

Signless Laplacian spectral radius and fractional matchings in graphs*

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May 28, 2019

Abstract

A fractional matching of a graph G is a function f giving each edge a number in $[0, 1]$ such that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each vertex $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to v . The fractional matching number of G , written $\alpha'_*(G)$, is the maximum value of $\sum_{e \in E(G)} f(e)$ over all fractional matchings. In this paper, we investigate the relations between the fractional matching number and the signless Laplacian spectral radius of a graph. Moreover, we give some sufficient spectral conditions for the existence of a fractional perfect matching.

1 Introduction

Graphs considered in this paper are simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. As usual, $d(u)$ stands for the degree of a vertex u in G . The *adjacent matrix* of G is $A(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if i and j are adjacent, and $a_{ij} = 0$ otherwise. The *diagonal matrix* of G is $D(G) = (d(i))_{n \times n}$, where $d(i)$ is the degree of vertex i . Let $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$, $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ and $q_1(G) \geq q_2(G) \geq \dots \geq q_n(G)$ be the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, where $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$. Particularly, the eigenvalues $\lambda_1(G)$, $\mu_1(G)$ and $q_1(G)$ are called the *spectral radius*, *Laplacian spectral radius* and *signless Laplacian spectral radius* of G , respectively. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S , and let $G - S$ be the graph obtained from G by deleting the vertices in S together with their incident edges. The *complement graph* G^c of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G . For any two vertex-disjoint graphs G_1 and G_2 , we use $G_1 \vee G_2$ to denote the *join* of graphs G_1 and G_2 and $G_1 \cup G_2$ to denote the *disjoint union* of graphs G_1 and G_2 .

An edge set M of G is called a *matching* if any two edges in M have no common vertices. If each vertex of G is incident with exactly one edge of M , then M is called a *perfect matching* of G . The *matching number* of a graph G , denoted by $\alpha'(G)$, is the number of edges in a maximum matching. A fractional matching of a graph G is a function f giving each edge a number in $[0, 1]$

*This work was supported by the National Natural Foundation of China [61773020]. Corresponding author: Yingui Pan(panygui@163.com)

such that $\sum_{e \in \Gamma(v)} f(e) \leq 1$ for each vertex $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to v . The fractional matching number of G , written $\alpha'_*(G)$, is the maximum value of $\sum_{e \in E(G)} f(e)$ over all fractional matchings f . A *fractional perfect matching* of a graph G is a fractional matching f with $\alpha'_*(G) = \sum_{e \in E(G)} f(e) = \frac{n}{2}$, and a fractional perfect matching f of a graph G is a perfect matching if it takes only the values 0 or 1.

Fractional matching has attracted many researchers' attention. Behrend et al. [3] established a lower bound on the fractional matching number of a graph with given some graph parameters and characterized the graphs whose fractional matching number attains the lower bound. Choi et al. [7] gave the tight upper bounds on the difference and ratio of the fractional matching number and matching number among all n -vertex graphs, and characterized the infinite family of graphs where equalities hold. O [8] investigated the relations between the spectral radius of a connected graph with minimum degree δ and its fractional matching number, and gave a lower bound on the fractional matching number in terms of the spectral radius and minimum degree. Xue [11] studied the connections between the fractional matching number and the Laplacian spectral radius of a graph, and obtained some lower bounds on the fractional matching number of a graph. Moreover, they presented some sufficient spectral conditions for the existence of a fractional perfect matching.

Motivated by [8, 11], we investigate the relations between the signless Laplacian spectral radius of a graph and its fractional matching number. In Section 2, we list some useful lemmas. In Section 3, we establish a lower bound on the fraction number of a graph in terms of its signless Laplacian spectral radius and minimum degree. In Section 4, we obtain some sufficient spectral conditions for the existence of a fractional perfect matching.

2 Preliminaries

In this section, we list some lemmas which will be used in our paper later. Some fundamental properties of fractional matching were obtained in [9].

Lemma 2.1. [9] *For any graph G , let $\alpha'_*(G)$ be the fractional matching number of G . Then*

- (i) $2\alpha'_*(G)$ is an integer.
- (ii) $\alpha'_*(G) = \frac{1}{2}(n - \max\{i(G - S) - |S|\})$, where the maximum is taken over all $S \subseteq V(G)$.

Lemma 2.2. [10] *Let G be a connected graph. If H is a subgraph of G , then $q_1(H) \leq q_1(G)$.*

Lemma 2.3. [6] *Let K_n be a complete graph of order n , where $n \geq 2$. Then $q_1(K_n) = 2n - 2$.*

We now explain the concepts of the equitable matrix and equitable partition.

Definition 2.4. [2] *Let M be a real matrix of order n described in the following block form*

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{pmatrix},$$

where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$ and $n = n_1 + \dots + n_t$. For $1 \leq i, j \leq t$, let b_{ij} denote the average row sum of M_{ij} , i.e. b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then $B(M) = (b_{ij})$ (simply by B) is called the quotient matrix of M . If for each pair i, j , M_{ij} has constant row sum, then B is called the equitable quotient matrix of M and the partition is called equitable.

Lemma 2.5. [2] Let M be a symmetric real matrix. If M has an equitable partition and B is the corresponding matrix, then each eigenvalue of B is also an eigenvalue of M .

The relation between $\lambda_1(B)$ and $\lambda_1(M)$ is obtained as below.

Lemma 2.6. [12] Let B be the equitable matrix of M as defined in Definition 2.4, and M be a nonnegative matrix. Then $\lambda_1(B) = \lambda_1(M)$.

O [8] constructed a family of connected bipartite graphs $\mathcal{H}(\delta, k)$, where δ and k are two positive integers. For each graph $G \in \mathcal{H}(\delta, k)$ with the bipartition $V(G) = V_1 \cup V_2$, G satisfies the following conditions:

- (i) every vertex in V_1 has degree δ ,
- (ii) $|V_1| = |V_2| + k$,
- (iii) the degrees of vertices in V_2 are equal.

The exact values of the fractional matching number and the spectral radius for graphs in $\mathcal{H}(\delta, k)$ are obtained as below.

Lemma 2.7. [8] If $H \in \mathcal{H}(\delta, k)$, then $\alpha'_*(H) = \frac{|V(H)|-k}{2}$ and $\lambda_1(H) = \delta \sqrt{1 + \frac{2k}{|V(H)|-k}}$.

We now determine the signless Laplacian spectral radius of graphs in $\mathcal{H}(\delta, k)$.

Lemma 2.8. If $H \in \mathcal{H}(\delta, k)$, then $q_1(H) = \frac{2\delta|V(H)|}{|V(H)|-k}$.

Proof. If $H \in \mathcal{H}(\delta, k)$, by the partition $V(H) = V_1 \cup V_2$, we can obtain the quotient matrix of $Q(H)$:

$$B = \begin{pmatrix} \delta & \delta \\ \frac{\delta|V_1|}{|V_2|} & \frac{\delta|V_1|}{|V_2|} \end{pmatrix}.$$

It is easy to calculate that $\lambda_1(B) = \delta(1 + \frac{|V_1|}{|V_2|}) = \delta(2 + \frac{k}{|V_2|})$. By the construction of $\mathcal{H}(\delta, k)$, the partition $V(H) = V_1 \cup V_2$ is equitable and $|V_2| = \frac{|V(H)|-k}{2}$. By Lemma 2.6, we have

$$q_1(H) = \lambda_1(Q(H)) = \lambda_1(B) = \delta \left(2 + \frac{k}{|V_2|} \right) = \delta \left(2 + \frac{2k}{|V(H)|-k} \right) = \frac{2\delta|V(H)|}{|V(H)|-k}$$

as desired. □

3 A relationship between $q_1(G)$ and $\alpha'_*(G)$

In this section, we investigate the relationship between the signless Laplacian spectral radius of a graph with minimum degree δ and its fractional matching number. Similar to the proof of Lemma 3.2 in [8], we can obtain the following lemma.

Lemma 3.1. *Let G be an n -vertex connected graph with minimum degree δ , and let k be a real number between 0 and n . If $q_1(G) < \frac{2n\delta}{n-k}$, then $\alpha'_*(G) > \frac{n-k}{2}$.*

Proof. If $\alpha'_*(G) \leq \frac{n-k}{2}$, by Lemma 2.1, there exists a vertex set $S \subseteq V(G)$ such that $i(G-S) - |S| \geq k$. Since $i(G-S)$ is an integer, then $i(G-S) - |S| \geq \lceil k \rceil$. Let A be the set of all isolated vertices in $G-S$. Then,

$$|A| = i(G-S) \geq |S| + \lceil k \rceil.$$

Consider the bipartite subgraph H with the partitions $V(H) = A \cup S$ such that $E(H)$ is the set of edges of G having one endpoint in A and the other in S . Let r be the number of edges in H . Then $r \geq \delta|A|$. For the partition $V(H) = A \cup S$, we can obtain a quotient matrix of $Q(H)$ as below:

$$B = \begin{pmatrix} \frac{r}{|A|} & \frac{r}{|A|} \\ \frac{r}{|S|} & \frac{r}{|S|} \end{pmatrix}.$$

It is easy to calculate that $\lambda_1(B) = \frac{r(|A|+|S|)}{|A||S|}$. Since the partition is equitable, by Lemma 2.5, we have

$$q_1(G) = \lambda_1(Q(G)) \geq \lambda_1(B) = \frac{r(|A|+|S|)}{|A||S|} \geq \delta \frac{|A|+|S|}{|S|} \geq \delta \frac{2|S| + \lceil k \rceil}{|S|} \geq \delta \left(2 + \frac{2\lceil k \rceil}{n-k} \right) \geq \frac{2n\delta}{n-k}$$

since $r \geq \delta|A|$, $|A| \geq |S| + \lceil k \rceil$, $n \geq |A| + |S| \geq 2|S| + k$ and $|S| \geq \delta$. \square

Theorem 3.2. *If G is an n -vertex graph with minimum degree δ , then we have*

$$\alpha'_*(G) \geq \frac{n\delta}{q_1(G)},$$

with equality if and only if $k = \frac{n(q_1(G)-2\delta)}{q_1(G)}$ is an integer and G is an element of $\mathcal{H}(\delta, k)$.

Proof. By Lemma 3.1, $\alpha'_*(G) > \frac{n-k}{2}$ if $q_1(G) < \frac{2n\delta}{n-k}$. Note that $\frac{2n\delta}{n-k}$ is an increasing function of k on $[0, n)$, thus $\frac{2n\delta}{n-k}$ decreases towards $q_1(G)$ as k decreases towards z , where $z = \frac{n(q_1(G)-2\delta)}{q_1(G)}$. Then for each value $k \in (z, n)$, we have $\alpha'_*(G) > \frac{n-k}{2}$ by Lemma 3.1. Let k tend to z and finally equal to z , we obtain $\alpha'_*(G) \geq \frac{n\delta}{q_1(G)}$ as desired.

If $k = \frac{n(q_1(G)-2\delta)}{q_1(G)}$ is an integer and $G \in \mathcal{H}(\delta, k)$, then by Lemma 2.7, we have $\alpha'_*(G) = \frac{n-k}{2} = \frac{n\delta}{q_1(G)}$. For the 'only if' part, assume that $\alpha'_*(G) = \frac{n\delta}{q_1(G)}$. Then $k = z$ and the inequalities in Lemma 3.1 become equality. Since $\lceil k \rceil = k$, k must be an integer. In addition, note that $r = \delta|A|$, $|A| = |S| + k$, $n = 2|S| + k$ and $|S| = \delta$, G must be included in $\mathcal{H}(\delta, k)$. \square

Let G be a bipartite graph with partition $V(G) = V_1 \cup V_2$. Then G is said to be semi-regular if all vertices in V_i have the same degree d_i for $i = 1, 2$.

Lemma 3.3. [5] *Let G be a connected graph. Then*

$$q_1(G) \leq \max\{d(u) + d(v) : uv \in E(G)\},$$

with equality if and only if G is a regular bipartite graph or a semi-regular bipartite graph.

Let $g(G)$ be the length of a shortest cycle in G , and let $\alpha(G)$ be the independence number of G which is the cardinality of the largest independent set of G . Similar to the proof of Theorem 2.6 in [11], we can obtain the following theorem.

Theorem 3.4. *Let G be a graph with independence number $\alpha(G)$. If $g(G) \geq 5$, then $q_1(G) < 2 + \alpha(G)$.*

Proof. Without loss of generality, assume that G is connected and $d(u_1) + d(v_1) = \max\{d(u) + d(v) : uv \in E(G)\}$. Let $A = N(u_1) \setminus \{v_1\}$ and $B = N(v_1) \setminus \{u_1\}$. Since $g(G) \geq 5$, then $|A| + |B| \leq \alpha(G)$ and thus $d(u_1) + d(v_1) = 2 + |A| + |B| \leq 2 + \alpha(G)$. By Lemma 3.3, $q_1(G) \leq 2 + \alpha(G)$. If $q_1(G) = 2 + \alpha(G)$, then $\alpha(G) = |A| + |B|$ and thus G is bipartite regular or semi-regular. Suppose that $|A| \geq |B|$ for convenience. Let w_1 be a vertex of B . Then $u_1, w_1 \in B$ since both u_1 and w_1 are adjacent to v_1 . Since G is regular or semi-regular, then $d(u_1) = d(w_1)$ and thus $|N(w_1) \setminus \{v_1\}| = |A|$. Note that $N(w_1) \cup A$ is an independent set of G , then $\alpha(G) \geq |N(w_1) \cup A| = 2|A| + 1$, a contradiction to the fact $\alpha(G) = |A| + |B|$. \square

Together with Theorems 3.2 and 3.4, we obtain a lower bound on the fractional matching number in terms of the independence number and minimum degree, which improves the lower bound obtained in [11].

Corollary 3.5. *Let G be a connected graph with independence number $\alpha(G)$ and minimum degree δ . If $g(G) \geq 5$, then*

$$\alpha'_*(G) > \frac{n\delta}{\alpha(G) + 2}.$$

4 Signless Laplacian spectral radius and fractional perfect matching

Some sufficient condition for the existence of a fractional perfect matching in a graph in terms of the spectral radius were obtained in [11]. In this section, we are devoted to give some sufficient conditions for a graph to have a fractional perfect matching from the viewpoint of signless Laplacian spectral radius.

Theorem 4.1. *Let G be an n -vertex connected graph with minimum degree δ . If $q_1(G) < \frac{2n\delta}{n-1}$, then G has a fractional perfect matching.*

Proof. If $q_1(G) < \frac{2n\delta}{n-1}$, then it follows from Lemma 3.1 that $\alpha'_*(G) > \frac{n-1}{2}$. By Lemma 2.1, $2\alpha'_*(G)$ is an integer, then $\alpha'_*(G) = \frac{n}{2}$, which means that G has a fractional perfect matching. \square

We now give a sufficient condition for the existence of a fractional perfect matching in a graph in terms of the signless Laplacian spectral radius of its complement.

Theorem 4.2. *Let G be an n -vertex connected graph with minimum degree δ and G^c be the complement of G . If $q_1(G^c) < 2\delta$, then G has a fractional perfect matching.*

Proof. Assume to the contrary that $\alpha'_*(G) < \frac{n}{2}$. By Lemma 2.1, there exists a vertex set $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Denote by A the set of isolated vertices in $G - S$. Note that the neighbours of each isolated vertex belong to S , then $|S| \geq \delta$, which implies that $|A| \geq |S| + 1 \geq \delta + 1$. Since $G^c[A]$ is a clique, by Lemmas 2.2 and 2.3, we have

$$q_1(G^c) \geq q_1(G^c[A]) = 2(|A| - 1) = 2\delta,$$

a contradiction. This completes the proof. \square

Theorem 4.3. *Let G be an n -vertex connected graph with minimum degree δ and G^c be the complement of G . If $q_1(G^c) < 2\delta + 1$, then G has a fractional perfect matching unless $G \cong H_1 \vee H_2$, where H_1 is a $(\delta + 1)$ -independent set and H_2 is any graph of order δ .*

Proof. Suppose that $\alpha'_*(G) < \frac{n}{2}$. By Lemma 2.1, there exists a vertex set $S \subseteq V(G)$ such that $i(G - S) - |S| > 0$. Let A be the set of isolated vertices in $G - S$. Then $|A| \geq |S| + 1 \geq \delta + 1$. If $|A| \geq \delta + 2$, then there is a clique of order $\delta + 2$ in G^c and thus $q_1(G^c) \geq 2(\delta + 1)$, a contradiction. Furthermore, we have $|A| = |S| + 1 = \delta + 1$. If $V(G) \neq A \cup S$, then there is a clique of order $\delta + 2$ in G^c and thus $q_1(G^c) \geq 2(\delta + 1)$, a contradiction. Hence, we have $V(G) = A \cup S$. Therefore, we have $G \cong H_1 \vee H_2$. This completes the proof. \square

For regular graphs, the authors in [1, 4] investigated the relations between the eigenvalues and the perfect matching. Here, we obtain the relations between the eigenvalues and the fractional perfect matching for regular graphs.

Theorem 4.4. *Let G be an n -vertex connected k -regular graph with eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. If*

$$\lambda_3 \leq \begin{cases} k - 1 + \frac{3}{k+1}, & \text{if } k \text{ is even;} \\ k - 1 + \frac{4}{k+2}, & \text{if } k \text{ is odd,} \end{cases}$$

then G has a fractional perfect matching.

Proof. Assume that there exists not a fractional perfect matching in G . Then G has no perfect matching. Similar to the proof of the Theorem 4.8.9 in [2], we can get a contradiction. \square

By Theorem 4.4, we can get the following corollary immediately.

Corollary 4.5. *A regular graph with algebraic connectivity at least one has a fractional perfect matching.*

Acknowledgement(s)

The authors would like to express their sincere gratitude to all the referees for their careful reading and insightful suggestions.

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