On degree sum conditions for 2-factors with a prescribed number of cycles

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Abstract

For a vertex subset X of a graph G, let $\Delta_t(X)$ be the maximum value of the degree sums of the subsets of X of size t. In this paper, we prove the following result: Let k be a positive integer, and let G be an m-connected graph of order $n \geq 5k - 2$. If $\Delta_2(X) \geq n$ for every independent set X of size $\lfloor m/k \rfloor + 1$ in G, then G has a 2-factor with exactly k cycles. This is a common generalization of the results obtained by Brandt et al. [Degree conditions for 2-factors, J. Graph Theory 24 (1997) 165–173] and Yamashita [On degree sum conditions for long cycles and cycles through specified vertices, Discrete Math. 308 (2008) 6584–6587], respectively.

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. The independence number and the connectivity of a graph G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. For a vertex x of a graph G, we denote by $d_G(x)$ and $N_G(x)$ the degree and the neighborhood of x in G. Let $\sigma_m(G)$ be the minimum degree sum of an independent set of m vertices in a graph G, i.e., if $\alpha(G) \geq m$, then

$$\sigma_m(G) = \min\left\{\sum_{x \in X} d_G(x) : X \text{ is an independent set of } G \text{ with } |X| = m\right\};$$

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otherwise, $\sigma_m(G) = +\infty$. If the graph G is clear from the context, we often omit the graph parameter G in the graph invariant. In this paper, "disjoint" always means "vertex-disjoint".

A graph having a *hamilton cycle*, i.e., a cycle containing all the vertices of the graph, is said to be *hamiltonian*. The hamiltonian problem has long been fundamental in graph theory. But, it is NP-complete, and so no easily verifiable necessary and sufficient condition seems to exist. Therefore, many researchers have focused on "better" sufficient conditions for graphs to be hamiltonian (see a survey [14]). In particular, the following degree sum condition, due to Ore (1960), is classical and well known.

Theorem A (Ore [15]) Let G be a graph of order $n \ge 3$. If $\sigma_2 \ge n$, then G is hamiltonian.

Chvátal and Erdős (1972) discovered the relationship between the connectivity, the independence number and the hamiltonicity.

Theorem B (Chvátal, Erdős [8]) Let G be a graph of order at least 3. If $\alpha \leq \kappa$, then G is hamiltonian.

Bondy [2] pointed out that the graph satisfying the Ore condition also satisfies the Chvátal-Erdős condition, that is, Theorem B implies Theorem A.

By Theorem B, we should consider the degree condition for the existence of a hamilton cycle in graphs G with $\alpha(G) \geq \kappa(G) + 1$. In fact, Bondy (1980) gave the following degree sum condition by extending Theorem B.

Theorem C (Bondy [3]) Let G be an m-connected graph of order $n \geq 3$. If $\sigma_{m+1} > \frac{1}{2}(m+1)(n-1)$, then G is hamiltonian.

In 2008, Yamashita [17] introduced a new graph invariant and further generalized Theorem C as follows. For a vertex subset X of a graph G with $|X| \ge t$, we define

$$\Delta_t(X) = \max\Big\{\sum_{x \in Y} d_G(x) : Y \subseteq X, \ |Y| = t\Big\}.$$

Let $m \ge t$, and if $\alpha(G) \ge m$, then let

$$\sigma_t^m(G) = \min\left\{\Delta_t(X) : X \text{ is an independent set of } G \text{ with } |X| = m\right\};$$

otherwise, $\sigma_t^m(G) = +\infty$. Note that $\sigma_t^m(G) \ge \frac{t}{m} \cdot \sigma_m(G)$.

Theorem D (Yamashita [17]) Let G be an m-connected graph of order $n \ge 3$. If $\sigma_2^{m+1} \ge n$, then G is hamiltonian.

This result suggests that the degree sum of non-adjacent "two" vertices is important for hamilton cycles.

On the other hand, it is known that a 2-factor is one of the important generalizations of a hamilton cycle. A 2-factor of a graph is a spanning subgraph in which every component is a cycle, and thus a hamilton cycle is a 2-factor with "exactly 1 cycle". As one of the studies concerning the difference between hamilton cycles and 2-factors, in this paper, we focus on 2-factors with "exactly k cycles". Similar to the situation for hamilton cycles, deciding whether a graph has a 2-factor with $k \ (\geq 2)$ cycles is also NP-complete. Therefore, the sufficient conditions for the existence of such a 2-factor also have been extensively studied in graph theory (see a survey [11]). In particular, the following theorem, due to Brandt, Chen, Faudree, Gould and Lesniak (1997), is interesting. (In the paper [5], the order condition is not " $n \ge 4k - 1$ " but " $n \ge 4k$ ". However, by using a theorem of Enomoto [9] and Wang [16] ("every graph G of order at least 3k with $\sigma_2(G) \ge 4k - 1$ contains k disjoint cycles") for the cycles packing problem, we can obtain the following. See the proof in [5, Lemma 1].)

Theorem E (Brandt et al. [5]) Let k be a positive integer, and let G be a graph of order $n \ge 4k - 1$. If $\sigma_2 \ge n$, then G has a 2-factor with exactly k cycles.

This theorem shows that the Ore condition guarantees the existence of a hamilton cycle but also the existence of a 2-factor with a prescribed number of cycles.

By considering the relation between Theorem A and Theorem E, Chen, Gould, Kawarabayashi, Ota, Saito and Schiermeyer [6] conjectured that the Chvátal-Erdős condition in Theorem B also guarantees the existence of a 2-factor with exactly kcycles (see [6, Conjecture 1]). Chen et al. also proved that if the order of a 2-connected graph G with $\alpha(G) = \alpha \leq \kappa(G)$ is sufficiently large compared with k and with the Ramsey number $r(\alpha + 4, \alpha + 1)$, then the graph G has a 2-factor with k cycles. In [12], Kaneko and Yoshimoto "almost" solved the above conjecture for k = 2 (see the comment after Theorem E in Chen et al. [6] for more details). Another related result can be found in [7]. But, the above conjecture is still open in general. In this sense, there is a big gap between hamilton cycles and 2-factors with exactly $k (\geq 2)$ cycles.

In this paper, by combining the techniques of the proof for hamiltonicity and the proof for 2-factors with a prescribed number of cycles, we give the following Yamashita-type condition for 2-factors with k cycles.

Theorem 1 Let k be a positive integer, and let G be an m-connected graph of order $n \ge 5k-2$. If $\sigma_2^{\lceil m/k \rceil+1} \ge n$, then G has a 2-factor with exactly k cycles.

This theorem implies the following:

Remark 2

• Theorem 1 is a generalization of Theorem D.

- Theorem 1 leads to the Bondy-type condition: If G is an m-connected graph of order $n \ge 5k - 2$ with $\sigma_{\lceil m/k \rceil + 1}(G) > \frac{1}{2}(\lceil m/k \rceil + 1)(n - 1)$, then G has a 2-factor with exactly k cycles. Therefore, Theorem 1 is also a generalization of Theorem E for sufficiently large graphs. (Recall that $\sigma_t^m(G) \ge \frac{t}{m} \cdot \sigma_m(G)$ and $\sigma_m(G) \ge \frac{m}{2} \cdot \sigma_2(G)$ for $m \ge t \ge 2$.)
- Theorem 1 leads to the Chvátal-Erdős-type condition: If G is a graph of order at least 5k 2 with $\alpha(G) \leq \lceil \kappa(G)/k \rceil$, then G has a 2-factor with exactly k cycles.

The complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ (*n* is odd) does not contain a 2factor, and hence the degree condition in Theorem 1 is best possible in this sense. The order condition in Theorem 1 comes from our proof techniques. Similar to the situation for the proof of Theorem E, we will use the order condition only for the cycles packing problem (see Lemma 5 and the proof of Theorem 1 in Section 3). The complete bipartite graph $K_{2k-1,2k-1}$ shows that $n \ge 4k - 1$ is necessary. In the last section (Section 4), we note that " $n \ge 5k - 2$ " can be replaced with " $n \ge 4k - 1$ " for the Bondy-type condition (and the Chvátal-Erdős-type condition) in Remark 2.

	hamilton cycle	2-factor with k cycles
Ore-type	$\sigma_2 \ge n$	$\sigma_2 \ge n$
	Theorem A (Ore)	Theorem E (Brandt et al.)
Chvátal-Erdős-type	$\alpha \leq \kappa$	$\alpha \leq \lceil \ \kappa/k \ \rceil$
	Theorem B (Chvátal and Erdős)	Remark 2
Bondy-type	$\sigma_{\kappa+1} > \frac{1}{2}(\kappa+1)(n-1)$	$\sigma_{\lceil \kappa/k\rceil+1} > \frac{1}{2}(\lceil \kappa/k\rceil+1)(n-1)$
	Theorem C (Bondy)	Remark 2
Yamashita-type	$\sigma_2^{\kappa+1} \ge n$	$\sigma_2^{\lceil \kappa/k\rceil+1} \geq n$
	Theorem D (Yamashita)	Theorem 1 (Main theorem)

Table 1 summarizes the conditions mentioned in the above.

Table 1: Comparison of the degree conditions

To prove Theorem 1, in the next section, we extend the concept of insertible vertices which was introduced by Ainouche [1], and we prove Theorem 1 in Section 3 by using it.

2 The concept of insertible vertices

In this section, we prepare terminology and notations and give some lemmas.

Let G be a graph. For $v \in V(G)$ and $X \subseteq V(G)$, we let $N_G(v; X) = N_G(v) \cap X$ and $d_G(v; X) = |N_G(v; X)|$. For $V, X \subseteq V(G)$, let $N_G(V; X) = \bigcup_{v \in V} N_G(v; X)$. For $X \subseteq V(G)$, we denote by G[X] the subgraph of G induced by X. An (x, y)-path in G is a path from a vertex x to a vertex y in G. We write a cycle (or a path) C with a given orientation by \overrightarrow{C} . If there exists no fear of confusion, we abbreviate \overrightarrow{C} by C. Let C be an oriented cycle (or path). We denote by \overleftarrow{C} the cycle C with a reverse orientation. For $x \in V(C)$, we denote the successor and the predecessor of x on \overrightarrow{C} by x^+ and x^- . For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ the (x, y)-path on \overrightarrow{C} . The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. In the rest of this paper, we consider that every cycle (path) has a fixed orientation, unless stated otherwise, and we often identify a subgraph H of G with its vertex set V(H).

The following lemma is obtained by using the standard crossing argument, and so we omit the proof.

Lemma 1 Let G be a graph of order n, and let P be an (x, y)-path of order at least 3 in G. If $d_G(x) + d_G(y) \ge n$, then G contains a cycle of order at least |P|.

In [1], Ainouche introduced the concept of insertible vertices, which has been used for the proofs of the results on hamilton cycles. In this paper, we modify it for 2-factors with k cycles, and it also plays a crucial role in our proof. Let G be a graph, and let $\mathcal{D} = \{D_1, \ldots, D_{r+s}\}$ $(r+s \ge 1)$ be the set of r cycles and s paths in G which are pairwise disjoint. For a vertex x in $G - \bigcup_{1 \le p \le r+s} D_p$, the vertex x is *insertible for* \mathcal{D} if there is an edge uv in $E(D_p)$ such that $xu, xv \in E(G)$ for some p with $1 \le p \le r+s$. In the following lemma, "partition" of a graph means a partition of the vertex set.

Lemma 2 Let G be a graph, and let $\mathcal{D} = \{D_1, \ldots, D_{r+s}\}$ $(r+s \ge 1)$ be the set of r cycles and s paths in G which are pairwise disjoint, and P be a path in $G - \bigcup_{1 \le p \le r+s} D_p$. If every vertex of P is insertible for \mathcal{D} , then $G[\bigcup_{1 \le p \le r+s} V(D_p) \cup V(P)]$ can be partitioned into r cycles and s paths.

Proof of Lemma 2. By choosing the following two vertices $u, v \in V(P)$ and the edge $ww^+ \in \bigcup_{1 \le p \le r+s} E(D_p)$ inductively, we can get the desired partition of $G[\bigcup_{1 \le p \le r+s} V(D_p) \cup V(P)]$. Let u be the first vertex along \overrightarrow{P} , and take an edge ww^+ in $E(D_i)$ ($\subseteq \bigcup_{1 \le p \le r+s} E(D_p)$) such that $uw, uw^+ \in E(G)$ for some i with $1 \le i \le r+s$ (since u is insertible for \mathcal{D} , we can take such an edge). We let v be the last vertex along \overrightarrow{P} such that $vw, vw^+ \in E(G)$ (may be u = v). Then, we can insert all vertices of $u\overrightarrow{P}v$ into D_i . In fact, by replacing the edge ww^+ by the path $wu\overrightarrow{P}vw^+$, we can obtain a spanning subgraph D'_i of $G[V(D_i \cup u\overrightarrow{P}v)]$ such that D'_i is a cycle if D_i is a cycle; otherwise, D'_i is a path. By the choice of u and v, we have $zw \notin E(G)$ or $zw^+ \notin E(G)$ for each vertex z of $P' := P - u\overrightarrow{P}v$, and hence every vertex of P' is insertible for $\mathcal{D}' = \{D_1, \ldots, D_{i-1}, D'_i, D_{i+1}, \ldots, D_{r+s}\}$. Thus, we can repeat this argument for the path P' and the set \mathcal{D}' , and we get then the desired partition. \Box

In the rest of this section, we fix the following. Let C_1, \ldots, C_k be k disjoint cycles in a graph G, and let $C^* = \bigcup_{1 \le p \le k} C_p$. Choose C_1, \ldots, C_k so that

$$|C^*| \left(= \sum_{1 \le p \le k} |C_i| \right)$$
 is as large as possible.

Suppose that C^* does not form a 2-factor of G. Let $H = G - C^*$, and let H_0 be a component of H and $x_0 \in V(H_0)$. Let

$$u_1, u_2, \ldots, u_l$$
 be l distinct vertices in $N_G(H_0; C_1)$, where $l \geq 2$

We assume that u_1, u_2, \ldots, u_l appear in this order on $\overrightarrow{C_1}$, and let $u_{l+1} = u_1$. Note that by the maximality of $|C^*|$, $u_i^+ \neq u_{i+1}$ for $1 \leq i \leq l$. We denote by $\overrightarrow{Q_i}$ and $\overrightarrow{Q_{i,j}}$ a (u_i, x_0) -path in $G[V(H_0) \cup \{u_i\}]$ and a (u_i, u_j) -path passing through a vertex of H_0 in $G[V(H_0) \cup \{u_i, u_j\}]$, respectively.

Lemma 3 For $1 \le i \le l$, $u_i^+ \overrightarrow{C_1} u_{i+1}^-$ contains a non-insertible vertex for $\{C_2, \ldots, C_k\}$. **Proof of Lemma 3.** Suppose that every vertex of $u_i^+ \overrightarrow{C_1} u_{i+1}^-$ is insertible for $\{C_2, \ldots, C_k\}$. Then, by Lemma 2, $G[\bigcup_{2\le p\le k} V(C_p)\cup V(u_i^+ \overrightarrow{C_1} u_{i+1}^-)]$ has a 2-factor with exactly k-1 cycles. With the cycle $u_{i+1}\overrightarrow{C_1} u_i\overrightarrow{Q_{i,i+1}} u_{i+1}$, we can get k disjoint cycles in G such that the sum of the orders is larger than $|C^*|$, a contradiction. \Box

For $1 \leq i \leq l$, let x_i be the first non-insertible vertex for $\{C_2, \ldots, C_k\}$ in $V(u_i^+ \overrightarrow{C_1} u_{i+1}^-)$ on $\overrightarrow{C_1}$, i.e., every vertex of $u_i^+ \overrightarrow{C_1} x_i^-$ is insertible for $\{C_2, \ldots, C_k\}$, but x_i is not insertible (Lemma 3 guarantees the existence of such a vertex x_i).

Lemma 4 Let i, j be integers with $1 \le i, j \le l$ and $i \ne j$. If $x \in V(u_i^+ \overrightarrow{C_1} x_i)$ and $x' \in \{x_0, u_j^+\}$, then (i) $xx' \notin E(G)$, and (ii) $d_G(x; H \cup C_1) + d_G(x'; H \cup C_1) \le |H \cup C_1| - 1$.

Proof of Lemma 4. Consider the path

$$\overrightarrow{P} = \begin{cases} x \overrightarrow{C_1} u_i \overrightarrow{Q_i} x_0 & \text{(if } x' = x_0) \\ x \overrightarrow{C_1} u_j \overleftarrow{Q_{i,j}} u_i \overleftarrow{C_1} u_j^+ & \text{(if } x' = u_j^+) \end{cases}$$

See Figure 1. Then, P is a path in $G[V(H \cup x\overrightarrow{C_1}u_i)]$ passing through all vertices of $x\overrightarrow{C_1}u_i$ and a vertex of H_0 . Recall that every vertex of $u_i^+\overrightarrow{C_1}x^-$ is insertible for $\{C_2, \ldots, C_k\}$, and hence $G[\bigcup_{2 \le p \le k} V(C_p) \cup V(u_i^+\overrightarrow{C_1}x^-)]$ has a 2-factor with exactly k-1 cycles (by Lemma 2). Hence, the maximality of $|C^*|$ and Lemma 1 yield that

$$xx' \notin E(G) \text{ and } d_G(x; H \cup x\overrightarrow{C_1}u_i) + d_G(x'; H \cup x\overrightarrow{C_1}u_i) \le |H \cup x\overrightarrow{C_1}u_i| - 1.$$

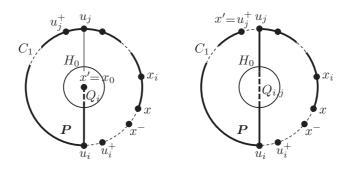


Figure 1: The path P

In particular, (i) holds. Then, by applying (i) for each vertex in $u_i^+ \overrightarrow{C_1} x^-$ and the vertex x', we have $N_G(x'; u_i^+ \overrightarrow{C_1} x^-) = \emptyset$. Combining this with the above inequality, we get,

$$d_G(x; H \cup C_1) + d_G(x'; H \cup C_1)$$

= $d_G(x; H \cup x\overrightarrow{C_1}u_i) + d_G(x'; H \cup x\overrightarrow{C_1}u_i) + d_G(x; u_i^+\overrightarrow{C_1}x^-)$
 $\leq (|H \cup x\overrightarrow{C_1}u_i| - 1) + |u_i^+\overrightarrow{C_1}x^-| = |H \cup C_1| - 1.$

Thus (ii) also holds. \Box

3 Proof of Theorem 1

Before proving Theorem 1, we will give the following lemma for the cycles packing problem.

Lemma 5 Let k, m, n and G be the same ones as in Theorem 1. Under the same degree sum condition as Theorem 1, G contains k disjoint cycles.

Proof of Lemma 5. If k = 1, then it is easy to check that G contains a cycle. If $\lceil m/k \rceil = 1$ or $\lceil m/k \rceil \ge 3$, then by a theorem of Enomoto [9], G contains k disjoint cycles (note that if $\lceil m/k \rceil \ge 3$, then G is (2k + 1)-connected, that is, the minimum degree $\delta(G)$ is at least 2k + 1). Thus, we may assume that $k \ge 2$ and $\lceil m/k \rceil = 2$. Then, we have $\delta(G) \ge m \ge k + 1$ and $\sigma_2^3(G) = \sigma_2^{\lceil m/k \rceil + 1}(G) \ge n \ge 5k - 2$. Note that, by the definition of $\sigma_2^3(G)$ and $\sigma_3(G)$, $\sigma_3(G) \ge \sigma_2^3(G) + \delta(G)$. Note also that $n \ge 5k - 2 \ge 3k + 2 \ge 8$ because $k \ge 2$. Hence, by a theorem of Fujita et al. [10] ("every graph G of order at least $3k + 2 \ge 8$ with $\sigma_3(G) \ge 6k - 2$ contains k disjoint cycles"), we can get the desired conclusion. \Box

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let G be an m-connected graph of order $n \ge 5k - 2$ such that $\sigma_2^{\lceil m/k \rceil + 1}(G) \ge n$. We show that G has a 2-factor with exactly k cycles. By Theorem E, we may assume that $\lceil m/k \rceil \ge 2$. By Lemma 5, G contains k disjoint cycles. Let C_i $(1 \le i \le k), C^*, H, H_0, x_0$ and u_i $(1 \le i \le l)$ be the same graphs and vertices as the ones described in the paragraph preceding Lemma 3 in Section 2. In particular, we may assume that $l = \lceil m/k \rceil$. Because, since G is m-connected, it follows that $|N_G(H_0; C^*)| \ge m$ (note that by the maximality of $|C^*|, |C^*| > m$), and hence, without loss of generality, we may assume that $|N_G(H_0; C_1)| \ge \lceil m/k \rceil$ (≥ 2).

We first consider the set

$$X = \{x_0\} \cup \{u_i^+ : 1 \le i \le l\}.$$

Then, Lemma 4 implies the following:

(1) X is an independent set of size l + 1.

(2)
$$d_G(x; H \cup C_1) + d_G(x'; H \cup C_1) \le |H \cup C_1| - 1$$
 for $x, x' \in X$ $(x \ne x')$.

On the other hand, by the maximality of $|C^*|$ and Lemma 2, x_0 is non-insertible for $\{C_2, \ldots, C_k\}$. This implies the following:

(3) $d_G(x_0; C_p) \le |C_p|/2$ for $2 \le p \le k$, and hence $d_G(x_0; C^* - C_1) \le |C^* - C_1|/2$.

Since $\sigma_2^{l+1}(G) \ge n$, it follows from (1) that there exist two distinct vertices x and x' in X such that $d_G(x) + d_G(x') \ge n$. Then, by (2), we get

$$d_G(x; C^* - C_1) + d_G(x'; C^* - C_1) \ge n - (|H \cup C_1| - 1) = |C^* - C_1| + 1.$$

Combining this with (3) and the definition of X, we may assume that

(4) $d_G(u_1^+; C^* - C_1) > |C^* - C_1|/2.$

Next, let x_1 be the first non-insertible vertex for $\{C_2, \ldots, C_k\}$ in the path $u_1^- \overleftarrow{C_1} u_l^+$ on $\overleftarrow{C_1}$ (we can take such a vertex by Lemma 3 and the symmetry of $\overrightarrow{C_1}$ and $\overleftarrow{C_1}$), and we consider the set

$$Y = \{x_0, x_1\} \cup \{u_i^- : 2 \le i \le l\}.$$

Then, by the symmetry of $\overrightarrow{C_1}$ and $\overleftarrow{C_1}$, Lemma 4, and since x_1 is non-insertible for $\{C_2, \ldots, C_k\}$, we have the following:

(5) Y is an independent set of size l + 1.

- (6) $d_G(y; H \cup C_1) + d_G(y'; H \cup C_1) \le |H \cup C_1| 1$ for $y, y' \in Y$ $(y \ne y')$.
- (7) $d_G(x_1; C_p) \le |C_p|/2$ for $2 \le p \le k$, and hence $d_G(x_1; C^* C_1) \le |C^* C_1|/2$.

Since $\sigma_2^{l+1}(G) \ge n$, it follows from (5) that there exist two distinct vertices y and y' in Y such that $d_G(y) + d_G(y') \ge n$. Then, by (6), we get

$$d_G(y; C^* - C_1) + d_G(y'; C^* - C_1) \ge n - (|H \cup C_1| - 1) = |C^* - C_1| + 1.$$

Combining this with (3), (7) and the definition of Y, we have the following:

(8) $d_G(u_i^-; C^* - C_1) > |C^* - C_1|/2$ for some *i* with $2 \le i \le l$.

By (4) and (8), we have

$$d_G(u_1^+; C^* - C_1) + d_G(u_i^-; C^* - C_1) > |C^* - C_1| = \sum_{2 \le p \le k} |C_p|.$$

Hence, there exists a cycle C_p $(2 \le p \le k)$, say p = 2, such that

$$d_G(u_1^+; C_2) + d_G(u_i^-; C_2) \ge |C_2| + 1.$$

This implies that there exists an edge uv in $E(C_2)$ such that $u_1^+u, u_i^-v \in E(G)$. By changing the orientation of C_2 if necessary, we may assume that $u^+ = v$. Note that $i \ge 2$, and consider two cycles

$$D_1 = u_i \overrightarrow{C_1} u_1 \overrightarrow{Q_{1,i}} u_i$$
 and $D_2 = u_1^+ \overrightarrow{C_1} u_i^- u^+ \overrightarrow{C_2} u u_1^+$ (see Figure 2).

Then, $D_1, D_2, C_3, \ldots, C_k$ are k disjoint cycles such that the sum of the orders is

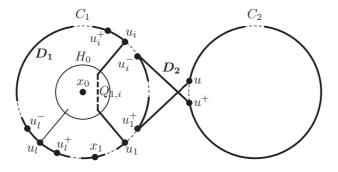


Figure 2: The cycles D_1 and D_2

larger than $|C^*|$, a contradiction.

This completes the proof of Theorem 1. \Box

4 Notes on the order condition

As shown in the argument of the previous section, in the proof of Theorem 1, the order condition " $n \ge 5k-2$ " is required only to show the existence of k disjoint cycles in a graph G (recall that the order condition in Theorem E is also). Therefore, the proof of Theorem 1 actually implies the following.

Theorem 3 Let k be a positive integer, and let G be an m-connected graph of order n. Suppose that G contains k disjoint cycles. If $\sigma_2^{\lceil m/k \rceil + 1} \ge n$, then G has a 2-factor with exactly k cycles.

From this theorem, if we can obtain better results on the cycles packing problem, then the order conditions in Theorem 1 and Remark 2 can be improved. In fact, by using the result of Kierstead, Kostochka and Yeager (2017) and modifying the proof of Lemma 5, we can obtain a sharp order condition for the result in Remark 2 (see Corollary 4).

Theorem F (Kierstead et al. [13]) Let k be an integer with $k \ge 2$, and let G be a graph of order $n \ge 3k$ with $\delta(G) \ge 2k - 1$. Then G contains k disjoint cycles if and only if (i) $\alpha(G) \le n - 2k$, and (ii) if k is odd and n = 3k, then $G \not\cong 2K_k \lor \overline{K_k}$ and if k = 2, then G is not a wheel.

Lemma 6 Let k be a positive integer, and let G be an m-connected graph of order $n \ge 4k-1$. If $\sigma_{\lceil m/k \rceil+1}(G) > \frac{1}{2}(\lceil m/k \rceil+1)(n-1)$, then G contains k disjoint cycles.

Proof of Lemma 6. By a similar argument as in the proof of Lemma 5, we have the following: If k = 1, then we can easily find a cycle; If $\lceil m/k \rceil = 1$ or $\lceil m/k \rceil \ge 3$, then by a theorem of Enomoto [9], G contains k disjoint cycles; If $\lceil m/k \rceil = 2$, and $k \ge 3$ or $n \ge 4k$, then by a theorem of Fujita et al. [10], G contains k disjoint cycles. Thus, we may assume that k = 2, $\lceil m/k \rceil = 2$ and n = 4k - 1 = 7. Then, $\delta(G) \ge m \ge k + 1 = 3 = 2k - 1$ and $\sigma_3(G) > \frac{3}{2}(n-1) = 6k - 3 = 9$. Since n = 7and $\sigma_3(G) > 9$, it follows that $\alpha(G) \le 3 = n - 2k$ and G is not a wheel. Hence, by Theorem F, G contains two disjoint cycles. Thus, the lemma follows. \Box

Recall that $\sigma_t^m(G) \geq \frac{t}{m} \cdot \sigma_m(G)$ for $m \geq t \geq 2$, and hence Theorem 3 and Lemma 6 lead to the following.

Corollary 4 Let k be a positive integer, and let G be an m-connected graph of order $n \ge 4k - 1$. If $\sigma_{\lceil m/k \rceil + 1}(G) > \frac{1}{2}(\lceil m/k \rceil + 1)(n - 1)$, then G has a 2-factor with exactly k cycles.

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