right]$ and $V\left(G^{\prime}\right)$ in $G$, this implies that $G^{\prime}$ has at most two isolated vertices, and, hence, by (11), it has an acyclic matching $M^{\prime}$ of size at least $\left(n\left(G^{\prime}\right)-2\right) / 4=n / 4-4$. Adding the edges $x v_{3}$, $u_{1} v_{1}, u_{2} u_{3}$, and $u_{5} u_{6}$ to $M^{\prime}$ yields a contradiction. Hence, we may assume that the graph $G^{\prime}=G-N_{G}\left[\left\{v_{1}, u_{3}, u_{5}, u_{6}\right\}\right]$ has no isolated vertex. By (11), the graph $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-3$. Adding the edges $u_{1} v_{1}, u_{2} u_{3}$, and $u_{5} u_{6}$ to $M^{\prime}$ yields a contradiction.
We are now in a position to complete the proof.
First, suppose that $g$ is odd. If the graph $G^{\prime}=G-N_{G}\left[\left\{u_{1}, \ldots, u_{g-2}\right\}\right]$ has an isolated vertex, then, by Claim [11, there is a cycle of length at most $\left\lfloor\frac{g}{3}\right\rfloor+4$. Since the last expression is less than $g$ for odd $g$ at least 7, it follows that $G^{\prime}$ has no isolated vertex. By (1), the graph $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-(g-1) / 2$. Adding the edges in $\left\{u_{2 i-1} u_{2 i}: i \in[(g-1) / 2]\right\}$ to $M^{\prime}$ yields a contradiction. Hence, we may assume that $g$ is even. Let $w_{1}$ and $w_{2}$ be the neighbors of $v_{1}$ distinct from $u_{1}$. By the choice of $C$, the vertices $v_{i}$ for $i \in[g], w_{1}$, and $w_{2}$ are distinct. If the graph $G^{\prime}=G-N_{G}\left[\left\{v_{1}, u_{1}, \ldots, u_{g-2}\right\}\right]$ has an
isolated vertex, then, by Claim 11, there is a cycle of length at most $\left\lfloor\frac{g}{3}\right\rfloor+5$. Since the last expression is less than $g$ for even $g$ at least 8 , it follows that $G^{\prime}$ has no isolated vertex. By (1), the graph $G^{\prime}$ has an acyclic matching $M^{\prime}$ of size at least $n\left(G^{\prime}\right) / 4=n / 4-g / 2$. Adding the edges in $\left\{u_{1} v_{1}\right\} \cup\left\{u_{2 i} u_{2 i+1}: i \in[(g-2) / 2]\right\}$ to $M^{\prime}$ yields a contradiction, which completes the proof.

## 3 Conclusion

We believe that Theorem 2 can be improved as follows.
Conjecture 3 There is a constant c such that $\nu_{a c}(G) \geq \frac{3 n(G)}{11}-c$ for every connected subcubic graph $G$.

Conjecture 3 would be asymptotically best possible. If $H$ arises from a copy of $K_{1,2}$, where $u(H)$ denotes the vertex of degree 2 , by replacing each endvertex with an endblock isomorphic to $K_{2,3}$, and, for some positive integer $k$, the connected subcubic graph $G_{k}$ arises from $k$ disjoint copies $H_{1}, \ldots, H_{k}$ of $H$ by adding, for every $i \in[k-1]$, an edge between $u\left(H_{i}\right)$ and some vertex of degree 2 in $H_{i+1}$ that is distinct from $u\left(H_{i+1}\right)$, then $\nu_{a c}\left(G_{k}\right)=3 n\left(G_{k}\right) / 11$.

For general maximum degree, we pose the following conjecture motivated by [13].
Conjecture 4 If $G$ is a graph of maximum degree $\Delta$ without isolated vertices, then

$$
\nu_{a c}(G) \geq \min \left\{\frac{2 n(G)}{\left(\left\lceil\frac{\Delta}{2}\right\rceil+1\right)\left(\left\lfloor\frac{\Delta}{2}\right\rfloor+1\right)}, \frac{n(G)}{2 \Delta}\right\} .
$$

There should be better lower bounds on the acyclic matching number for graphs of large girth, and methods from [3,5,10] might be useful. Moreover, a lower bound as Conjecture 4, which is essentially tight for all possible densities of a graph $G$ of bounded maximum degree, would be interesting, yet very challenging.

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