# On degree sum conditions for 2-factors with a prescribed number of cycles 

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#### Abstract

For a vertex subset $X$ of a graph $G$, let $\Delta_{t}(X)$ be the maximum value of the degree sums of the subsets of $X$ of size $t$. In this paper, we prove the following result: Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n \geq 5 k-2$. If $\Delta_{2}(X) \geq n$ for every independent set $X$ of size $\lceil m / k\rceil+1$ in $G$, then $G$ has a 2 -factor with exactly $k$ cycles. This is a common generalization of the results obtained by Brandt et al. [Degree conditions for 2 -factors, J. Graph Theory 24 (1997) 165-173] and Yamashita [On degree sum conditions for long cycles and cycles through specified vertices, Discrete Math. 308 (2008) 6584-6587], respectively.


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## 1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. The independence number and the connectivity of a graph $G$ are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. For a vertex $x$ of a graph $G$, we denote by $d_{G}(x)$ and $N_{G}(x)$ the degree and the neighborhood of $x$ in $G$. Let $\sigma_{m}(G)$ be the minimum degree sum of an independent set of $m$ vertices in a graph $G$, i.e., if $\alpha(G) \geq m$, then

$$
\sigma_{m}(G)=\min \left\{\sum_{x \in X} d_{G}(x): X \text { is an independent set of } G \text { with }|X|=m\right\} ;
$$

[^0]otherwise, $\sigma_{m}(G)=+\infty$. If the graph $G$ is clear from the context, we often omit the graph parameter $G$ in the graph invariant. In this paper, "disjoint" always means "vertex-disjoint".

A graph having a hamilton cycle, i.e., a cycle containing all the vertices of the graph, is said to be hamiltonian. The hamiltonian problem has long been fundamental in graph theory. But, it is NP-complete, and so no easily verifiable necessary and sufficient condition seems to exist. Therefore, many researchers have focused on "better" sufficient conditions for graphs to be hamiltonian (see a survey [14]). In particular, the following degree sum condition, due to Ore (1960), is classical and well known.

Theorem A (Ore [15]) Let $G$ be a graph of order $n \geq 3$. If $\sigma_{2} \geq n$, then $G$ is hamiltonian.

Chvátal and Erdős (1972) discovered the relationship between the connectivity, the independence number and the hamiltonicity.

Theorem B (Chvátal, Erdős [8]) Let $G$ be a graph of order at least 3. If $\alpha \leq \kappa$, then $G$ is hamiltonian.

Bondy [2] pointed out that the graph satisfying the Ore condition also satisfies the Chvátal-Erdős condition, that is, Theorem B implies Theorem A,

By Theorem B, we should consider the degree condition for the existence of a hamilton cycle in graphs $G$ with $\alpha(G) \geq \kappa(G)+1$. In fact, Bondy (1980) gave the following degree sum condition by extending Theorem B.

Theorem C (Bondy [3]) Let $G$ be an m-connected graph of order $n \geq 3$. If $\sigma_{m+1}>\frac{1}{2}(m+1)(n-1)$, then $G$ is hamiltonian.

In 2008, Yamashita [17] introduced a new graph invariant and further generalized Theorem [] as follows. For a vertex subset $X$ of a graph $G$ with $|X| \geq t$, we define

$$
\Delta_{t}(X)=\max \left\{\sum_{x \in Y} d_{G}(x): Y \subseteq X,|Y|=t\right\}
$$

Let $m \geq t$, and if $\alpha(G) \geq m$, then let

$$
\sigma_{t}^{m}(G)=\min \left\{\Delta_{t}(X): X \text { is an independent set of } G \text { with }|X|=m\right\}
$$

otherwise, $\sigma_{t}^{m}(G)=+\infty$. Note that $\sigma_{t}^{m}(G) \geq \frac{t}{m} \cdot \sigma_{m}(G)$.
Theorem D (Yamashita [17]) Let $G$ be an $m$-connected graph of order $n \geq 3$. If $\sigma_{2}^{m+1} \geq n$, then $G$ is hamiltonian.

This result suggests that the degree sum of non-adjacent "two" vertices is important for hamilton cycles.

On the other hand, it is known that a 2 -factor is one of the important generalizations of a hamilton cycle. A 2-factor of a graph is a spanning subgraph in which every component is a cycle, and thus a hamilton cycle is a 2 -factor with "exactly 1 cycle". As one of the studies concerning the difference between hamilton cycles and 2 -factors, in this paper, we focus on 2 -factors with "exactly $k$ cycles". Similar to the situation for hamilton cycles, deciding whether a graph has a 2 -factor with $k(\geq 2)$ cycles is also NP-complete. Therefore, the sufficient conditions for the existence of such a 2 -factor also have been extensively studied in graph theory (see a survey [11]). In particular, the following theorem, due to Brandt, Chen, Faudree, Gould and Lesniak (1997), is interesting. (In the paper [5], the order condition is not " $n \geq 4 k-1$ " but " $n \geq 4 k$ ". However, by using a theorem of Enomoto [9] and Wang [16] ("every graph $G$ of order at least $3 k$ with $\sigma_{2}(G) \geq 4 k-1$ contains $k$ disjoint cycles") for the cycles packing problem, we can obtain the following. See the proof in [5, Lemma 1].)

Theorem E (Brandt et al. [5]) Let $k$ be a positive integer, and let $G$ be a graph of order $n \geq 4 k-1$. If $\sigma_{2} \geq n$, then $G$ has a 2 -factor with exactly $k$ cycles.

This theorem shows that the Ore condition guarantees the existence of a hamilton cycle but also the existence of a 2 -factor with a prescribed number of cycles.

By considering the relation between Theorem A and Theorem E Chen, Gould, Kawarabayashi, Ota, Saito and Schiermeyer [6] conjectured that the Chvátal-Erdős condition in Theorem B also guarantees the existence of a 2 -factor with exactly $k$ cycles (see [6, Conjecture 1]). Chen et al. also proved that if the order of a 2 -connected graph $G$ with $\alpha(G)=\alpha \leq \kappa(G)$ is sufficiently large compared with $k$ and with the Ramsey number $r(\alpha+4, \alpha+1)$, then the graph $G$ has a 2 -factor with $k$ cycles. In [12], Kaneko and Yoshimoto "almost" solved the above conjecture for $k=2$ (see the comment after Theorem E in Chen et al. [6] for more details). Another related result can be found in [7]. But, the above conjecture is still open in general. In this sense, there is a big gap between hamilton cycles and 2-factors with exactly $k(\geq 2)$ cycles.

In this paper, by combining the techniques of the proof for hamiltonicity and the proof for 2 -factors with a prescribed number of cycles, we give the following Yamashita-type condition for 2-factors with $k$ cycles.

Theorem 1 Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n \geq 5 k-2$. If $\sigma_{2}^{[m / k]+1} \geq n$, then $G$ has a 2 -factor with exactly $k$ cycles.

This theorem implies the following:

## Remark 2

- Theorem 1 is a generalization of Theorem D .
- Theorem 1 leads to the Bondy-type condition: If $G$ is an $m$-connected graph of order $n \geq 5 k-2$ with $\sigma_{\lceil m / k\rceil+1}(G)>\frac{1}{2}(\lceil m / k\rceil+1)(n-1)$, then $G$ has a 2 -factor with exactly $k$ cycles. Therefore, Theorem 1 is also a generalization of Theorem E for sufficiently large graphs. (Recall that $\sigma_{t}^{m}(G) \geq \frac{t}{m} \cdot \sigma_{m}(G)$ and $\sigma_{m}(G) \geq \frac{m}{2} \cdot \sigma_{2}(G)$ for $m \geq t \geq 2$.)
- Theorem 1 leads to the Chvátal-Erdős-type condition: If $G$ is a graph of order at least $5 k-2$ with $\alpha(G) \leq\lceil\kappa(G) / k\rceil$, then $G$ has a 2-factor with exactly $k$ cycles.

The complete bipartite graph $K_{(n-1) / 2,(n+1) / 2}(n$ is odd) does not contain a 2factor, and hence the degree condition in Theorem 1 is best possible in this sense. The order condition in Theorem 1 comes from our proof techniques. Similar to the situation for the proof of Theorem E we will use the order condition only for the cycles packing problem (see Lemma 5 and the proof of Theorem 1 in Section 3). The complete bipartite graph $K_{2 k-1,2 k-1}$ shows that $n \geq 4 k-1$ is necessary. In the last section (Section (4), we note that " $n \geq 5 k-2$ " can be replaced with " $n \geq 4 k-1$ " for the Bondy-type condition (and the Chvátal-Erdős-type condition) in Remark 2 ,

Table 1 summarizes the conditions mentioned in the above.

|  | hamilton cycle | 2-factor with $k$ cycles |
| :---: | :---: | :---: |
| Ore-type | $\begin{gathered} \sigma_{2} \geq n \\ \text { Theorem } \boldsymbol{\triangle}(\text { Ore }) \end{gathered}$ | $\begin{gathered} \sigma_{2} \geq n \\ \text { Theorem } \text { (Brandt et al.) } \end{gathered}$ |
| Chvátal-Erdős-type | $\begin{gathered} \alpha \leq \kappa \\ \text { Theorem } \quad \text { (Chvátal and Erdős) } \end{gathered}$ | $\begin{gathered} \alpha \leq\lceil\kappa / k\rceil \\ \text { Remark } 27 \end{gathered}$ |
| Bondy-type | $\overline{\sigma_{\kappa+1}>\frac{1}{2}(\kappa+1)(n-1)}$ <br> Theorem (Bondy) | $\begin{gathered} \sigma_{\lceil\kappa / k\rceil+1}> \\ \frac{1}{2}(\lceil\kappa / k\rceil+1)(n-1) \\ \text { Remark 2 } \end{gathered}$ |
| Yamashita-type | $\begin{gathered} \sigma_{2}^{\kappa+1} \geq n \\ \text { Theorem } \square \text { (Yamashita) } \end{gathered}$ | $\sigma_{2}^{[\kappa / k]+1} \geq n$ <br> Theorem (Main theorem) |

Table 1: Comparison of the degree conditions

To prove Theorem [1, in the next section, we extend the concept of insertible vertices which was introduced by Ainouche [1] and we prove Theorem 1 in Section 3 by using it.

## 2 The concept of insertible vertices

In this section, we prepare terminology and notations and give some lemmas.
Let $G$ be a graph. For $v \in V(G)$ and $X \subseteq V(G)$, we let $N_{G}(v ; X)=N_{G}(v) \cap X$ and $d_{G}(v ; X)=\left|N_{G}(v ; X)\right|$. For $V, X \subseteq V(G)$, let $N_{G}(V ; X)=\bigcup_{v \in V} N_{G}(v ; X)$. For
$X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. An $(x, y)$-path in $G$ is a path from a vertex $x$ to a vertex $y$ in $G$. We write a cycle (or a path) $C$ with a given orientation by $\vec{C}$. If there exists no fear of confusion, we abbreviate $\vec{C}$ by $C$. Let $C$ be an oriented cycle (or path). We denote by $\overleftarrow{C}$ the cycle $C$ with a reverse orientation. For $x \in V(C)$, we denote the successor and the predecessor of $x$ on $\vec{C}$ by $x^{+}$and $x^{-}$. For $x, y \in V(C)$, we denote by $x \vec{C} y$ the $(x, y)$-path on $\vec{C}$. The reverse sequence of $x \vec{C} y$ is denoted by $y \overleftarrow{C} x$. In the rest of this paper, we consider that every cycle (path) has a fixed orientation, unless stated otherwise, and we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$.

The following lemma is obtained by using the standard crossing argument, and so we omit the proof.

Lemma 1 Let $G$ be a graph of order $n$, and let $P$ be an $(x, y)$-path of order at least 3 in $G$. If $d_{G}(x)+d_{G}(y) \geq n$, then $G$ contains a cycle of order at least $|P|$.

In [1], Ainouche introduced the concept of insertible vertices, which has been used for the proofs of the results on hamilton cycles. In this paper, we modify it for 2 -factors with $k$ cycles, and it also plays a crucial role in our proof. Let $G$ be a graph, and let $\mathcal{D}=\left\{D_{1}, \ldots, D_{r+s}\right\}(r+s \geq 1)$ be the set of $r$ cycles and $s$ paths in $G$ which are pairwise disjoint. For a vertex $x$ in $G-\bigcup_{1 \leq p \leq r+s} D_{p}$, the vertex $x$ is insertible for $\mathcal{D}$ if there is an edge $u v$ in $E\left(D_{p}\right)$ such that $x u, x v \in E(G)$ for some $p$ with $1 \leq p \leq r+s$. In the following lemma, "partition" of a graph means a partition of the vertex set.

Lemma 2 Let $G$ be a graph, and let $\mathcal{D}=\left\{D_{1}, \ldots, D_{r+s}\right\}(r+s \geq 1)$ be the set of $r$ cycles and $s$ paths in $G$ which are pairwise disjoint, and $P$ be a path in $G-$ $\bigcup_{1 \leq p \leq r+s} D_{p}$. If every vertex of $P$ is insertible for $\mathcal{D}$, then $G\left[\bigcup_{1 \leq p \leq r+s} V\left(D_{p}\right) \cup V(P)\right]$ can be partitioned into $r$ cycles and $s$ paths.

Proof of Lemma 2. By choosing the following two vertices $u, v \in V(P)$ and the edge $w w^{+} \in \bigcup_{1 \leq p \leq r+s} E\left(D_{p}\right)$ inductively, we can get the desired partition of $G\left[\bigcup_{1 \leq p \leq r+s} V\left(D_{p}\right) \cup V(P)\right]$. Let $u$ be the first vertex along $\vec{P}$, and take an edge $w w^{+}$in $E\left(D_{i}\right)\left(\subseteq \bigcup_{1 \leq p \leq r+s} E\left(D_{p}\right)\right)$ such that $u w, u w^{+} \in E(G)$ for some $i$ with $1 \leq i \leq r+s$ (since $u$ is insertible for $\mathcal{D}$, we can take such an edge). We let $v$ be the last vertex along $\vec{P}$ such that $v w, v w^{+} \in E(G)$ (may be $u=v$ ). Then, we can insert all vertices of $u \vec{P} v$ into $D_{i}$. In fact, by replacing the edge $w w^{+}$by the path $w u \vec{P} v w^{+}$, we can obtain a spanning subgraph $D_{i}^{\prime}$ of $G\left[V\left(D_{i} \cup u \vec{P} v\right)\right]$ such that $D_{i}^{\prime}$ is a cycle if $D_{i}$ is a cycle; otherwise, $D_{i}^{\prime}$ is a path. By the choice of $u$ and $v$, we have $z w \notin E(G)$ or $z w^{+} \notin E(G)$ for each vertex $z$ of $P^{\prime}:=P-u \vec{P} v$, and hence every vertex of $P^{\prime}$ is insertible for $\mathcal{D}^{\prime}=\left\{D_{1}, \ldots, D_{i-1}, D_{i}^{\prime}, D_{i+1}, \ldots, D_{r+s}\right\}$. Thus, we can repeat this argument for the path $P^{\prime}$ and the set $\mathcal{D}^{\prime}$, and we get then the desired
partition.
In the rest of this section, we fix the following. Let $C_{1}, \ldots, C_{k}$ be $k$ disjoint cycles in a graph $G$, and let $C^{*}=\bigcup_{1 \leq p \leq k} C_{p}$. Choose $C_{1}, \ldots, C_{k}$ so that

$$
\left|C^{*}\right|\left(=\sum_{1 \leq p \leq k}\left|C_{i}\right|\right) \text { is as large as possible. }
$$

Suppose that $C^{*}$ does not form a 2-factor of $G$. Let $H=G-C^{*}$, and let $H_{0}$ be a component of $H$ and $x_{0} \in V\left(H_{0}\right)$. Let

$$
u_{1}, u_{2}, \ldots, u_{l} \text { be } l \text { distinct vertices in } N_{G}\left(H_{0} ; C_{1}\right), \text { where } l \geq 2 \text {. }
$$

We assume that $u_{1}, u_{2}, \ldots, u_{l}$ appear in this order on $\overrightarrow{C_{1}}$, and let $u_{l+1}=u_{1}$. Note that by the maximality of $\left|C^{*}\right|, u_{i}^{+} \neq u_{i+1}$ for $1 \leq i \leq l$. We denote by $\overrightarrow{Q_{i}}$ and $\overrightarrow{Q_{i, j}}$ a $\left(u_{i}, x_{0}\right)$-path in $G\left[V\left(H_{0}\right) \cup\left\{u_{i}\right\}\right]$ and a $\left(u_{i}, u_{j}\right)$-path passing through a vertex of $H_{0}$ in $G\left[V\left(H_{0}\right) \cup\left\{u_{i}, u_{j}\right\}\right]$, respectively.

Lemma 3 For $1 \leq i \leq l, u_{i}^{+} \overrightarrow{C_{1}} u_{i+1}^{-}$contains a non-insertible vertex for $\left\{C_{2}, \ldots, C_{k}\right\}$.
Proof of Lemma 3. Suppose that every vertex of $u_{i}^{+} \overrightarrow{C_{1}} u_{i+1}^{-}$is insertible for $\left\{C_{2}, \ldots\right.$, $\left.C_{k}\right\}$. Then, by Lemma 2, $G\left[\bigcup_{2 \leq p \leq k} V\left(\xrightarrow{\left.C_{p}\right) \cup V}\left(u_{i}^{+} \overrightarrow{C_{1}} u_{i+1}^{-}\right)\right]\right.$has a 2 -factor with exactly $k-1$ cycles. With the cycle $u_{i+1} \overrightarrow{C_{1}} u_{i} \overrightarrow{Q_{i, i+1}} u_{i+1}$, we can get $k$ disjoint cycles in $G$ such that the sum of the orders is larger than $\left|C^{*}\right|$, a contradiction.

For $1 \leq i \leq l$, let $x_{i}$ be the first non-insertible vertex for $\left\{C_{2}, \ldots, C_{k}\right\}$ in $V\left(u_{i}^{+} \overrightarrow{C_{1}} u_{i+1}^{-}\right)$on $\overrightarrow{C_{1}}$, i.e., every vertex of $u_{i}^{+} \overrightarrow{C_{1}} x_{i}^{-}$is insertible for $\left\{C_{2}, \ldots, C_{k}\right\}$, but $x_{i}$ is not insertible (Lemma 3 guarantees the existence of such a vertex $x_{i}$ ).

Lemma 4 Let $i, j$ be integers with $1 \leq i, j \leq l$ and $i \neq j$. If $x \in V\left(u_{i}^{+} \overrightarrow{C_{1}} x_{i}\right)$ and $x^{\prime} \in$ $\left\{x_{0}, u_{j}^{+}\right\}$, then (i) $x x^{\prime} \notin E(G)$, and (ii) $d_{G}\left(x ; H \cup C_{1}\right)+d_{G}\left(x^{\prime} ; H \cup C_{1}\right) \leq\left|H \cup C_{1}\right|-1$.

Proof of Lemma 4. Consider the path

$$
\vec{P}=\left\{\begin{array}{ll}
x \overrightarrow{C_{1}} u_{i} \overrightarrow{Q_{i}} x_{0} & \left(\text { if } x^{\prime}=x_{0}\right) \\
x \vec{C}_{1} u_{j} \overleftarrow{Q_{i, j}} u_{i} \overleftarrow{C_{1}} u_{j}^{+} & \text {(if } \left.x^{\prime}=u_{j}^{+}\right)
\end{array} .\right.
$$

See Figure 1. Then, $P$ is a path in $G\left[V\left(H \cup x \overrightarrow{C_{1}} u_{i}\right)\right]$ passing through all vertices of $x \overrightarrow{C_{1}} u_{i}$ and a vertex of $H_{0}$. Recall that every vertex of $u_{i}^{+} \overrightarrow{C_{1}} x^{-}$is insertible for $\left\{C_{2}, \ldots, C_{k}\right\}$, and hence $G\left[\bigcup_{2 \leq p \leq k} V\left(C_{p}\right) \cup V\left(u_{i}^{+} \overrightarrow{C_{1}} x^{-}\right)\right]$has a 2-factor with exactly $k-1$ cycles (by Lemma (2). Hence, the maximality of $\left|C^{*}\right|$ and Lemma 1 yield that

$$
x x^{\prime} \notin E(G) \text { and } d_{G}\left(x ; H \cup x \overrightarrow{C_{1}} u_{i}\right)+d_{G}\left(x^{\prime} ; H \cup x \overrightarrow{C_{1}} u_{i}\right) \leq\left|H \cup x \overrightarrow{C_{1}} u_{i}\right|-1 .
$$



Figure 1: The path $P$

In particular, (i) holds. Then, by applying (i) for each vertex in $u_{i}^{+} \overrightarrow{C_{1}} x^{-}$and the vertex $x^{\prime}$, we have $N_{G}\left(x^{\prime} ; u_{i}^{+} \overrightarrow{C_{1}} x^{-}\right)=\emptyset$. Combining this with the above inequality, we get,

$$
\begin{aligned}
& d_{G}\left(x ; H \cup C_{1}\right)+d_{G}\left(x^{\prime} ; H \cup C_{1}\right) \\
= & d_{G}\left(x ; H \cup x \overrightarrow{C_{1}} u_{i}\right)+d_{G}\left(x^{\prime} ; H \cup x \overrightarrow{C_{1}} u_{i}\right)+d_{G}\left(x ; u_{i}^{+} \overrightarrow{C_{1}} x^{-}\right) \\
\leq & \left(\left|H \cup x \overrightarrow{C_{1}} u_{i}\right|-1\right)+\left|u_{i}^{+} \overrightarrow{C_{1}} x^{-}\right|=\left|H \cup C_{1}\right|-1 .
\end{aligned}
$$

Thus (ii) also holds.

## 3 Proof of Theorem 1

Before proving Theorem [1 we will give the following lemma for the cycles packing problem.

Lemma 5 Let $k, m, n$ and $G$ be the same ones as in Theorem 1. Under the same degree sum condition as Theorem 1, $G$ contains $k$ disjoint cycles.

Proof of Lemma [5. If $k=1$, then it is easy to check that $G$ contains a cycle. If $\lceil m / k\rceil=1$ or $\lceil m / k\rceil \geq 3$, then by a theorem of Enomoto [9], $G$ contains $k$ disjoint cycles (note that if $\lceil m / k\rceil \geq 3$, then $G$ is $(2 k+1)$-connected, that is, the minimum degree $\delta(G)$ is at least $2 k+1$ ). Thus, we may assume that $k \geq 2$ and $\lceil m / k\rceil=2$. Then, we have $\delta(G) \geq m \geq k+1$ and $\sigma_{2}^{3}(G)=\sigma_{2}^{[m / k\rceil+1}(G) \geq n \geq 5 k-2$. Note that, by the definition of $\sigma_{2}^{3}(G)$ and $\sigma_{3}(G), \sigma_{3}(G) \geq \sigma_{2}^{3}(G)+\delta(G)$. Note also that $n \geq 5 k-2 \geq 3 k+2 \geq 8$ because $k \geq 2$. Hence, by a theorem of Fujita et al. [10] ("every graph $G$ of order at least $3 k+2 \geq 8$ with $\sigma_{3}(G) \geq 6 k-2$ contains $k$ disjoint cycles"), we can get the desired conclusion.

Now we are ready to prove Theorem (1)

Proof of Theorem 1. Let $G$ be an $m$-connected graph of order $n \geq 5 k-2$ such that $\sigma_{2}^{[m / k]+1}(G) \geq n$. We show that $G$ has a 2 -factor with exactly $k$ cycles. By Theorem E, we may assume that $\lceil m / k\rceil \geq 2$. By Lemma [5, $G$ contains $k$ disjoint cycles. Let $C_{i}(1 \leq i \leq k), C^{*}, H, H_{0}, x_{0}$ and $u_{i}(1 \leq i \leq l)$ be the same graphs and vertices as the ones described in the paragraph preceding Lemma 3 in Section 2. In particular, we may assume that $l=\lceil m / k\rceil$. Because, since $G$ is $m$-connected, it follows that $\left|N_{G}\left(H_{0} ; C^{*}\right)\right| \geq m$ (note that by the maximality of $\left|C^{*}\right|,\left|C^{*}\right|>m$ ), and hence, without loss of generality, we may assume that $\left|N_{G}\left(H_{0} ; C_{1}\right)\right| \geq\lceil m / k\rceil(\geq 2)$.

We first consider the set

$$
X=\left\{x_{0}\right\} \cup\left\{u_{i}^{+}: 1 \leq i \leq l\right\} .
$$

Then, Lemma 4 implies the following:
(1) $X$ is an independent set of size $l+1$.
(2) $d_{G}\left(x ; H \cup C_{1}\right)+d_{G}\left(x^{\prime} ; H \cup C_{1}\right) \leq\left|H \cup C_{1}\right|-1$ for $x, x^{\prime} \in X\left(x \neq x^{\prime}\right)$.

On the other hand, by the maximality of $\left|C^{*}\right|$ and Lemma 2, $x_{0}$ is non-insertible for $\left\{C_{2}, \ldots, C_{k}\right\}$. This implies the following:
(3) $d_{G}\left(x_{0} ; C_{p}\right) \leq\left|C_{p}\right| / 2$ for $2 \leq p \leq k$, and hence $d_{G}\left(x_{0} ; C^{*}-C_{1}\right) \leq\left|C^{*}-C_{1}\right| / 2$.

Since $\sigma_{2}^{l+1}(G) \geq n$, it follows from (1) that there exist two distinct vertices $x$ and $x^{\prime}$ in $X$ such that $d_{G}(x)+d_{G}\left(x^{\prime}\right) \geq n$. Then, by (2), we get

$$
d_{G}\left(x ; C^{*}-C_{1}\right)+d_{G}\left(x^{\prime} ; C^{*}-C_{1}\right) \geq n-\left(\left|H \cup C_{1}\right|-1\right)=\left|C^{*}-C_{1}\right|+1 .
$$

Combining this with (3) and the definition of $X$, we may assume that

$$
\begin{equation*}
d_{G}\left(u_{1}^{+} ; C^{*}-C_{1}\right)>\left|C^{*}-C_{1}\right| / 2 . \tag{4}
\end{equation*}
$$

 on ${\overleftarrow{C_{1}}}_{1}$ (we can take such a vertex by Lemma 3 and the symmetry of $\overrightarrow{C_{1}}$ and $\overleftarrow{C_{1}}$ ), and we consider the set

$$
Y=\left\{x_{0}, x_{1}\right\} \cup\left\{u_{i}^{-}: 2 \leq i \leq l\right\}
$$

Then, by the symmetry of $\overrightarrow{C_{1}}$ and $\overleftarrow{C_{1}}$, Lemma 四, and since $x_{1}$ is non-insertible for $\left\{C_{2}, \ldots, C_{k}\right\}$, we have the following:
(5) $Y$ is an independent set of size $l+1$.
(6) $d_{G}\left(y ; H \cup C_{1}\right)+d_{G}\left(y^{\prime} ; H \cup C_{1}\right) \leq\left|H \cup C_{1}\right|-1$ for $y, y^{\prime} \in Y\left(y \neq y^{\prime}\right)$.
(7) $d_{G}\left(x_{1} ; C_{p}\right) \leq\left|C_{p}\right| / 2$ for $2 \leq p \leq k$, and hence $d_{G}\left(x_{1} ; C^{*}-C_{1}\right) \leq\left|C^{*}-C_{1}\right| / 2$.

Since $\sigma_{2}^{l+1}(G) \geq n$, it follows from (5) that there exist two distinct vertices $y$ and $y^{\prime}$ in $Y$ such that $d_{G}(y)+d_{G}\left(y^{\prime}\right) \geq n$. Then, by (6), we get

$$
d_{G}\left(y ; C^{*}-C_{1}\right)+d_{G}\left(y^{\prime} ; C^{*}-C_{1}\right) \geq n-\left(\left|H \cup C_{1}\right|-1\right)=\left|C^{*}-C_{1}\right|+1 .
$$

Combining this with (3), (7) and the definition of $Y$, we have the following:
(8) $d_{G}\left(u_{i}^{-} ; C^{*}-C_{1}\right)>\left|C^{*}-C_{1}\right| / 2$ for some $i$ with $2 \leq i \leq l$.

By (4) and (8), we have

$$
d_{G}\left(u_{1}^{+} ; C^{*}-C_{1}\right)+d_{G}\left(u_{i}^{-} ; C^{*}-C_{1}\right)>\left|C^{*}-C_{1}\right|=\sum_{2 \leq p \leq k}\left|C_{p}\right| .
$$

Hence, there exists a cycle $C_{p}(2 \leq p \leq k)$, say $p=2$, such that

$$
d_{G}\left(u_{1}^{+} ; C_{2}\right)+d_{G}\left(u_{i}^{-} ; C_{2}\right) \geq\left|C_{2}\right|+1 .
$$

This implies that there exists an edge $u v$ in $E\left(C_{2}\right)$ such that $u_{1}^{+} u, u_{i}^{-} v \in E(G)$. By changing the orientation of $C_{2}$ if necessary, we may assume that $u^{+}=v$. Note that $i \geq 2$, and consider two cycles

$$
D_{1}=u_{i} \overrightarrow{C_{1}} u_{1} \overrightarrow{Q_{1, i}} u_{i} \text { and } D_{2}=u_{1}^{+} \overrightarrow{C_{1}} u_{i}^{-} u^{+} \overrightarrow{C_{2}} u u_{1}^{+} \text {(see Figure (2). }
$$

Then, $D_{1}, D_{2}, C_{3}, \ldots, C_{k}$ are $k$ disjoint cycles such that the sum of the orders is


Figure 2: The cycles $D_{1}$ and $D_{2}$
larger than $\left|C^{*}\right|$, a contradiction.
This completes the proof of Theorem 1 .

## 4 Notes on the order condition

As shown in the argument of the previous section, in the proof of Theorem (1) the order condition " $n \geq 5 k-2$ " is required only to show the existence of $k$ disjoint cycles in a graph $G$ (recall that the order condition in Theorem 国 is also). Therefore, the proof of Theorem 1 actually implies the following.

Theorem 3 Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n$. Suppose that $G$ contains $k$ disjoint cycles. If $\sigma_{2}^{[m / k]+1} \geq n$, then $G$ has a 2-factor with exactly $k$ cycles.

From this theorem, if we can obtain better results on the cycles packing problem, then the order conditions in Theorem 1 and Remark 2 can be improved. In fact, by using the result of Kierstead, Kostochka and Yeager (2017) and modifying the proof of Lemma 5. we can obtain a sharp order condition for the result in Remark 2 (see Corollary (4).

Theorem F (Kierstead et al. [13]) Let $k$ be an integer with $k \geq 2$, and let $G$ be a graph of order $n \geq 3 k$ with $\delta(G) \geq 2 k-1$. Then $G$ contains $k$ disjoint cycles if and only if (i) $\alpha(G) \leq n-2 k$, and (ii) if $k$ is odd and $n=3 k$, then $G \not \not ⿻ 2 K_{k} \vee \overline{K_{k}}$ and if $k=2$, then $G$ is not a wheel.

Lemma 6 Let $k$ be a positive integer, and let $G$ be an m-connected graph of order $n \geq 4 k-1$. If $\sigma_{\lceil m / k\rceil+1}(G)>\frac{1}{2}(\lceil m / k\rceil+1)(n-1)$, then $G$ contains $k$ disjoint cycles.

Proof of Lemma 6. By a similar argument as in the proof of Lemma 5, we have the following: If $k=1$, then we can easily find a cycle; If $\lceil m / k\rceil=1$ or $\lceil m / k\rceil \geq 3$, then by a theorem of Enomoto [9], $G$ contains $k$ disjoint cycles; If $\lceil m / k\rceil=2$, and $k \geq 3$ or $n \geq 4 k$, then by a theorem of Fujita et al. [10], $G$ contains $k$ disjoint cycles. Thus, we may assume that $k=2,\lceil m / k\rceil=2$ and $n=4 k-1=7$. Then, $\delta(G) \geq m \geq k+1=3=2 k-1$ and $\sigma_{3}(G)>\frac{3}{2}(n-1)=6 k-3=9$. Since $n=7$ and $\sigma_{3}(G)>9$, it follows that $\alpha(G) \leq 3=n-2 k$ and $G$ is not a wheel. Hence, by Theorem ( $\mathrm{F}, G$ contains two disjoint cycles. Thus, the lemma follows.

Recall that $\sigma_{t}^{m}(G) \geq \frac{t}{m} \cdot \sigma_{m}(G)$ for $m \geq t \geq 2$, and hence Theorem 3 and Lemma 6 lead to the following.

Corollary 4 Let $k$ be a positive integer, and let $G$ be an m-connected graph of order $n \geq 4 k-1$. If $\sigma_{\lceil m / k\rceil+1}(G)>\frac{1}{2}(\lceil m / k\rceil+1)(n-1)$, then $G$ has a 2 -factor with exactly $k$ cycles.

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