The formula for Turán number of spanning linear forests

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Abstract

Let \mathcal{F} be a family of graphs. The Turán number $ex(n; \mathcal{F})$ is defined to be the maximum number of edges in a graph of order n that is \mathcal{F} -free. In 1959, Erdős and Gallai determined the Turán number of M_{k+1} (a matching of size k+1) as follows:

$$ex(n; M_{k+1}) = \max\left\{ \binom{2k+1}{2}, \binom{n}{2} - \binom{n-k}{2} \right\}.$$

Since then, there has been a lot of research on Turán number of linear forests.

A linear forest is a graph whose connected components are all paths or isolated vertices. Let $\mathcal{L}_{n,k}$ be the family of all linear forests of order n with k edges. In this paper, we prove that

$$ex(n; \mathcal{L}_{n,k}) = \max \left\{ {k \choose 2}, {n \choose 2} - {n - \lfloor \frac{k-1}{2} \rfloor \choose 2} + c \right\},$$

where c = 0 if k is odd and c = 1 otherwise. This determines the maximum number of edges in a non-Hamiltonian graph with given Hamiltonian completion number and also solves two open problems in [22] as special cases.

Moreover, we show that our main theorem implies Erdős-Gallai Theorem and also gives a short new proof for it by the closure and counting techniques. Finally, we generalize our theorem to a conjecture which implies the famous Erdős Matching Conjecture.

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1 Introduction

A graph G is a pair G = (V, E), where V is a finite vertex set and E is a family of 2-element subsets of V. Let \mathcal{F} be a family of graphs. A graph G is called \mathcal{F} -free if for any $F \in \mathcal{F}$, there is no subgraph of G isomorphic to F. The $Tur\'{a}n$ number $ex(n;\mathcal{F})$ is defined to be the maximum number of edges in a graph on n vertices that is \mathcal{F} -free. Tur\'{a}n introduced this problem in [21], and we recommend [12,20] for surveys on Tur\'{a}n problems for graphs and hypergraphs.

A matching M in a graph G is a collection of disjoint edges of G. We denote by $\nu(G)$ the number of edges in a maximum matching of G. For an integer s, K_s , M_s and E_s

denote the complete graph on s vertices, the matching of s edges, and the empty graph on s vertices, respectively. The symbol " \vee " means the join operation. In 1959, Erdős and Gallai [4] determined the Turán number $ex(n; M_{k+1})$. The constructions K_{2k+1} and $K_k \vee E_{n-k}$ show that the bound given below is tight.

Theorem 1.1 (Erdős and Gallai [4]). Let G be a graph on n vertices. If $\nu(G) \leq k$, then

$$e(G) \le \max\left\{ {2k+1 \choose 2}, {n \choose 2} - {n-k \choose 2} \right\}.$$

A matching can also be viewed as a forest of paths with length one. Let P_t be a path on t vertices and $k \cdot P_t$ be k vertex-disjoint copies of P_t . Confirming a conjecture of Gorgol [10], Bushaw and Kettle [2] determined the exact values of $ex(n; k \cdot P_t)$ for n appropriately large relative to k and t. Later, Yuan and Zhang [23] determined the values of $ex(n; k \cdot P_3)$ and characterized all extremal graphs for all k and n. Denote by $L(s_1, \ldots, s_k)$ a linear forest consisting of k vertex-disjoint paths with s_1, \ldots, s_k edges. Lidicky, Liu and Palmer [16] determined Turán number $ex(n; L(s_1, \ldots, s_k))$ when n is sufficiently large. But, the number of vertices in the forbidden linear forest is independent of the order n.

Compared with the shortest path, i.e., an edge, the possible longest path is a Hamiltonian path. Ore [17] proved that $ex(n; C_n) = \binom{n-1}{2} + 1$, where C_n is the cycle of order n. By Ore's theorem, it is easy to prove that $ex(n; P_n) = \binom{n-1}{2}$. The Hamiltonian completion number of a graph G, denoted by h(G), is defined to be the minimum number of edges we have to add to G to make it Hamiltonian. This type of parameter was introduced by Goodman and Hedetniemi in the 1970s, and was studied in the algorithmic and structural aspects, see [8, 9, 13]. In the view of extremal graph theory, a natural generalization of Ore's theorem is the following.

Problem 1.2. What is the maximum number of edges in a graph G on n vertices with $h(G) \ge k \ge 1$?

Throughout the left part, we define a linear forest to be a graph consisting of vertexdisjoint paths or isolated vertices. This type of linear forests is also well studied (see Hu et al. [11]). Let $\mathcal{L}_{n,\geq k}^{-1}$ be the set of all linear forests of order n with at least k edges, and $\mathcal{L}_{n,k}$ be the set of all linear forests of order n with exactly k edges. By the definitions, one can see that " $\mathcal{L}_{n,\geq k}$ -free" is equivalent to " $\mathcal{L}_{n,k}$ -free", and so $ex(n;\mathcal{L}_{n,\geq k}) = ex(n;\mathcal{L}_{n,k})$. In [22], the second author and Yang have pointed out that the solution to Problem 1.2 is equivalent to determining the Turán number $ex(n;\mathcal{L}_{n,k})$. In the same paper, the authors proved that when $n \geq 3k$,

$$ex(n; \mathcal{L}_{n,n-k}) = \binom{n-k}{2} + O(k^2).$$

They also asked the Turán number $ex(n; \mathcal{L}_{n,k})$ for some special cases.

¹The symbol used here is somewhat different from the one in [22]. But we think it is more natural.

Problem 1.3 (Problem 4.1 in [22]). Determine the exact value of $ex(n; \mathcal{L}_{n,n-k+1})$ for k = o(n).

Problem 1.4 (Problem 4.2 in [22]). Let c be a constant satisfying 0 < c < 1. Determine the value of $ex(n; \mathcal{L}_{n,n-k+1})$ for k = cn.

In this paper, we completely determine the Turán number $ex(n; \mathcal{L}_{n,k})$, which solves all these problems above.

Theorem 1.5. For any integers n and k with $1 \le k \le n-1$, we have

$$ex(n; \mathcal{L}_{n,k}) = \max \left\{ \binom{k}{2}, \binom{n}{2} - \binom{n - \left\lfloor \frac{k-1}{2} \right\rfloor}{2} + c \right\},$$

where c = 0 if k is odd, and c = 1 otherwise.

We first show that Theorem 1.5 implies Theorem 1.1. Let G be a graph on n vertices with $\nu(G) \leq k$. Obviously, a linear forest with at least 2k+1 edges has a matching of size at least k+1. Thus, G is $\mathcal{L}_{n,2k+1}$ -free. Therefore, by Theorem 1.5 we have

$$e(H) \le ex(n; \mathcal{L}_{n,2k+1}) = \max\left\{ {2k+1 \choose 2}, {n \choose 2} - {n-k \choose 2} \right\}.$$

The second immediate corollary is Ore's theorem by putting k = n - 1 in Theorem 1.5.

Theorem 1.6 (Ore [17]).
$$ex(n; P_n) = ex(n; \mathcal{L}_{n,n-1}) = {n-1 \choose 2}$$
.

Notations Let G be a graph. For any $S \subset V(G)$, let e(S) be the number of edges with two endpoints in S. Let $\bar{S} = V(G) - S$ and $e(S, \bar{S})$ be the number of edges with one endpoint in S and the other endpoint in \bar{S} . For any $x \in V(G)$ and $S \subset V(G)$, let $d_S(x)$ be the number of neighbours of x in S. Let H_1 and H_2 be two disjoint graphs. The *join* of H_1 and H_2 , denoted by $H_1 \vee H_2$, is defined as $V(H_1 \vee H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \vee H_2) = E(H_1) \cup E(H_2) \cup \{xy : x \in V(H_1), y \in V(H_2)\}$.

The rest of the paper is organized as follows. In Section 2, we determine the exact Turán number of $\mathcal{L}_{n,k}$. In Section 2.1, we give a new proof of Theorem 1.1. In the last section, we give a conjecture which implies the famous Erdős Matching Conjecture.

2 The exact Turán number of $\mathcal{L}_{n,k}$

Our proof of Theorem 1.5 is mainly based on the closure technique, which is initiated by Bondy and Chvátal [3] in 1976. But, the key ingredient is motivated by the counting technique from [15]. For more references on closure technique, we refer to [14, 15, 18, 19].

Let G be a graph of order n, P a property defined on G, and k a positive integer. A property P is said to be k-stable, if whenever G+uv has the property P and $d_G(u)+d_G(v) \ge k$, then G itself has the property P. We define the k-closure of G, denoted by $cl_k(G)$, to be the graph H obtained by iteratively joining non-adjacent vertices with degree sum at

least k until $d_H(u) + d_H(v) < k$ for all $uv \notin E(H)$. Then it is easy to see that: if P is k-stable and $cl_k(G)$ has P then G has P. Since the Turán number $ex(n; \mathcal{F})$ is defined to be the maximum number of edges in all graphs with the property \mathcal{F} -free, if " \mathcal{F} -free" is k-stable for some \mathcal{F} , then we can determine $ex(n; \mathcal{F})$ by finding the maximum number of edges in all k-closures. We call this approach the closure technique for Turán problems. In the rest of this section, we determine the Turán number of $\mathcal{L}_{n,k}$ by this approach exactly.

2.1 The property $\mathcal{L}_{n,k}$ -free is k-stable

In this subsection, we prove that the property $\mathcal{L}_{n,k}$ -free is k-stable. For simplicity, we view isolated vertices as paths of length zero, whose end vertices are the same.

Lemma 2.1 (Wang and Yang [22]). Suppose that G is a graph that contains a linear forest F with k-1 edges. If u and v are vertices that are endpoints of different paths in F and $d(u) + d(v) \ge k$, then G contains a linear forest with k edges.

Lemma 2.2. Let G be a graph on n vertices. Suppose that $u, v \in V(G)$ with $d(u) + d(v) \ge k$. Then G is $\mathcal{L}_{n,k}$ -free if and only if G + uv is $\mathcal{L}_{n,k}$ -free.

Proof. If G + uv is $\mathcal{L}_{n,k}$ -free, then clearly G is $\mathcal{L}_{n,k}$ -free. Therefore we only need to verify the other direction.

Suppose G + uv is not $\mathcal{L}_{n,k}$ -free. Then G + uv contains a linear forest F with k edges. If uv is not in F, then F is also a linear forest in G, which contradicts the fact that G is $\mathcal{L}_{n,k}$ -free. If uv is in F, then F - uv is a linear forest with k-1 edges in G. Moreover, u and v are endpoints of different paths in F - uv. Since $d(u) + d(v) \geq k$, by Lemma 2.1 we can find a linear forest with k edges in G, completing the proof.

2.2 The proof of Turán number of $\mathcal{L}_{n,k}$

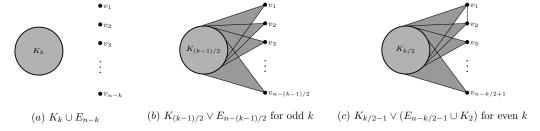


Fig. 1. Extremal graphs for the Turán number of $\mathcal{L}_{n,k}$.

Proof of Theorem 1.5. It is easy to see that when k is odd, $K_k \cup E_{n-k}$ and $K_{(k-1)/2} \vee E_{n-(k-1)/2}$ are two extremal graphs for the Turán number of $\mathcal{L}_{n,k}$, as shown in Fig. 1. (a) and (b). When k is even, $K_k \cup E_{n-k}$ and $K_{k/2-1} \vee (E_{n-k/2-1} \cup K_2)$ are two extremal graphs for the Turán number of $\mathcal{L}_{n,k}$, as shown in Fig. 1. (a) and (c). Thus, we only need to show that the result is the upper bound.

Let G be an $\mathcal{L}_{n,k}$ -free graph on n vertices and G' the k-closure of G. By Lemma 2.2, G' is also $\mathcal{L}_{n,k}$ -free. Let C be the set of all vertices in G' with degree at least $\lceil \frac{k}{2} \rceil$. Then

C forms a clique in G'. Let S be the set of vertices in a maximal clique that contains C in G'. Denote s = |S|. It is easy to see that $s \leq k$; otherwise G' contains a linear forest with k edges, which contradicts the fact that G' is $\mathcal{L}_{n,k}$ -free.

Let $\bar{S} = V(G') - S$. For any $x \in \bar{S}$, we want to give an upper bound on $d_{G'}(x)$. On one hand, since x is not in C, we have $d_{G'}(x) \leq \lceil \frac{k}{2} \rceil - 1$. On the other hand, since S is a maximal clique and x is not in S, there must exist a vertex $v \in S$ such that $xv \notin E(G')$. It follows that $d_{G'}(x) + d_{G'}(v) \leq k - 1$. As $d_{G'}(v) \geq s - 1$, we have that $d_{G'}(x) \leq k - s$. Consequently, $d_{G'}(x) \leq \min\{\lceil \frac{k}{2} \rceil - 1, k - s\}$. The proof splits into two cases, depending on the size of s.

Case 1. $s \leq \lceil \frac{k-1}{2} \rceil$.

For any $x \in \overline{S}$, it follows that $d_{G'}(x) \leq \lceil \frac{k}{2} \rceil - 1$. Since S is a maximal clique, we have $d_S(x) \leq s - 1$. The following equality depends on a trick to estimate the edges outside the clique S, which will be used for several times in the following:

$$e(\bar{S}) + e(\bar{S}, S) = \sum_{x \in \bar{S}} d_S(x) + \frac{1}{2} \sum_{x \in \bar{S}} d_{\bar{S}}(x)$$

$$= \frac{1}{2} \sum_{x \in \bar{S}} d_S(x) + \frac{1}{2} \sum_{x \in \bar{S}} (d_S(x) + d_{\bar{S}}(x))$$

$$= \frac{1}{2} \sum_{x \in \bar{S}} (d_S(x) + d_{G'}(x)).$$

Thus, the number of edges in G' can be bounded as follows:

$$e(G') = e(S) + e(\bar{S}) + e(\bar{S}, S)$$

$$= e(S) + \frac{1}{2} \sum_{x \in \bar{S}} (d_S(x) + d_{G'}(x))$$

$$\leq {s \choose 2} + \frac{1}{2} \left(s - 1 + \left\lceil \frac{k}{2} \right\rceil - 1 \right) (n - s).$$

Let $f(s) = {s \choose 2} + \frac{1}{2} (s - 1 + \lceil \frac{k}{2} \rceil - 1) (n - s)$. Then

$$f'(s) = \frac{n+1}{2} - \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil > 0.$$

It follows that the function f(s) is monotonically increasing. Thus we get

$$e(G') \le f\left(\left\lceil \frac{k-1}{2} \right\rceil\right) = \left(\left\lceil \frac{k-1}{2} \right\rceil\right) + \left(\frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil\right) - 1\right)\left(n - \left\lceil \frac{k-1}{2} \right\rceil\right).$$

If k is odd, then

$$e(G') \le {k-1 \choose 2} + {k \choose 2} + {k \choose 2} - 1 \left(n - {k-1 \over 2}\right) < {k-1 \choose 2} + {k-1 \over 2} \left(n - {k-1 \over 2}\right).$$

If k is even, then

$$e(G') \le {k \choose 2} + {k \choose 2} + {k \choose 2} - 1 \left(n - \frac{k}{2}\right) = {k \choose 2} + {k \choose 2} + {k \choose 2} - 1 \left(n - \frac{k}{2} + 1\right).$$

Case 2. $\lceil \frac{k+1}{2} \rceil \le s \le k$.

It follows that $d_{G'}(x) \leq k - s$ for $x \in \bar{S}$. Therefore,

$$e(G') \le e(S) + \sum_{x \in \bar{S}} d_{G'}(x) \le {s \choose 2} + (k-s)(n-s).$$

Let $f(s) = \binom{s}{2} + (k-s)(n-s)$. Since $f''(s) = 3 \ge 0$, f(s) is a convex function. Thus, we can bound the number of edges of G' as follows:

$$e(G') \le \max \left\{ f(k), f\left(\left\lceil \frac{k+1}{2} \right\rceil\right) \right\}$$
$$= \max \left\{ \binom{k}{2}, \binom{\left\lceil \frac{k+1}{2} \right\rceil}{2} + \left\lfloor \frac{k-1}{2} \right\rfloor \left(n - \left\lceil \frac{k+1}{2} \right\rceil\right) \right\}.$$

If k is odd, then

$$e(G') \le \max\left\{ \binom{k}{2}, \binom{\frac{k+1}{2}}{2} + \frac{k-1}{2} \left(n - \frac{k+1}{2}\right) \right\}$$

= $\max\left\{ \binom{k}{2}, \binom{\frac{k-1}{2}}{2} + \frac{k-1}{2} \left(n - \frac{k-1}{2}\right) \right\}.$

If k is even, then

$$e(G') \le \max\left\{ \binom{k}{2}, \binom{\frac{k}{2}+1}{2} + \left(\frac{k}{2}-1\right) \left(n - \frac{k}{2} - 1\right) \right\}$$

$$= \max\left\{ \binom{k}{2}, \binom{\frac{k}{2}-1}{2} + \left(\frac{k}{2} - 1\right) \left(n - \frac{k}{2} + 1\right) + 1 \right\}.$$

Combining the two cases above, we complete the proof of Theorem 1.5.

3 A short proof of Turán number of matchings

In [1], Akiyama and Frankl gave a short proof of Theorem 1.1 by using the shifting method. Here we shall give a short and new proof for it. Our motivation is to focus on the powerful closure technique.

The proof of the following lemma is easy and short, see [3].

Lemma 3.1 (Bondy and Chvátal [3]). Let G be a graph on n vertices. For any two nonadjacent vertices $u, v \in V(G)$, if whenever $\nu(G+uv) = k+1$ and $d_G(u)+d_G(v) \geq 2k+1$, then $\nu(G) = k+1$.

A new proof of Theorem 1.1. Let G be a graph on n vertices with $\nu(G) = k$ and G' the (2k+1)-closure of G. By Lemma 3.1, we have $\nu(G') = k$. Let G be the set of all vertices in G' with degree at least k+1, and let G be the set of vertices in a maximal clique that contains G in G'. Denote S = |S|. Obviously, $S \leq 2k+1$; otherwise $V(G') \geq k+1$, a contradiction.

Let $\bar{S} = V(G') - S$. For any $x \in \bar{S}$, on one hand, $d_{G'}(x) \leq k$ since x is not in C. On the other hand, as S is a maximal clique and $x \notin S$, there exists a vertex $v \in S$ such that $xv \notin E(G')$. It follows that $d_{G'}(x) + d_{G'}(v) \leq 2k$. As $d_{G'}(v) \geq s - 1$, we have $d_{G'}(x) \leq 2k - s + 1$. Consequently, $d_{G'}(x) \leq \min\{k, 2k - s + 1\}$. The proof is divided into two cases.

Case 1. s < k + 1.

Recall that $d_{G'}(x) \leq k$. For any $x \in \bar{S}$, since S is a maximal clique, we also have $d_S(x) \leq s - 1$. Thus

$$e(G') = e(S) + e(\bar{S}) + e(\bar{S}, S) \le {s \choose 2} + \frac{1}{2} \sum_{x \in \bar{S}} (d_S(x) + d_{G'}(x)) \le {s \choose 2} + \frac{1}{2} (s - 1 + k)(n - s).$$

Let $f(s) = {s \choose 2} + \frac{1}{2}(s-1+k)(n-s)$. As f(s) is monotonically increasing, we obtain

$$e(G') < f(k+1) = {k \choose 2} + k(n-k).$$

Case 2. $k+1 \le s \le 2k+1$.

Recall $d_{G'}(x) \leq 2k - s + 1$ for $x \in \bar{S}$. Thus

$$e(G') \le e(S) + \sum_{x \in \bar{S}} d_{G'}(x) \le {s \choose 2} + (2k - s + 1)(n - s).$$

Let $f(s) = {s \choose 2} + (2k - s + 1)(n - s)$. As f(s) is a convex function, we can obtain

$$e(G') \le \max\{f(2k+1), f(k+1)\} = \max\left\{\binom{2k+1}{2}, \binom{k}{2} + k(n-k)\right\}.$$

Combining these two cases, we complete the proof of Theorem 1.1.

4 Concluding remarks

Let $M_k^{(r)}$ be an r-graph with exact k disjoint edges. The famous Erdős Matching Conjecture can be expressed as a Turán function as follows:

$$ex_r(n; M_{k+1}^{(r)}) = \max\left\{ \binom{rk+r-1}{r}, \binom{n}{r} - \binom{n-k}{r} \right\}.$$

The case k = 1 is the classic Erdős-Ko-Rado Theorem [5]; the case r = 1 is trivial and the case r = 2 is the Erdős-Gallai Theorem [4]. The current best record on Erdős Matching Conjecture is due to Frankl, see [7]. For references on this topic, see ones within [7].

Define a tight linear forest to be an r-graph consisting of vertex-disjoint tight paths or isolated vertices. When r=2, it reduced to the linear forest in graphs. Let $\mathcal{L}_{n,k}^r$ be the family of all tight linear forests of order n with at least k edges.

We can view the tight linear forest as an intermediate concept between matching and Hamilton tight cycle. Motivated by this fact, the second author proposed the following conjecture which implies Erdős Matching Conjecture.

Conjecture 4.1 (Wang). Let $\mathcal{L}_{n,k}^r$ be the set of all r-linear forests of order n with at least k edges. For k = mr + 1 and $m \ge 1$,

$$ex_r(n; \mathcal{L}_{n,k}^{(r)}) = \max\left\{ \binom{k+r-2}{r}, \binom{n}{r} - \binom{n-(k-1)/r}{r} \right\}.$$

Our main result in this paper shows that Conjecture 4.1 is true for r=2.

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