

Strong fractional choice number of series-parallel graphs

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Abstract

The strong fractional choice number of a graph G is the infimum of those real numbers r such that G is $(\lceil rm \rceil, m)$ -choosable for every positive integer m . The strong fractional choice number of a family \mathcal{G} of graphs is the supremum of the strong fractional choice number of graphs in \mathcal{G} . We denote by \mathcal{Q}_k the class of series-parallel graphs with girth at least k . This paper proves that for $k = 4q - 1, 4q, 4q + 1, 4q + 2$, the strong fractional number of \mathcal{Q}_k is exactly $2 + \frac{1}{q}$.

Keywords: strong fractional choice number; series-parallel graph

1 Introduction

A b -fold colouring of a graph G is a mapping ϕ which assigns to each vertex v of G a set $\phi(v)$ of b colours so that adjacent vertices receive disjoint colour sets. An (a, b) -colouring of G is a b -fold colouring ϕ of G such that $\phi(v) \subseteq \{1, 2, \dots, a\}$ for each vertex v . The *fractional chromatic number* of G is

$$\chi_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-colourable} \right\}.$$

An a -list assignment of G is a mapping L which assigns to each vertex v a set $L(v)$ of a permissible colours. A b -fold L -colouring of G is a b -fold colouring ϕ of G such that $\phi(v) \subseteq L(v)$ for each vertex v . We say G is (a, b) -choosable if for any a -list assignment L of G , there is a b -fold L -colouring of G . The *fractional choice number* of G is

$$ch_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-choosable} \right\}.$$

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It was proved by Alon, Tuza and Voigt [1] that for any finite graph G , $\chi_f(G) = ch_f(G)$ and moreover the infimum in the definition of $ch_f(G)$ is attained and hence can be replaced by minimum. This implies that if G is (a, b) -colourable, then for some integer m , G is (am, bm) -choosable. The integer m depends on G and is usually a large integer. A natural question is for which (a, b) , G is (am, bm) -choosable for any positive integer m . This motivated the definition of strong fractional choice number of a graph [5].

Definition 1 Assume G is a graph and r is a real number. We say G is strongly fractional r -choosable if for any positive integer m , G is $(\lceil rm \rceil, m)$ -choosable. The strong fractional choice number of G is

$$ch_f^*(G) = \inf\{r : G \text{ is strongly fractional } r\text{-choosable}\}.$$

The strong fractional choice number of a class \mathcal{G} of graphs is

$$ch_f^*(\mathcal{G}) = \sup\{ch_f^*(G) : G \in \mathcal{G}\}.$$

It follows from the definition that for any graph G , $ch_f^*(G) \geq ch(G) - 1$. The variant $ch_f^*(G)$ is intended to be a refinement for $ch(G)$. However, currently we do not have a good upper bound for $ch_f^*(G)$ in terms of $ch(G)$. It was conjectured by Erdős, Rubin and Taylor [3] that if G is k -choosable, then G is (km, m) -choosable for any positive integer m . If this conjecture were true, then we would have $ch_f^*(G) \leq ch(G)$. But this conjecture is refuted recently by Dvořák, Hu and Sereni in [2]. Nevertheless, it is possible that for any k -choosable graph G , for any positive integer m , G is $(km + 1, m)$ -choosable. If this is true, we also have $ch_f^*(G) \leq ch(G)$. In any case, $ch_f^*(G)$ is an interesting graph invariant and there are many challenging problems concerning this parameter. The strong fractional choice number of planar graphs were studied in [5] and [4]. Let \mathcal{P} denote the family of planar graphs and for a positive integer k , let \mathcal{P}_k be the family of planar graphs containing no cycles of length k . It was proved in [5] that $5 \geq ch_f^*(\mathcal{P}) \geq 4 + 2/9$ and prove in [4] that $4 \geq ch_f^*(\mathcal{P}_3) \geq 3 + 1/17$.

In this paper, we consider the strong fractional choice number of series-parallel graphs. For a positive integer k , let

$$\mathcal{Q}_k = \{G : G \text{ is a series-parallel graph with girth at least } k\}.$$

This paper proves the following result.

Theorem 1 Assume q is a positive integer. For $k \in \{4q - 1, 4q, 4q + 1, 4q + 2\}$, $ch_f^*(\mathcal{Q}_k) = 2 + \frac{1}{q}$.

2 The proof of Theorem 1

Series-parallel graphs is a well studied family of graphs and there are many equivalent definitions for this class of graphs. For the purpose of using induction, we adopt the definition that recursively construct series-parallel graphs from K_2 by series parallel constructions.

Definition 2 A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.
- (The parallel construction) Let $(G; x, y)$ and $(G'; x', y')$ be two vertex disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying x and x' into a single vertex x'' , and identifying y and y' into a single vertex y'' . Then $(G''; x'', y'')$ is a two-terminal series-parallel graph.
- (The series construction) Let again $(G; x, y)$ and $(G'; x', y')$ be two vertex disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying y and x' into a single vertex x'' . Then $(G''; x, y')$ is a two-terminal series-parallel graph.

A graph is a series-parallel graph if there exist some two vertices x, y such that $(G; x, y)$ is a two-terminal series-parallel graph.

Lemma 1 Assume m, l are positive integers, $\epsilon > 0$ is a real number, $P_l = (v_0, v_1, \dots, v_l)$ is a path and L is a list assignment of path P_l with $|L(v_0)| = m$, $|L(v_i)| = 2m + \epsilon m$ ($1 \leq i \leq l$). For $0 \leq j \leq l$, there is a subset T_j of $L(v_j)$, for which the following holds:

- If $j = 2t + 1$ is odd, then $|T_j| = m + \epsilon m$; If $j = 2t$ is even, then $|T_j| = m$.
- For any m -subset B_j of $L(v_j)$ for which $|B_j \cap T_j| \geq (1 - t\epsilon)m$, there exists an m -fold L -colouring ϕ for $P_j = (v_0, v_1, \dots, v_j)$ such that $\phi(v_j) = B_j$.

Proof. By induction on j . If $j = 0$, then let $T_0 = L(v_0)$; if $j = 1$, then let $T_1 = L(v_1) - T_0$. The conclusion is obviously true.

Assume $j \geq 2$ and the lemma holds for $j' < j$.

Case 1 $j = 2t$ is even.

By induction hypothesis, there is an $(m + \epsilon m)$ -subset T_{2t-1} of $L(v_{2t-1})$, such that for any m -subset B_{2t-1} of $L(v_{2t-1})$ for which $|B_{2t-1} \cap T_{2t-1}| \geq (1 - (t-1)\epsilon)m$, there exists an m -fold L -colouring ϕ for P_{2t-1} such that $\phi(v_{2t-1}) = B_{2t-1}$.

As $|L(v_{2t})| = (2 + \epsilon)m$, we have $|L(v_{2t}) - T_{2t-1}| \geq m$. Let T_{2t} be any m -subset of $L(v_{2t}) - T_{2t-1}$. Assume B_{2t} is an m -subset of $L(v_{2t})$ with $|B_{2t} \cap T_{2t}| \geq (1 - t\epsilon)m$. We shall show that there exists an m -fold L -colouring ϕ for P_{2t} such that $\phi(v_{2t}) = B_{2t}$.

Note that $|B_{2t} \cap T_{2t-1}| \leq m - (1 - t\epsilon)m = t\epsilon m$. So $|T_{2t-1} - B_{2t}| \geq (1 - (t-1)\epsilon)m$. Let B_{2t-1} be an m -subset of $L(v_{2t-1}) - B_{2t}$ containing at least $(1 - (t-1)\epsilon)m$ colours from T_{2t-1} . By induction hypothesis, there exists an m -fold L -colouring ϕ of P_{2t-1} with $\phi(v_{2t-1}) = B_{2t-1}$. Now ϕ extends to an m -fold L -colouring ϕ of P_{2t} with $\phi(v_{2t}) = B_{2t}$.

Case 2 $j = 2t + 1$ is odd.

By induction hypothesis, there is an m -subset T_{2t} of $L(v_{2t})$, such that for any m -subset B_{2t} of $L(v_{2t})$ for which $|B_{2t} \cap T_{2t}| \geq (1 - t\epsilon)m$, there exists an m -fold L -colouring ϕ for P_{2t} such that $\phi(v_{2t}) = B_{2t}$.

As $|L(v_{2t+1})| = (2 + \epsilon)m$, we have $|L(v_{2t+1}) - T_{2t}| \geq (1 + \epsilon)m$. Let T_{2t+1} be any $(1 + \epsilon)m$ -subset of $L(v_{2t+1}) - T_{2t}$. Assume B_{2t+1} is an m -subset of $L(v_{2t+1})$ with $|B_{2t+1} \cap T_{2t+1}| \geq (1 - t\epsilon)m$. We shall show that there exists an m -fold L -colouring ϕ for P_{2t+1} such that $\phi(v_{2t+1}) = B_{2t+1}$.

Note that $|B_{2t+1} \cap T_{2t}| \leq m - (1 - t\epsilon)m = t\epsilon m$. So $|T_{2t} - B_{2t+1}| \geq (1 - t\epsilon)m$. Let B_{2t} be an m -subset of $L(v_{2t}) - B_{2t+1}$ containing at least $(1 - t\epsilon)m$ colours from T_{2t} . By induction hypothesis, there exists an m -fold L -colouring ϕ of P_{2t} with $\phi(v_{2t}) = B_{2t}$. Now ϕ extends to an m -fold L -colouring ϕ of P_{2t+1} with $\phi(v_{2t+1}) = B_{2t+1}$. ■

Corollary 1 Assume m, l are positive integers. Let

$$\epsilon = \begin{cases} \frac{2}{l-1}, & \text{if } l \text{ is odd} \\ \frac{2}{l}, & \text{if } l \text{ is even.} \end{cases}$$

If $P_l = (v_0, v_1, \dots, v_l)$ is a path of length l and L is a list assignment of path P_l with $|L(v_0)| = |L(v_l)| = m$, $|L(v_i)| = 2m + \epsilon m$ for $1 \leq i \leq l-1$, then there is an m -fold L -colouring of P_l .

Proof. We divide the proof into two cases.

Case 1 $l = 2t$ is even.

By Lemma 1, there is an $(m + \epsilon m)$ -subset T_{l-1} of $L(v_{l-1})$, such that for any m -subset B_{l-1} of $L(v_{l-1})$, for which $|B_{l-1} \cap T_{l-1}| \geq (1 - (t-1)\epsilon)m$, there exists an m -fold L -colouring ϕ for $P_{l-1} = (v_0, v_1, \dots, v_{l-1})$, such that $\phi(v_{l-1}) = B_{l-1}$.

As $t\epsilon m = m$, we have $|T_{l-1} - L(v_l)| \geq \epsilon m = (1 - (t-1)\epsilon)m$. So there is an m -subset B_{l-1} of $L(v_{l-1}) - L(v_l)$ containing at least $(1 - (t-1)\epsilon)m$ colours from T_{l-1} . By Lemma 1, there exists an m -fold L -colouring ϕ of P_{l-1} with $\phi(v_{l-1}) = B_{l-1}$. Now ϕ can be extended to an m -fold L -colouring ϕ of P_l with $\phi(v_l) = L(v_l)$.

Case 2 $l = 2t + 1$ is odd.

Let B be an m -subset of $L(v_{l-1}) - L(v_l)$. Let $L'(v_i) = L(v_i)$ for $1 \leq i \leq l-2$ and $L'(v_{l-1}) = B$. By Case 1, $P_{l-1} = (v_0, v_1, \dots, v_{l-1})$ has an m -fold L' -colouring ϕ with $\phi(v_{l-1}) = B$. Now ϕ can be extended to an m -fold L -colouring ϕ of P_l with $\phi(v_l) = L(v_l)$. ■

Lemma 2 If $(G; x, y)$ is a series-parallel graph of girth k and $l = \lceil k/2 \rceil$, then either G itself is a path or G contains a path $P = (v_0, v_1, \dots, v_l)$ of length l such that all the vertices v_1, v_2, \dots, v_{l-1} are degree 2 vertices of G , and none of them is a terminal vertex x or y .

Proof. Assume G contains a cycle C . If $G = C$, then the conclusion is true as C has length at least k . Otherwise, $(G; x, y)$ is obtained from $(G_1; x_1, y_1)$ and $(G_2; x_2, y_2)$ by a series construction or a parallel construction. If one of $(G_1; x_1, y_1)$ and $(G_2; x_2, y_2)$ contains a cycle, then G_1 or G_2 contains a required path. Otherwise, since G contains a cycle, we conclude that $(G; x, y)$ is obtained from $(G_1; x_1, y_1)$ and $(G_2; x_2, y_2)$ by a parallel construction, and for $i = 1, 2$, G_i is a path connecting x_i and y_i . Then G is a cycle, and the conclusion holds. ■

Theorem 2 Assume q, m are positive integers, for any series-parallel graph G with girth at least k , where $k \in \{4q-1, 4q, 4q+1, 4q+2\}$, G is $(\lceil (2 + \frac{1}{q})m \rceil, m)$ -choosable.

Proof. Assume L is a $\lceil (2 + \frac{1}{q})m \rceil$ -list assignment of G . We need to show that G has an m -fold L -colouring. The proof is by induction on the number of vertices of G . If G is a path, then G is $(2m, m)$ -choosable, and we are done. Assume G is not a path. By Lemma 2, G has a path $P = (v_0, v_1, \dots, v_l)$ of length l (where $l = 2q$ when $k \in \{4q-1, 4q\}$; $l = 2q+1$ when $k \in \{4q+1, 4q+2\}$), such that all the vertices v_1, v_2, \dots, v_{l-1} are degree 2 vertices of G . Let $G' = G - \{v_1, v_2, \dots, v_{l-1}\}$. Then G' is also a series-parallel graph of girth at least $4q-1$ or G' is a path. If G' is a series-parallel graph of girth at least $4q-1$, then by induction hypothesis, G' has an m -fold L -colouring ϕ ; If G' is a path, as path is $(2m, m)$ -choosable, so G' also has an m -fold L -colouring ϕ . Let L' be the list assignment of the path $P = (v_0, v_1, \dots, v_l)$ with $L'(v_0) = \phi(v_0)$, $L'(v_l) = \phi(v_l)$ and $L'(v_i) = L(v_i)$ for $1 \leq i \leq l-1$. By Corollary 1, P has an m -fold L' -colouring ψ . Then the union of ϕ and ψ is an m -fold L -colouring of G . ■

By Theorem 2, for $k \in \{4q-1, 4q, 4q+1, 4q+2\}$, the strong fractional choice number of \mathcal{Q}_k is at most $2 + \frac{1}{q}$. In order to show that equality holds, we need to construct, for each positive integer m , a graph belongs to \mathcal{Q}_k , which is not $((2 + \frac{1}{q})m - 1, m)$ -choosable.

Lemma 3 Assume m, l are positive integers and assume that ϵ is a positive real number such that ϵm is an integer and

$$\epsilon < \begin{cases} \frac{2}{l-1}, & \text{if } l \text{ is odd} \\ \frac{2}{l}, & \text{if } l \text{ is even.} \end{cases}$$

Let $P_l = (v_0, v_1, \dots, v_l)$ be a path. Let M_1, M_2 be m -sets. Then there exists a list assignment L of P_l for which the following holds:

- $L(v_0) = M_1$ and $L(v_l) = M_2$.
- $|L(v_i)| = 2m + \epsilon m$ ($1 \leq i \leq l-1$),
- there is no m -fold L -colouring of P_l .

Proof. Let A_r (for $r = 1, 3, 5, \dots, 2q - 3$), B_s (for $s = 2, 4, 6, \dots, 2q - 2$), Z_t (for $t = 1, 3, 5, \dots, 2q - 1$) be disjoint colour sets, where $|A_r| = |B_s| = m$, $|Z_t| = \epsilon m$. Let L be the list assignment of P_l defined as follows:

- $L(v_0) = M_1$, $L(v_l) = M_2$, $L(v_1) = M_1 \cup A_1 \cup Z_1$.
- $|L(v_{2i+1})| = B_{2i} \cup A_{2i+1} \cup Z_{2i+1}$, for $i = 1, 2, 3, \dots, q - 2$.
- $|L(v_{2j})| = B_{2j} \cup A_{2j-1} \cup Z_{2j-1}$, for $j = 1, 2, 3, \dots, q - 1$.
- If $l = 2q$, then $L(v_{2q-1}) = M_2 \cup B_{2q-2} \cup Z_{2q-1}$; if $l = 2q + 1$, then $L(v_{2q}) = M_2 \cup A_{2q-1} \cup Z_{2q-1}$.

We shall show that P_l is not L -colourable.

Claim 1 For any $j \in \{2, 3, 4, \dots, q\}$, if ϕ is an m -fold L -colouring of P_{2j-2} , then $|\phi(v_{2j-2}) \cap B_{2j-2}| \geq m - (j - 1)\epsilon m$.

Proof. We shall prove the claim by induction on the index j . Assume $j = 2$ and ϕ is an m -fold L -colouring of P_2 . As $\phi(v_0) = M_1$, we conclude that $\phi(v_1) \subseteq A_1 \cup Z_1$. Therefore $|\phi(v_2) \cap (A_1 \cup Z_1)| \leq \epsilon m$. Hence $|\phi(v_2) \cap B_2| \geq m - \epsilon m$.

Assume $j \geq 3$ and the claim holds for $j' < j$ and ϕ is an m -fold L -colouring of P_{2j-2} . Apply induction hypothesis to the restriction of ϕ to P_{2j-4} , we conclude that

$$|\phi(v_{2j-4}) \cap B_{2j-4}| \geq m - (j - 2)\epsilon m.$$

Hence $|\phi(v_{2j-3}) \cap B_{2j-4}| \leq (j - 2)\epsilon m$. This implies that

$$|\phi(v_{2j-3}) \cap (A_{2j-3} \cup Z_{2j-3})| \geq m - (j - 2)\epsilon m.$$

Hence

$$|\phi(v_{2j-2}) \cap (A_{2j-3} \cup Z_{2j-3})| \leq (j - 1)\epsilon m.$$

So $|\phi(v_{2j-2}) \cap B_{2j-2}| \geq m - (j - 1)\epsilon m$. ■

Assume $l = 2q$ is even and ϕ is an m -fold L -colouring of P_{2q} . Then $|\phi(v_{2q-2}) \cap B_{2q-2}| \geq m - (q - 1)\epsilon m$. As $\phi(v_{2q}) = M_2$, we conclude that $\phi(v_{2q-1}) \subseteq (B_{2q-2} - \phi(v_{2q-2})) \cup Z_{2q-1}$. But $|(B_{2q-2} - \phi(v_{2q-2})) \cup Z_{2q-1}| \leq (q - 1)\epsilon m + \epsilon m = q\epsilon m < m$, a contradiction.

Assume $l = 2q + 1$ is odd and ϕ is an m -fold L -colouring of P_{2q+1} . By claim 1, $|\phi(v_{2q-2}) \cap B_{2q-2}| \geq m - (q - 1)\epsilon m$. Hence $|\phi(v_{2q-1}) \cap B_{2q-2}| \leq (q - 1)\epsilon m$. This implies that $|\phi(v_{2q-1}) \cap (A_{2q-1} \cup Z_{2q-1})| \geq m - (q - 1)\epsilon m$. As $\phi(v_{2q+1}) = M_2$, we conclude that $\phi(v_{2q}) \subseteq (A_{2q-1} \cup Z_{2q-1}) - \phi(v_{2q-1})$. But $|(A_{2q-1} \cup Z_{2q-1}) - \phi(v_{2q-1})| \leq m + \epsilon m - (m - (q - 1)\epsilon m) = q\epsilon m < m$, a contradiction. ■

For disjoint two-terminal series-parallel graphs $(G_1; l_1, r_1)$ and $(G_2; l_2, r_2)$, we use $G_1 \parallel G_2$ to denote the parallel composition of G_1 and G_2 and $G_1 \bullet G_2$ to denote the series composition of G_1 and G_2 , respectively. For a two-terminal series-parallel graph G , we let $G^{<n>}$ denote the series composition of n copies of G , and let $G_{<n>}$ denote the parallel composition of n copies of G .

Theorem 3 Assume m, q are positive integers and assume that ϵ is a positive real number such that ϵm is an integer and $\epsilon < \frac{1}{q}$. For $k \in \{4q - 1, 4q, 4q + 1, 4q + 2\}$, there exists a graph $G \in \mathcal{Q}_k$, such that G is not $((2 + \epsilon)m, m)$ -choosable.

Proof. Let $p = \binom{(2+\epsilon)m}{m}^2$.

Let graph G be obtained by making parallel composition of p paths with length $\lceil \frac{k}{2} \rceil$. Denote the two terminals by x and y . Then $G \in \mathcal{Q}_k$.

We shall show that G is not $((2 + \epsilon)m, m)$ -choosable. Let X and Y be two $(2m + \epsilon m)$ -sets. Let $L(x) = X$ and $L(y) = Y$. There are p possible m -fold L -colourings of x and y . Each such a colouring ϕ corresponds to one path with length $\lceil \frac{k}{2} \rceil$. In that path, define the list assignment L as in the proof of Lemma 3, by replacing M_1 with $\phi(x)$ and M_2 with $\phi(y)$. Then Lemma 3 implies that no m -fold L -colouring of x and y can be extended to G . ■

By theorem 3, for $k \in \{4q - 1, 4q, 4q + 1, 4q + 2\}$, the strong fractional choice number of \mathcal{Q}_k is at least $2 + \frac{1}{q}$. Combining with theorem 2, this completes the proof of theorem 1.

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