ON HAAR DIGRAPHICAL REPRESENTATIONS OF GROUPS

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ABSTRACT. In this paper we extend the notion of digraphical regular representations in the context of Haar digraphs. Given a group G, a Haar digraph Γ over G is a bipartite digraph having a bipartition $\{X, Y\}$ such that G is a group of automorphisms of Γ acting regularly on X and on Y. We say that G admits a Haar digraphical representation (HDR for short), if there exists a Haar digraph over G such that its automorphism group is isomorphic to G. In this paper, we classify finite groups admitting a HDR.

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1. INTRODUCTION

By a digraph Γ , we mean an ordered pair (V, A) where the vertex set V is a non-empty set and the arc set $A \subseteq V \times V$ is a binary relation on V. The elements of V and A are called vertices and arcs of Γ , respectively. For simplicity, we write $V(\Gamma) := V$ and $A(\Gamma) := A$. An automorphism of Γ is a permutation σ of V fixing A setwise, that is, $(x^{\sigma}, y^{\sigma}) \in A$ for every $(x, y) \in A$. The digraph Γ is a graph if the binary relation A is symmetric.

A digraph is called *regular* if each vertex has the same out-valency and the same in-valency. Throughout this paper, all groups and digraphs are finite, and all digraphs are regular.

Let G be a group and let S be a subset of G. The Cayley digraph $\Gamma := \operatorname{Cay}(G, R)$ is the digraph with $V(\Gamma) := G$ and with $A(\Gamma) := \{(g, rg) \mid g \in G, r \in R\}$. The right regular representation of G gives rise to an embedding of G into Aut(Γ) and we identify G with its image under this permutation representation. We say that a group admits a (di)graphical regular representation (resp. GRR or DRR for short) if there exists a Cayley (di)graph Γ over G such that Aut(Γ) = G. Babai [2] proved that, except for $Q_8, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4$ and \mathbb{Z}_3^2 , every finite group admits a DRR. It is clear that, if a group G admits a GRR, then G admits a DRR, however the converse is not true. Indeed, despite the natural argument used by Babai for the classification of groups admitting a DRR, the classification of groups admitting a GRR has required considerable more work. For some of the most influential papers along the way we refer to [15, 16, 20, 21]. Watkins [25] observed that there are two infinite families of graphs admitting no GRR: generalised dicyclic groups, and abelian groups of exponent greater than two. Then, Hetzel [12] has proved that besides these two infinite families, among soluble groups, there are only 13 more groups admitting no GRR. Finally, Godsil [11] has put the last piece into the puzzle and has shown that every non-solvable group admits a GRR, and so completed the classification of groups admitting a GRR.

Once the classification of DRRs and GRRs was completed, researchers proposed and investigated various natural generalisations. For instance, Babai and Imrich [3] have classified finite groups admitting a tournament regular representation, TRR for short. Morris and Spiga [18, 19, 22], answering a question of Babai [2], have classified the finite groups admitting an oriented regular representation, ORR for short. For more results, generalising the classical DRR and GRR classification in various direction, we refer to [5, 7, 6, 17, 23, 24, 26].

We now describe the generalisation we intend to investigate in this paper. Let G be a permutation group on a set Ω and let $\omega \in \Omega$. Denote by G_{ω} the stabilizer of ω in G, that is, the subgroup of

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G fixing ω . We say that G is semiregular on Ω if $G_{\omega} = 1$ for every $\omega \in \Omega$, and regular if it is semiregular and transitive. An *m*-Cayley (di)graph Γ over a group G is defined as a (di)graph which has a semiregular group of automorphisms isomorphic to G with m orbits on its vertex set. When m = 1, 1-Cayley (di)graphs are the usual Cayley (di)graphs. We say that a group G admits a (di)graphical m-semiregular representation (DmSR and GmSR, for short), if there exists a regular m-Cayley (di)graph Γ over G such that $\operatorname{Aut}(\Gamma) \cong G$. In particular, D1SRs and G1SRs are the usual GRRs and DRRs. For each $m \in \mathbb{N}$, we have classified in [6] the finite groups admitting a DmSR and the finite groups admitting a GmSR. In this paper we propose a natural variant of this problem.

A bipartite 2-Cayley (di)graph (over a group G, where the two parts of the bipartition are the two orbits of G) is known as Haar (di)graph in the literature. We say that a finite group G admits a Haar (di)graphical representation (resp. HDR or HGR for short), if there exists a Haar (di)graph over G such that its automorphism group isomorphic to G.

Theorem 1.1. With the only exceptions of \mathbb{Z}_1 , \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_2^2 and \mathbb{Z}_2^3 , every finite group admits a HDR.

Du *et al* [8, Lemma 2.6(i)] have shown that Haar graphs over abelian groups are Cayley graphs. Hence, abelian groups do not admit HGRs. Estélyi [9, Proposition 11] has proved that the dihedral group of order 2n admits a HGR if and only if $n \ge 8$. To end this section, we propose the following problem.

Problem 1.2. Classify finite groups admitting a HGR.

We are not sure what the answer to this problem might be, but besides the finite abelian groups we are aware of no infinite family of groups admitting no HGR. For instance, every generalised quaternion group of order 4n with $4 \le n \le 100$ admits a HGR.

2. Preliminaries and notation

In what follows, we describe some preliminary results which will be used later. We start by recalling Babai's classification of DRRs.

Theorem 2.1. [2, Theorem 2.1] A finite group G admits a DRR if and only if G is not isomorphic to one of the following five groups Q_8 , \mathbb{Z}_2^2 , \mathbb{Z}_2^3 , \mathbb{Z}_2^4 or \mathbb{Z}_3^2 .

We recall that a tournament is a digraph Γ such that, for every two distinct vertices $x, y \in V(\Gamma)$, exactly one of (x, y) and (y, x) is in $A(\Gamma)$. Observe that the Cayley digraph Cay(G, R) is a tournament if and only if $R \cap R^{-1} = \emptyset$ and $R \cup R^{-1} = G \setminus \{1\}$. In particular, finite groups of even order have no TRR.

Theorem 2.2. [3, Theorem 1.5] A finite group of odd order admits a TRR if and only if it is not isomorphic to \mathbb{Z}_3^2 .

Let G be a group. Consistently throughout the whole paper, for not making our notation too cumbersome to use, we denote the element (g, i) of the Cartesian product $G \times \{0, 1\}$ simply by g_i . In particular, we write $G_0 = G \times \{0\} = \{g_0 \mid g \in G\}$ and $G_0 = G \times \{1\} = \{g_1 \mid g \in G\}$.

Let S and T be subsets of G. We define

$$\operatorname{Haar}(G, S, T)$$

to be the digraph having vertex set $G \times \{0,1\} = G_0 \cup G_1$ and having arc set the union of $\{(g_0, (sg)_1) \mid g \in G, s \in S\}$ and $\{(g_1, (tg)_0) \mid g \in G, t \in T\}$. Now, G induces a subgroup of Aut(Haar(G, S, T)) by defining:

 $(h_i)^g = (hg)_i$, for every $g, h \in G$ and $i \in \{0, 1\}$.

For not making the notation too cumbersome, we identify G with this subgroup of Aut(Haar(G, S, T)). Clearly, G acts semiregularly with two orbits G_0 and G_1 on V(Haar(G, S, T)). In particular, Haar(G, S, T) is a Haar digraph over G. It is not hard to see that every Haar digraph over Gis isomophic to Haar(G, S, T), for some suitable subsets S and T of G.

For every automorphism α of G and for every $x, y \in G$, we define two permutations $\delta_{\alpha,x}$ and $\sigma_{\alpha,y}$ of $G_0 \cup G_1$ by setting

(2.1)
$$\delta_{\alpha,x} : \begin{cases} g_0 \mapsto (g^{\alpha})_0, & \forall g \in G, \\ g_1 \mapsto (xg^{\alpha})_1, & \forall g \in G, \end{cases}$$
$$\sigma_{\alpha,y} : \begin{cases} g_0 \mapsto (g^{\alpha})_1, & \forall g \in G, \\ g_1 \mapsto (yg^{\alpha})_0, & \forall g \in G. \end{cases}$$

The permutation $\delta_{\alpha,x}$ will play little role in this paper, but $\sigma_{\alpha,y}$ will be rather important. Then, we define

$$X := \{ \delta_{\alpha,x} \mid S^{\alpha} = x^{-1}S \text{ and } T^{\alpha} = Tx \},$$

$$Y := \{ \sigma_{\alpha,y} \mid S^{\alpha} = y^{-1}T \text{ and } T^{\alpha} = Sy \}.$$

We conclude this section by reporting a result describing the normaliser in Aut(Haar(G, S, T)) of G.

Proposition 2.3. ([1, Theorem 1] and [13, Lemma 2.1]) Let G be a finite group and let S and T be subsets of G, then

$$\mathbf{N}_{\operatorname{Aut}(\operatorname{Haar}(G,S,T))}(G) = GL = \{g\ell \mid g \in G, \ell \in L\},\$$

where $L = X \cup Y$ and $L \cap G = 1$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Lemma 3.1. Let G be a finite group and let S be a subset of G. The Haar digraph Haar(G, S, S) is vertex transitive and hence Haar(G, S, S) is not a HDR.

Proof. Let $\Gamma := \text{Haar}(G, S, S)$ and let ϕ be the permutation of $V(\Gamma) = G_0 \cup G_1$ with $g_0 \mapsto g_1$ and $g_1 \mapsto g_0$, for each $g \in G$.

For every $g \in G$ and $s \in S$, $(g_0, (sg)_1)^{\phi} = (g_1, (sg)_0)$ and $(g_1, (sg)_0)^{\phi} = (g_0, (sg)_1)$ are arcs of Γ and hence ϕ is an automorphism of Γ interchanging G_0 and G_1 . As G is transitive on G_0 and G_1 , we deduce that $\langle G, \phi \rangle$ is transitive on $V(\Gamma)$. Hence Γ is vertex transitive and Γ is not a HDR. \Box

Notation 3.1. Let G be a finite group and let $\phi \in \text{Sym}(G)$ be a permutation of G. We let ϕ' be the permutation of $G_0 \cup G_1$ defined by

 $(g_i)^{\phi'} = (g^{\phi})_i$, for each $g \in G$ and for each $i \in \{0, 1\}$.

Lemma 3.2. Let G be a finite group and let $\phi \in \text{Sym}(G)$. Then, $\phi' \in \text{Aut}(\text{Haar}(G, S, T))$ if and only if $\phi \in \text{Aut}(\text{Cay}(G, S)) \cap \text{Aut}(\text{Cay}(G, T))$.

Proof. Let $\Sigma_1 := \operatorname{Cay}(G, S), \Sigma_2 := \operatorname{Cay}(G, T)$ and $\Gamma := \operatorname{Haar}(G, S, T)$. The permutation ϕ lies in $\operatorname{Aut}(\operatorname{Cay}(G, S)) \cap \operatorname{Aut}(\operatorname{Cay}(G, T))$ if and only if

$$(g, sg)^{\phi} = (g^{\phi}, (sg)^{\phi}) \in A(\Sigma_1) \text{ and } (g, tg)^{\phi} = (g^{\phi}, (tg)^{\phi}) \in A(\Sigma_2),$$

for each $g \in G$, $s \in S$ and $t \in T$. This happens if and only if, for each $s \in S$ and $t \in T$, there exist $s' \in S$ and $t' \in T$ with

$$(sg)^{\phi} = s'g^{\phi}$$
 and $(tg)^{\phi} = t'g^{\phi}$.

In turn, this happens if and only if $(g_0, (sg)_1)^{\phi'} = ((g^{\phi})_0, ((sg)^{\phi})_1) = ((g^{\phi})_0, (s'g^{\phi})_1) \in A(\Gamma)$ and $(g_1, (tg)_0)^{\phi'} = ((g^{\phi})_1, ((tg)^{\phi})_0) = ((g^{\phi})_1, (t'g^{\phi})_0) \in A(\Gamma)$, that is, $\phi' \in \operatorname{Aut}(\Gamma)$.

Lemma 3.3. Let G be a finite group admitting no DRR. Then G admits a HDR except when G is isomorphic to either \mathbb{Z}_2^2 or \mathbb{Z}_2^3 .

Proof. By Theorem 2.1, G is isomorphic to one of the following groups: Q_8 , \mathbb{Z}_3^2 , \mathbb{Z}_2^2 , \mathbb{Z}_2^3 or \mathbb{Z}_2^4 . It can be verified with the computer algebra system MAGMA [4] that \mathbb{Z}_2^2 and \mathbb{Z}_2^3 admit no HDR.

When $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \cong \mathbb{Z}_2^4$, it can be verified with MAGMA that

 $Haar(G, \{1, a, b, c, d, ab\}, \{1, a, c, bd, abc, bcd\})$

is a HDR. Similarly, when $G = \langle a, b \mid a^4 = b^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle \cong Q_8$,

Haar
$$(G, \{1, a, b\}, \{a^2, b^3, ab\})$$

is a HDR and, when $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_3^2$,

Haar $(G, \{1, a, b\}, \{a, b^2, ab\})$

is a HDR.

Notation 3.2. Let Γ be a digraph and let v be a vertex of Γ . We denote by $\Gamma^+(v)$ and by $\Gamma^-(v)$ the out-neighbourhood and the in-neighbourhood of v in Γ .

Lemma 3.4. Let G be a finite group and let R be a subset of G with Cay(G, R) a DRR of G, $1 \notin R$ and |R| < |G|/2. Let L be a subset of $G \setminus (R^{-1} \cup \{1\})$ with |L| = |R| and let $\Gamma := Haar(G, R \cup \{1\}, L \cup \{1\})$. Then

- (1) $\Gamma^+(g_i) \cap \Gamma^-(g_i) = \{g_{1-i}\}, \text{ for every } g \in G \text{ and for every } i \in \{0,1\},$
- (2) $|\operatorname{Aut}(\Gamma) : G| \le 2$,
- (3) Γ is a HDR if and only if $R^{\alpha} \neq L$ for each $\alpha \in \operatorname{Aut}(G)$, and
- (4) the subgroup of $\operatorname{Aut}(\Gamma)$ fixing G_0 and G_1 setwise is G.

Proof. From the definition of the arc set of $\text{Haar}(G, R \cup \{1\}, L \cup \{1\})$, for every $g \in G$, we have

$$\Gamma^+(g_0) = (Rg \cup \{g\})_1 = \{(rg)_1 \mid r \in R \cup \{1\}\},\$$

$$\Gamma^-(g_0) = (L^{-1}g \cup \{g\})_1 = \{(l^{-1}g)_1 \mid l \in L \cup \{1\}\}$$

Applying this with g := 1, we obtain

$$\Gamma^+(1_0) = \{r_1 \mid r \in R \cup \{1\}\} \text{ and } \Gamma^-(1_0) = \{(l^{-1})_1 \mid l \in L \cup \{1\}\}.$$

Since $L \subseteq G \setminus (R^{-1} \cup \{1\})$, we have $(R \cup \{1\}) \cap (L^{-1} \cup \{1\}) = \{1\}$ and hence

$$\Gamma^+(1_0) \cap \Gamma^-(1_0) = \{1_1\}.$$

With a similar argument, we have $\Gamma^+(1_1) \cap \Gamma^-(1_1) = \{1_0\}$. Now, since G is transitive on G_0 and on G_1 , we deduce (1). In particular, each automorphism of Γ fixing g_i must fix also g_{1-i} .

Let $A := \operatorname{Aut}(\Gamma)$ and let A^+ be the subgroup of A fixing G_0 and G_1 setwise. Clearly, $|A : A^+| \leq 2$. Observe that each element φ of A^+ is uniquely determined by a pair (φ_0, φ_1) of permutations of G, where φ_0 and φ_1 are defined by the rules $(g^{\varphi_0})_0 = (g_0)^{\varphi}$ and $(g^{\varphi_1})_1 = (g_1)^{\varphi}$, for each $g \in G$. From (1), we deduce that, for each $\varphi \in A^+$, we have $\varphi_0 = \varphi_1$ and hence, using Notation 3.1, every element of A^+ is of the form ϕ' , for some $\phi \in \operatorname{Sym}(G)$.

Let $\phi' \in A^+$, for some $\phi \in \text{Sym}(G)$. By Lemma 3.2, ϕ induces an automorphism of $\text{Cay}(G, R \cup \{1\})$ and hence $\phi \in \text{Aut}(\text{Cay}(G, R \cup \{1\})) = \text{Aut}(\text{Cay}(G, R)) = G$, because Cay(G, R) is a DRR. Therefore $A^+ \leq G$ and hence $A^+ = G$. This proves (2) and (4).

Suppose there exists $\alpha \in \operatorname{Aut}(G)$ with $R^{\alpha} = L$. Then the mapping $\sigma_{\alpha,1}$ defined in (2.1) is an automorphism of Γ interchanging G_0 and G_1 . Hence $A = \langle G, \sigma_{\alpha,1} \rangle > G$ and Γ is not a HDR. Conversely, suppose Γ is not a HDR. Since $A^+ = G$ and $|A : A^+| \leq 2$, we deduce $|A : A^+| = 2$, Γ is vertex transitive and $G \leq A$. In particular, there exists $\phi \in A$ with $1_0^{\phi} = 1_1$. From (1), we

deduce $1_1^{\phi} = 1_0$. As $\phi \in A = \mathbf{N}_A(G)$, by Proposition 2.3, there exist $y \in G$ and $\alpha \in \operatorname{Aut}(G)$ with $\phi = \sigma_{\alpha,y}$. Now, $1_0 = 1_1^{\phi} = 1_1^{\sigma_{\alpha,y}} = y_0$ and hence y = 1. Furthermore, the definition of $\sigma_{\alpha,y}$ in (2.1) gives $(R \cup \{1\})^{\alpha} = y^{-1}(L \cup \{1\}) = L \cup \{1\}$ and hence $R^{\alpha} = L$. Now, (3) is also proven. \Box

Lemma 3.5. Let G be a finite group of order at least 4 admitting a DRR. Then G has a subset R with Cay(G, R) a DRR, $1 \notin R$ and |R| < (|G| - 1)/2.

Proof. Let R be a subset of G of cardinality as small as possible with Cay(G, R) a DRR. Since

$$Aut(Cay(G, R \cup \{1\})) = Aut(Cay(G, R)) = G,$$

we have $1 \notin R$. Similarly, since

$$\operatorname{Aut}(\operatorname{Cay}(G, G \setminus (R \cup \{1\}))) = \operatorname{Aut}(\operatorname{Cay}(G, R)) = G,$$

we have $|R| \leq |G \setminus (R \cup \{1\})|$, that is, $1 \leq |R| < |G|/2$. If |G| is even, then |R| < (|G| - 1)/2. Therefore, we may assume |G| is odd and $|G| \geq 5$. In particular, G is solvable by the Odd Order Theorem [10].

If G is cyclic (generated by a say), then $Cay(G, \{a\})$ is a directed cycle. Thus $Cay(G, \{a\})$ a DRR over G and $1 = |\{a\}| < (|G| - 1)/2$.

Suppose G is not cyclic. Let M be a maximal normal subgroup of G. As G is solvable, G/M is cyclic of order p, for some odd prime p. Let $g \in G \setminus M$ and observe that

$$G = \langle M, g \rangle.$$

Assume $M \cong \mathbb{Z}_3^2$. Then $G = \langle a, b, g \rangle$ with o(a) = o(b) = 3, ab = ba and p dividing o(g). From [2, Lemma 3.4] and from the proof of [2, Lemma 3.1], G has a subset R with $\operatorname{Cay}(G, R)$ a DRR, $1 \notin R$ and |R| = 9. Clearly, |R| = 9 < (|G| - 1)/2, because $|G| = 9p \ge 27$.

Assume $M \not\cong \mathbb{Z}_3^2$. By Proposition 2.2, M has a subset S such that $\operatorname{Cay}(M, S)$ is a TRR. In particular, |S| = (|M| - 1)/2 and $S \cap S^{-1} = \emptyset$. Let $R := S \cup \{g\}$, let $\Sigma := \operatorname{Cay}(G, R)$ and let $B := \operatorname{Aut}(\Sigma)$. For every $s \in S$, neither (g, s) nor (s, g) is an arc of Σ and, for every $s_1, s_2 \in S$, exactly one of (s_1, s_2) and (s_2, s_1) is an arc of Σ . Therefore, g is the unique isolated vertex in the neighbourhood of 1 in Σ . Then, the vertex stabiliser B_1 fixes g and fixes S setwise. Therefore, B_1 fixes $M = S \cup S^{-1} \cup \{1\}$ setwise and hence B_1 induces a group of automorphisms on $\Sigma[M]$ (the subgraph induced by Σ on M). Since $\Sigma[M] = \operatorname{Cay}(M, S)$ is a TRR, we deduce $B_1 = 1$ and hence Σ is a DRR over G with |R| = (|M| - 1)/2 + 1 < (|G| - 1)/2.

Proof of Theorem 1.1. We divide the proof in various cases.

CASE 1: G has no DRR.

By Lemma 3.3, G has a HDR except when G is isomorphic to \mathbb{Z}_2^2 or \mathbb{Z}_2^3 .

For the rest of the proof, we may suppose that G admits a DRR.

CASE 2: G is a elementary abelian 2-group, that is, $G \cong \mathbb{Z}_2^m$, for some $m \ge 0$.

By Proposition 2.1, $m \in \{0,1\}$ or $m \geq 5$. A direct inspection shows that $\mathbb{Z}_2^0 = \mathbb{Z}_1$ and $\mathbb{Z}_2^1 = \mathbb{Z}_2$ admit no HDR. In particular, we may suppose that $G = \langle a_1, \ldots, a_m \rangle$ with $m \geq 5$.

When m = 5, a computation with MAGMA shows that

Haar
$$(G, \{1, a_1, a_2, a_3, a_4, a_1a_2, a_5\}, \{1, a_1, a_3, a_2a_4, a_1a_2a_3, a_2a_3a_4, a_5\})$$

is a HDR. Suppose then $m \ge 6$ and let

 $R := \{a_1, a_2, \dots, a_m\} \cup \{a_1a_2, a_2a_3, \dots, a_{m-1}a_m\} \cup \{a_1a_2a_{m-2}a_{m-1}, a_1a_2a_{m-1}a_m\}.$

By [14], the Cayley graph Cay(G, R) is a GRR over G with |R| = m + (m-1) + 2 = 2m + 1. Let $H := \langle a_2, \ldots, a_m \rangle$ and observe that

$$|H \setminus R| = 2^{m-1} - (2m-2) > 2m+1,$$

because $m \ge 6$. Therefore, there exists a subset $L \subseteq H \setminus (R \cup \{1\}) \subseteq G \setminus (R \cup \{1\}) = G \setminus (R^{-1} \cup \{1\})$ with |L| = |R|.

Let $\Gamma := \text{Haar}(G, R \cup \{1\}, L \cup \{1\})$. Since $\langle L \rangle \neq G$ and $\langle R \rangle = G$, we have $R^{\alpha} \neq L$ for each $\alpha \in \operatorname{Aut}(G)$. In particular, Lemma 3.4 gives that Γ is a HDR.

In what follows, we assume G is not an elementary abelian 2-group and hence G has an element of order at least 3.

CASE 3: G is cyclic of order 3.

An easy inspection shows that G admits no HDR.

For the remaining cases, from Lemma 3.5, we see that G admits a DRR $\operatorname{Cay}(G, R)$ with $1 \notin R$ and $1 \leq |R| < (|G|-1)/2$. We partition the set R into two subsets. We let $J := \{x \in R \mid x^{-1} \notin R\}$ and $K := R \setminus J$. Observe that $R \setminus J = K$ is inverse-closed, that is, $K^{-1} = \{x^{-1} \mid x \in K\} = K$. Summing up,

$$R = J \cup K, \ R \cap R^{-1} = K = K^{-1} \text{ and } J \cap J^{-1} = J \cap K = \emptyset.$$

CASE 4: There exists a subset L of $G \setminus (R^{-1} \cup \{1\})$ with |L| = |R| and with $R^{\alpha} \neq L$, for every $\alpha \in \operatorname{Aut}(G).$

By Lemma 3.4, Haar $(G, R \cup \{1\}, L \cup \{1\})$ is a HDR.

For the rest of the proof, we may suppose that, for every subset L of $G \setminus (R^{-1} \cup \{1\})$ with |L| = |R|, there exists $\alpha \in \operatorname{Aut}(G)$ with $R^{\alpha} = L$. Let

$$H := G \setminus (\{1\} \cup R \cup R^{-1})$$

Observe that $G = \{1\} \cup (R \cup R^{-1}) \cup H$ is a partition of G and

$$|H| = |G| - 1 - |R \cup R^{-1}| = |G| - 1 - (|R| + |R^{-1}| - |R \cap R^{-1}|) = |G| - 1 - (2|R| - |K|)$$

Since 2|R| < |G| - 1, we deduce |H| > |K|.

CASE 5: There exists $x \in H$ with o(x) > 3.

Let U be any subset of H with $x \in U$ and $x^{-1} \notin U$ (observe that this is possible because |H| > |K|) and let $L := J \cup U$. Then |L| = |R| and $L \subseteq G \setminus (R^{-1} \cup \{1\})$. Since

$$\begin{split} |\{y \in R \mid y^{-1} \notin R\}| &= |J|, \\ |\{y \in L \mid y^{-1} \notin L\}| \geq |J \cup \{x\}| > |J|, \end{split}$$

there is no automorphism α of G with $R^{\alpha} = L$, which is a contradiction.

CASE 6: No element in H as order at least 3, that is, each element in H has order 2.

Suppose that K contains an element x having order at least 3. Let U be any subset of H with |U| = |K| and let $L := J \cup U$. Then |L| = |R| and $L \subseteq G \setminus (R^{-1} \cup \{1\})$. No element in J has order 2 and hence

$$|\{y \in R \mid o(y) = 2\}| = |\{y \in K \mid o(y) = 2\}| \le |K \setminus \{x\}| = |K| - 1.$$

On the other hand, $\{y \in L \mid o(y) = 2\} = U$ and hence $|\{y \in L \mid o(y) = 2\}| = |U| = |K|$. Therefore, there is no automorphism α of G with $R^{\alpha} = L$, which is a contradiction.

Suppose that every element in K has order 2. Since G is not an elementary abelian 2-group and

$$G = (R \cup R^{-1}) \cup H \cup \{1\} = J \cup J^{-1} \cup K \cup H \cup \{1\},\$$

we have $J \neq \emptyset$. Let $x \in J$, let U be any subset of H with |U| = |K| + 1 (observe that this is possible because |H| > |K| and let $L := U \cup (J \setminus \{x\})$. Then |L| = |R| and $L \subseteq G \setminus (R^{-1} \cup \{1\})$. However, since L has more involutions than R, there is no automorphism α of G with $R^{\alpha} = L$, which is our final contradiction. \square

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ON HDR

References

- [1] M. Arezoomand, B. Taeri, Normality of 2-Cayley digraphs, Discrete Math. 338 (2015), 41–47.
- [2] L. Babai, Finite digraphs with given regular automorphism groups, Period. Math. Hungar. 11 (1980), 257–270.
- [3] L. Babai, W. Imrich, Tournaments with given regular group, Aequationes Math. 19 (1979), 232–244.
- [4] W. Bosma, C. Cannon, C. Playoust, The MAGMA algebra system I: The user language, J. Symbolic Comput. 24 (1997), 235–265.
- [5] E. Dobson, P. Spiga, G. Verret, Cayley graphs on abelian groups, Combinatorica 36 (2016), 371–393.
- [6] J. -L. Du, Y. -Q. Feng, P. Spiga, A classification of the graphical m-semiregular representations of finite groups, J. Combin. Theory Ser. A, to appear.
- [7] J. -L. Du, Y. -Q. Feng, P. Spiga, On the existence and the enumeration of bipartite regular representations of Cayley graphs over abelian groups, submitted.
- [8] S. F. Du, M. Y. Xu, A classification of semi-symmetric graphs of order 2pq, Comm. Algebra 28 (2000), 2685–2715.
- [9] I. Estélyi, T. Pisanski, Which Haar graphs are Cayley graphs, Electron J. Combin. 23 (2016) #P3.10.
- [10] W. Feit, J. G. Thomphson, Solvability of groups of odd order, Pacific J. Math. 13 (1963), 755–1029.
- [11] C. D. Godsil, GRR's for non-solvable groups, in Algebraic Methods in Graph theory (Proc. Conf. Szeged 1978 L. Lovász and V. T. Sós, eds), Coll. Math. Soc. J. Bolyai 25, North-Holland, Amsterdam, 1981, 221–239.
- [12] D. Hetzel, Über reguläre graphische Darstellung von auflösbaren Gruppen, Technische Universität, Berlin, 1976.
- [13] A. Hujdurović, K. Kutnar, D. Marušič, On normality of n-Cayley graphs, Appl. Math. Comput. 332 (2018), 469–476.
- [14] W. Imrich, Graphs with transitive Abelian automorphism group in Combinatorial Theory and Its Applications, Coll. Soc. Janos Bolyai 4, Balatonfued, Hungary, (1969), 651–656.
- [15] W. Imrich, Graphical regular representations of groups odd order, in: Combinatorics, Coll. Math. Soc. János. Bolayi 18 (1976), 611–621.
- [16] W. Imrich, M.E. Watkins, On graphical regular representations of cyclic extensions of groups, Pac. J. Math. 55 (1974), 461–477.
- [17] J. Morris, P. Spiga, G. Verret, Automorphisms of Cayley graphs on generalised dicyclic groups, European J. Combin. 43 (2015), 68–81.
- [18] J. Morris, P. Spiga, Every finite non-solvable group admits an oriented regular representation, J. Combin. Theory Ser. B 126 (2017), 198–234.
- [19] J. Morris, P. Spiga, Classification of finite groups that admit an oriented regular representation, Bulletin of the London Math. Soc. (2018), 811–831.
- [20] L. A. Nowitz, M. E. Watkins, Graphical regular representations of non-abelain groups, I, Canad. J. Math. 24 (1972), 994–1008.
- [21] L. A. Nowitz, M. E. Watkins, Graphical regular representations of non-abelain groups, II, Canad. J. Math. 24 (1972), 1009–1018.
- [22] P. Spiga, Finite groups admitting an oriented regular representation, J. Combin. Theory Ser. A 153 (2018), 76–97.
- [23] P. Spiga, On the Existence of Frobenius Digraphical Representations, Electron. J. Comb. 25 (2018), #P2.6
- [24] P. Spiga, Cubic graphical regular representations of finite non-abelian simple groups, Commu. Algebra, 46 (2018), 2440–2450.
- [25] M. E. Watkins, On the action of non-abelian groups on graphs, J. Combin. Theory 11 (1971), 95–104.
- [26] B. Z. Xia, T. Fang, Cubic graphical regular representations of PSL₂(q), Discrete Math. 339 (2016), 2051–2055.

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