

Restricted extension of sparse partial edge colorings of hypercubes

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October 19, 2018

Abstract

We consider the following type of question: Given a partial proper d -edge coloring of the d -dimensional hypercube Q_d , and lists of allowed colors for the non-colored edges of Q_d , can we extend the partial coloring to a proper d -edge coloring using only colors from the lists? We prove that this question has a positive answer in the case when both the partial coloring and the color lists satisfy certain sparsity conditions.

1 Introduction

The chromatic index $\chi'(G)$ of a (simple) graph G is far simpler in terms of its possible values than the chromatic number; Vizing's theorem [Viz64] tells us that in order to properly color the edges of G we need either $\Delta(G)$ or $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of G , and by König's edge coloring theorem, $\chi'(G) = \Delta(G)$ if G is bipartite [Kön16]. This simplicity quickly disappears in many of the natural variations on the basic edge coloring problem, e.g. the precoloring extension problem, where some of the edges of a graph have been (properly) colored and we want to determine if this partial coloring can be extended to a proper edge coloring of the full graph using no extra colors; indeed this problem is NP-complete already for 3-regular bipartite graphs [Fia03].

One of the earlier references explicitly discussing the problem of extending a partial edge coloring is [MS90]; there a simple necessary condition for the existence of an extension is given and the authors find a class of graphs where this condition is also sufficient. More recently the question of extending a precoloring where the precolored edges form a matching has gathered interest; in [EGv⁺14] a number of positive results and conjectures are given. In particular, it is conjectured that for every graph G , if φ is an edge precoloring of a matching M in G using $\Delta(G) + 1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G) + 1)$ -edge coloring of G ; this was first conjectured in [AM01], but then with distance 3 instead. By the *distance* between two edges e and e' here we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e' ; a *distance- t matching* is a matching where any two edges are at distance at least t from each other. The *t -neighborhood* of an edge e is the graph induced by all edges of distance at most t from e .

Note that the conjecture in [EGv⁺14] on distance-2 matchings is sharp both with respect to the distance between precolored edges, and in the sense that $\Delta(G) + 1$ can in general not be replaced by $\Delta(G)$, even if any two precolored edges are at arbitrarily large distance from each other [EGv⁺14]. In [EGv⁺14], it is proved that this conjecture hold for e.g. bipartite multigraphs and subcubic multigraphs, and in [GK16] it is proved that a version of the conjecture with the distance increased to 9 holds for general graphs.

However, for one specific family of graphs, the balanced complete bipartite graphs $K_{n,n}$, the edge precoloring extension problem was studied far earlier than in the above-mentioned references. Here the extension problem corresponds to asking whether a partial latin square can be completed to a latin square. In this form the problem appeared already in 1960, when Evans [Eva60] stated his now classic conjecture that for every positive integer n , if $n - 1$ edges in $K_{n,n}$ have been (properly) colored, then this partial coloring can be extended to a proper n -edge-coloring of $K_{n,n}$. This conjecture was solved for large n by Häggkvist [Häg78] and later for all n by Smetaniuk [Sme81], and independently by Andersen and Hilton [AH83]. Generalizing this problem, Daykin and Häggkvist [DH84] proved several results on extending partial edge colorings of $K_{n,n}$, and they also conjectured that much denser partial colorings can be extended, as long as the colored edges are spread out in a specific sense: a partial n -edge coloring of $K_{n,n}$ is ϵ -dense if there are at most ϵn colored edges from $\{1, \dots, n\}$ at any vertex and each color in $\{1, \dots, n\}$ is used at most ϵn times in the partial coloring. Daykin and Häggkvist [DH84] conjectured that for every positive integer n , every $\frac{1}{4}$ -dense partial proper n -edge coloring can be extended to a proper n -edge coloring of $K_{n,n}$, and proved a version of the conjecture for $\epsilon = o(1)$ (as $n \rightarrow \infty$) and n divisible by 16. Bartlett [Bar13] proved that this conjecture holds for a fixed positive ϵ , and recently a different proof which improves the value of ϵ was given in [BKL⁺16].

For general edge colorings of balanced complete bipartite graphs, Dinitz conjectured, and Galvin proved [Gal95], that if each edge of $K_{n,n}$ is given a list of n colors, then there is a proper edge coloring of $K_{n,n}$ with support in the lists. Indeed, Galvin's result was a complete solution of the well-known List Coloring Conjecture for the case of bipartite multigraphs (see e.g. [HC92] for more background on this conjecture and its relation to the Dinitz' conjecture).

Motivated by the Dinitz' problem, Häggkvist [Häg89] introduced the notion of βn -arrays, which correspond to list assignments L of forbidden colors for $E(K_{n,n})$, such that each edge e of $K_{n,n}$ is assigned a list $L(e)$ of at most βn forbidden colors from $\{1, \dots, n\}$, and at every vertex v each color is forbidden on at most βn edges adjacent to v ; we call such a list assignment for $K_{n,n}$ β -sparse. If L is a list assignment for $E(K_{n,n})$, then a proper n -edge coloring φ of $K_{n,n}$ avoids the list assignment L if $\varphi(e) \notin L(e)$ for every edge e of $K_{n,n}$; if such a coloring exists, then L is avoidable. Häggkvist conjectured that there exists a fixed $\beta > 0$, in fact also that $\beta = \frac{1}{3}$, such that for every positive integer n , every β -sparse list assignment for $K_{n,n}$ is avoidable. That such a $\beta > 0$ exists was proved for even n by Andrén in her PhD thesis [And10], and later for all n in [ACÖ13].

Combining the notions of extending a sparse precoloring and avoiding a sparse list assignment, Andrén et al. [ACM16] proved that there are constants $\alpha > 0$ and $\beta > 0$, such that for every positive integer n , every α -dense partial edge coloring of $K_{n,n}$ can be extended to a proper n -edge-coloring avoiding any given β -sparse list assignment L , provided that no edge e is precolored by a color that appears in $L(e)$. In contrast to this, it was proved in [EGv⁺14] that there are bipartite graphs G with a precolored matching of size 2, which is not

extendable to a proper $\Delta(G)$ -edge coloring. These examples have edge densities converging to some constant $0 < c \leq \frac{1}{2}$, and many of the proof methods used in the papers mentioned above rely on the high edge density of the complete bipartite graph. It is thus natural to ask if the good behaviour seen for $K_{n,n}$ will hold for well-structured graphs of lower densities.

The aim of this paper is to show that some generalizations of this type are possible. We will demonstrate that results similar to those from [ACM16] hold for the family of d -dimensional hypercubes Q_d ; these graphs have a degree which is logarithmic in the number of vertices, rather than linear, so we are now looking at a family of graphs with vanishing asymptotic density. Instead of bounding the global density of precolored edges, as in the case of complete bipartite graphs, for hypercubes we shall bound the number of precolored edges appearing in neighborhoods of given size. Our results are stated in terms of 27-neighborhoods; this size of neighborhoods is solely due to proof technical reasons.

To state our main theorem, we need some terminology: a *dimensional matching* M of Q_d is a perfect matching of Q_d such that $Q_d - M$ is isomorphic to two copies of Q_{d-1} ; evidently there are precisely d dimensional matchings in Q_d . An edge precoloring of Q_d with colors $1, \dots, d$ is called α -dense if

- (i) there are at most αd precolored edges at each vertex;
- (ii) for every 27-neighborhood W of an edge e of Q_d , there are at most αd precolored edges with color i in W , $i = 1, \dots, d$;
- (iii) for every 27-neighborhood W , and every dimensional matching M , at most αd edges of M are precolored in W .

Here, and in the following, all t -neighborhoods are taken with respect to edges. A list assignment L for $E(Q_d)$ is β -sparse if the list of each edge is a (possibly empty) subset of $\{1, \dots, d\}$, and

- (i) $|L(e)| \leq \beta d$ for each edge $e \in E(Q_d)$;
- (ii) for every vertex $v \in V(Q_d)$, each color in $\{1, \dots, d\}$ occurs in at most βd lists of edges incident to v ;
- (iii) for every 27-neighborhood W , and every dimensional matching M , any color appears at most βd times in lists of edges of M contained in W .

Our main result is the following.

Theorem 1.1. *There are constants $\alpha > 0$ and $\beta > 0$ such that for every positive integer d , if φ is an α -dense d -edge precoloring of Q_d , L a β -sparse list assignment for Q_d , and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is a proper d -edge coloring of Q_d which agrees with φ on any precolored edge and which avoids L .*

As a corollary of our main theorem we note that a version of the conjecture on precolored distance-2 matchings from [EGv⁺14], with $\Delta(G)$ in place of $\Delta(G) + 1$, but with a weaker distance requirement, holds for the family of hypercubes.

Corollary 1.2. *There is a constant $\beta > 0$ such that if L is a β -sparse list assignment L for Q_d and φ is a d -edge precoloring of a distance- t matching in Q_d , where $t > 55$, then φ can be extended to a proper d -edge-coloring of Q_d which avoids L .*

This follows from the fact that a precolored distance- t matching is an α -dense precoloring if $t > 55$. A precolored matching has a much more restricted structure than a general α -dense d -edge precoloring, so this corollary is most likely far from optimal in terms of the lower bound on t ; it would be interesting to see how far this can be improved.

If we place both the precolored edges and those with a list of forbidden colors on them on a matching, then our proof method in fact trivially yields the following.

Theorem 1.3. *Let φ be a d -edge precoloring and L a list assignment for the edges of Q_d . If every edge e which is either precolored or satisfies $L(e) \neq \emptyset$ belongs to a distance-3 matching M in Q_d , then there is a d -edge coloring which agrees with φ on any precolored edge, and which avoids L .*

This proves that a slightly stronger version, with d rather than $d + 1$ colors, of the earlier mentioned conjecture from [AM01] holds for the family of hypercubes.

The rest of the paper is organized as follows. In Section 2 we introduce some terminology and notation and also outline the proof of Theorem 1.1. Section 3 contains the proof of a slightly reformulated version of Theorem 1.1; we also indicate how Theorem 1.3 can be deduced from the proof of Theorem 1.1. In Section 4 we give some concluding remarks; in particular, we give an example indicating what numerical values of α and β in Theorem 1.1 might be best possible. At the beginning of Section 3 we shall present numerical values of α and β for which our main theorem holds, provided that n is large enough. Finally, throughout the paper, the base of the natural logarithm is denoted by e .

2 Terminology, notation and proof outline

Given an edge precoloring φ (or just precoloring, or partial edge coloring) of a graph G with $\Delta(G)$ colors, an *extension* of φ is a proper $\Delta(G)$ -edge coloring of G which agrees with φ on every precolored edge; if such a coloring of G exists, then φ is *extendable*.

For a vertex $u \in Q_d$, we denote by E_u the set of edges with one endpoint being u , and for a (partial) edge coloring f of Q_d , let $f(u)$ denote the set of colors on edges in E_u under f . If two edges xy and zt of Q_d are in a cycle of length 4 in Q_d , and all the vertices x, y, z, t are distinct, then the edges xy and zt are *parallel*.

As noted above, Q_d decomposes into precisely d dimensional matchings. Note that a dimensional matching in Q_d contains precisely 2^{d-1} edges. For a d -edge coloring f of Q_d , a dimensional matching M is called a *dimensional matching of color c (under f)* if there are more edges of M colored c than by any other color.

For the proof of Theorem 1.1, we shall use the *standard* d -edge coloring h of Q_d where all edges of the i th dimensional matching in Q_d is colored i , $i = 1, \dots, d$. A cycle is *2-colored* if its edges are colored by two distinct colors from $\{1, \dots, d\}$. The following property is crucial for our proof of Theorem 1.1.

Property 2.1. *In the standard d -edge coloring h , every edge of Q_d is in exactly $d-1$ 2-colored 4-cycles.*

Let φ be an α -dense precoloring of Q_d . Edges of Q_d which are colored under φ , are called *prescribed (with respect to φ)*. For the edge coloring h (or an edge coloring obtained from h), an edge e of Q_d is called *requested (under h with respect to φ)* if $h(e) = c$ and e is adjacent to an edge e' such that $\varphi(e') = c$.

Consider a β -sparse list assignment L for Q_d . For the edge coloring h (or an edge coloring obtained from h), an edge e of Q_d is called a *conflict edge (of h with respect to L)* if $h(e) \in L(e)$. An *allowed cycle (under h with respect to L)* of Q_d is a 4-cycle $\mathcal{C} = uvztu$ in Q_d that is 2-colored under h , and such that interchanging colors on \mathcal{C} yields a proper d -edge coloring h_1 of Q_d where none of uv, vz, zt, tu is a conflict edge. We call such an interchange a *swap in h* .

Instead of proving Theorem 1.1 we shall in fact prove the following theorem, which is easily seen so imply Theorem 1.1.

Theorem 2.2. *There are constants $\alpha > 0$, $\beta > 0$ and d_0 , such that for every positive integer $d \geq d_0$, if φ is an α -dense precoloring of Q_d and L a β -sparse list assignment for Q_d , and $\varphi(e) \notin L(e)$ for every edge $e \in E(Q_d)$, then there is an extension of φ which avoids L .*

Below we outline the proof of Theorem 2.2. Let h be the standard proper d -edge coloring of Q_d defined above, φ an α -dense precoloring of Q_d , and L a β -sparse list assignment for $E(Q_d)$.

- Step I. Given the standard d -edge coloring h of Q_d , find a permutation ρ of the elements of the set $\{1, \dots, d\}$ such that in the proper d -edge-coloring h' obtained by applying ρ to the colors used in h , locally, each dimensional matching in Q_d contains “sufficiently few” conflict edges with L , as well as “sufficiently few” requested edges with respect to φ . Moreover, we require that each vertex u of Q_d satisfies that E_u contains “sufficiently few” conflict and requested edges, and that each edge of Q_d belongs to “many” allowed cycles under h' . These conditions shall be more precisely articulated below.
- Step II. From the precoloring φ of Q_d , define a new edge precoloring φ' such that an edge e of Q_d is colored under φ' if and only if e is colored under φ or e is a conflict edge of h' with respect to L . We shall also require that locally, each of the colors in $\{1, \dots, d\}$ is used a bounded number of times under φ' .
- Step III. From h' , construct a proper d -edge coloring h'' of Q_d such that under h'' , no edge in Q_d is both requested and prescribed (with respect to φ'); this is done by swapping on a set of disjoint allowed 4-cycles. We also require that each requested edge e of h'' is adjacent to at most one edge e' such that $h''(e) = \varphi'(e')$.
- Step IV. For each edge e of Q_d that is prescribed with respect to φ' , construct a subset $T_e \subseteq E(Q_d)$, such that performing a series of swaps in h'' on allowed cycles, all edges of which are in T_e , yields a coloring h''_1 where $h''_1(e) = \varphi'(e)$. Moreover, if e and e' are prescribed edges of Q_d , then the sets $T_{e'}$ and T_e will be disjoint. Thus, performing the series of swaps on all the sets T_e associated with prescribed edges e yields a proper d -edge coloring \hat{h} which is an extension of φ' (and thus φ), and which avoids L .

3 Proofs

In this section we prove Theorem 2.2. In the proof we shall verify that it is possible to perform Steps I-IV described above to obtain a proper d -edge-coloring of Q_d that is an extension of φ and which avoids L . This is done by proving a lemma in each step.

We will not specify the value of d_0 in the proof but rather assume that d is large enough whenever necessary. Since the proof will contain a finite number of inequalities that are valid if d is large enough, this suffices for proving Theorem 2.2.

The proof of Theorem 2.2 involves a number of functions and parameters:

$$\alpha, \beta, \gamma, \kappa, \epsilon, \epsilon_0, \tau,$$

and a number of inequalities that they must satisfy. For the reader's convenience, explicit choices for which the proof holds are presented here:

$$\begin{aligned} \alpha &= 10^{-622}, \beta = 2 \cdot 10^{-622}, \gamma = 2^{-11}, \kappa = 9/2^{11} \\ \epsilon &= 2^{-3}, \epsilon_0 = 2^{-8}, \tau = 2^{-7}. \end{aligned}$$

We remark that since the numerical values of α and β are not anywhere near what we expect to be optimal, we have not put an effort into choosing optimal values for these parameters. Finally, for simplicity of notation, we shall omit floor and ceiling signs whenever these are not crucial.

Proof of Theorem 2.2. Let φ be an α -dense precoloring of Q_d , and let L be a β -sparse list assignment for Q_d . Moreover, let h be the standard d -edge coloring defined above.

Step I: We use the following lemma for constructing a required d -edge-coloring h' from h .

Lemma 3.1. *Let $\gamma, \tau < 1$ be constants such that $0 < \alpha, \beta \leq \gamma$, $2^{\frac{1}{\gamma}+1}e\alpha < \frac{\gamma}{3}$, $2^{\frac{1}{\gamma}}e\beta < \frac{\gamma}{3}$ and $2^{\frac{2}{\tau-2\beta}+1}e\beta < \tau - 2\beta$. There is a permutation ρ of $\{1, \dots, d\}$, such that applying ρ to the set of colors $\{1, \dots, d\}$ used in h , we obtain a d -edge coloring h' of Q_d satisfying the following:*

- (a) *For every 26-neighborhood W , and every dimensional matching M , at most γd edges of $M \cap E(W)$ are requested.*
- (b) *For every 27-neighborhood W , and every dimensional matching M , at most γd edges of $M \cap E(W)$ are conflict.*
- (c) *No vertex u in Q_d satisfies that E_u contains more than γd requested edges.*
- (d) *No vertex u in Q_d satisfies that E_u contains more than γd conflict edges.*
- (e) *Each edge in Q_d belongs to at least $(1 - \tau)d$ allowed cycles.*

Proof. Let A, B, C, D and E be the number of permutations which do not fulfill the conditions (a), (b), (c), (d) and (e), respectively. Let X be the number of permutations satisfying the five conditions (a), (b), (c), (d) and (e). There are $d!$ ways to permute the colors, so we have

$$X \geq d! - A - B - C - D - E$$

We will now prove that X is greater than 0.

- Recall that an edge e is requested if e is adjacent to an edge e' such that $h(e) = \varphi(e')$. Let M' be a dimensional matching, and consider a subset $M \subseteq M'$ of all edges in M' that are contained in a given 26-neighborhood W_1 . Then every edge of M and every edge adjacent to an edge of M is contained in a 27-neighborhood W_2 containing W_1 . Since all edges in M have the same color in any edge coloring obtained from h by permuting colors, and there are at most αd precolored edges with color i in W_2 , $i = 1, \dots, d$, the maximum number of requested edges in M is αd . In other words, no subset of a dimensional matching contained in a 26-neighborhood contains more than αd requested edges. Since $\gamma \geq \alpha$, this means that all permutations satisfy condition (a) or $A = 0$.
- Since all edges that are in the same dimensional matching have the same color under h and for every 27-neighborhood W , and every dimensional matching M , any color appears at most βd times in lists of edges of M contained in W , we have that the maximum number of conflict edges in a subset of a given dimensional matching contained in a 27-neighborhood is βd . Since $\gamma \geq \beta$, this means that all permutations satisfy condition (b) or $B = 0$.
- To estimate C , let u be a fixed vertex of Q_d , and let S be a set of size γd of edges of E_u . There are $\binom{d}{\gamma d}$ ways to choose S . For a vertex v adjacent to u , if uv is a requested edge, then the colors used in h should be permuted in such a way that in the resulting coloring h' , uv is colored by some color in the set $\{(\varphi(u) \cup \varphi(v)) \setminus \varphi(uv)\}$. Since $|\varphi(u)| \leq \alpha d$ and $|\varphi(v)| \leq \alpha d$, there are at most $(2\alpha d)^{\gamma d}$ ways to choose which colors from $1, 2, \dots, d$ to assign to the edges in S so that all edges in S are requested. The rest of the colors can be arranged in any of the $(d - \gamma d)!$ possible ways. In total this gives at most

$$\binom{d}{\gamma d} (2\alpha d)^{\gamma d} (d - \gamma d)! = \frac{d! (2\alpha d)^{\gamma d}}{(\gamma d)!}$$

permutations that do not satisfy condition (c) on vertex u .

There are 2^d vertices in Q_d , so we have

$$C \leq 2^d \frac{d! (2\alpha d)^{\gamma d}}{(\gamma d)!}$$

- To estimate D , let u be a fixed vertex of Q_d , and let S be a set of size γd ($|S| = \gamma d$) of edges from E_u . For a vertex v adjacent to u , if uv is a conflict edge, then the colors used in h should be permuted in such a way that in the resulting coloring h' , the color of uv is in $L(uv)$. Since $|L(uv)| \leq \beta d$, there are at most $(\beta d)^{\gamma d}$ ways to choose which colors from $\{1, 2, \dots, d\}$ to assign to the edges in S so that all edges in S are conflict. The rest of the colors can be arranged in any of the $(d - \gamma d)!$ possible ways. In total this gives at most

$$\binom{d}{\gamma d} (\beta d)^{\gamma d} (d - \gamma d)! = \frac{d! (\beta d)^{\gamma d}}{(\gamma d)!}$$

permutations that do not satisfy condition (d) on vertex u . There are 2^d vertices in Q_d , so we have

$$D \leq 2^d \frac{d! (\beta d)^{\gamma d}}{(\gamma d)!}$$

- To estimate E , let uv be a fixed edge of Q_d . Each cycle $\mathcal{C} = uvztu$ containing uv is uniquely defined by an edge zt which is parallel with uv . Moreover, a permutation ς is in E if and only if there are more than τd choices for zt so that \mathcal{C} is not allowed. We shall count the number of ways ς could be constructed for this to happen. First, note that for each choice of color c_1 from $\{1, \dots, d\}$, for the dimensional matching which contains uv , there are up to $2\beta d$ cycles that are not allowed because of this choice. This follows from the fact that there are at most βd choices for t (or z) such that $L(ut)$ (or $L(vz)$) contains c_1 . So for a permutation ς to belong to E , ς must satisfy that at least $(\tau - 2\beta)d$ cycles containing uv are forbidden because of the color assigned to the dimensional matching containing ut and vz .

Let S be a set of edges, $|S| = (\tau - 2\beta)d$, such that for every edge $zt \in S$, the cycle $\mathcal{C} = uvztu$ is not allowed because of colors assigned to ut and vz . There are $\binom{d-1}{(\tau-2\beta)d}$ ways to choose S . Furthermore, $L(uv)$ and $L(zt)$ contain at most βd colors each, so there are at most $2\beta d$ choices for a color for the dimensional matching containing ut and vz that would make \mathcal{C} disallowed because of the color assigned to this dimensional matching. The remaining colors can be permuted in $(d - 1 - (\tau - 2\beta)d)! = ((1 - \tau + 2\beta)d - 1)!$ ways.

Hence, the total number of permutations σ with not enough allowed cycles for a given edge is bounded from above by

$$d \binom{d-1}{(\tau-2\beta)d} (2\beta d)^{(\tau-2\beta)d} ((1-\tau+2\beta)d-1)! = \frac{d!(2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

and the total number of permutation σ that have too few allowed cycles for at least one edge is bounded from above by

$$2^{d-1} d \frac{d!(2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

Hence,

$$X \geq d! - 2^d \frac{d!(2\alpha d)^{\gamma d}}{(\gamma d)!} - 2^d \frac{d!(\beta d)^{\gamma d}}{(\gamma d)!} - 2^{d-1} d \frac{d!(2\beta d)^{(\tau-2\beta)d}}{((\tau-2\beta)d)!}$$

Using Stirling's approximation, $n! \geq n^n e^{-n}$ and $2^{d-1} d < \frac{2^{2d}}{3}$, we have

$$\begin{aligned} X &\geq d! \left(1 - 2^d \frac{e^{\gamma d} (2\alpha d)^{\gamma d}}{(\gamma d)^{\gamma d}} - 2^d \frac{e^{\gamma d} (\beta d)^{\gamma d}}{(\gamma d)^{\gamma d}} - 2^{2d} \frac{e^{(\tau-2\beta)d} (2\beta d)^{(\tau-2\beta)d}}{3((\tau-2\beta)d)^{(\tau-2\beta)d}} \right) \\ X &> d! \left(1 - \left(\frac{2^{\frac{1}{\gamma}+1} e \alpha}{\gamma} \right)^{\gamma d} - \left(\frac{2^{\frac{1}{\gamma}} e \beta}{\gamma} \right)^{\gamma d} - \frac{1}{3} \left(\frac{2^{\frac{2}{\tau-2\beta}+1} e \beta}{\tau-2\beta} \right)^{\tau-2\beta d} \right) \end{aligned}$$

Using the conditions $\frac{2^{\frac{1}{\gamma}+1} e \alpha}{\gamma} < \frac{1}{3}$, $\frac{2^{\frac{1}{\gamma}} e \beta}{\gamma} < \frac{1}{3}$, and $\frac{2^{\frac{2}{\tau-2\beta}+1} e \beta}{\tau-2\beta} < 1$, we have

$$\left(\frac{2^{\frac{1}{\gamma}+1} e \alpha}{\gamma} \right)^{\gamma d} < \frac{1}{3} \quad \text{and} \quad \left(\frac{2^{\frac{1}{\gamma}} e \beta}{\gamma} \right)^{\gamma d} < \frac{1}{3} \quad \text{and} \quad \frac{1}{3} \left(\frac{2^{\frac{2}{\tau-2\beta}+1} e \beta}{\tau-2\beta} \right)^{\tau-2\beta d} < \frac{1}{3}$$

This implies $X > 0$.

□

Step II: Let h' be the proper d -edge coloring satisfying conditions (a)-(e) of Lemma 3.1 obtained in the previous step.

We use the following lemma for extending φ to a proper d -edge precoloring φ' of Q_d , such that an edge e of Q_d is colored under φ' if and only if e is precolored under φ or e is a conflict edge of h' with L .

Lemma 3.2. *Let $\alpha', \epsilon_0, \gamma, \kappa$ be constants such that $\alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$, $\kappa \geq \max(\alpha + \gamma, \alpha + \epsilon_0, \gamma + \epsilon_0)$ and*

$$d - \beta d - 2\alpha d - 2\gamma d - \frac{4\gamma}{\epsilon_0}d - \frac{\alpha}{\epsilon_0}d \geq 1.$$

There is a proper d -edge precoloring φ' of Q_d satisfying the following:

- (a) $\varphi'(uv) = \varphi(uv)$ for any edge uv of Q_d that is precolored under φ .
- (b) For every conflict edge uv of h' , uv is colored under φ' and $\varphi'(uv) \notin L(uv)$.
- (c) There are at most $\alpha'd$ precolored edges at each vertex of Q_d under φ' .
- (d) For every 12-neighborhood W in Q_d , there are at most $\alpha'd$ precolored edges with color i in W , $i = 1, \dots, d$, under φ' .
- (e) For every 12-neighborhood W in Q_d , and every dimensional matching M , at most $\alpha'd$ edges of M are precolored under φ' in W .

Furthermore, the edge coloring h' of Q_d and the precoloring φ' of Q_d satisfy that

- (f) For every 11-neighborhood W in Q_d , and every dimensional matching M , at most κd edges of $M \cap E(W)$ are requested.
- (g) No vertex x in Q_d satisfies that E_x contains more than κd requested edges.

Proof. Consider the edge coloring h' and the precoloring φ ; for each 26-neighborhood W , no dimensional matching in Q_d contains more than γd requested edges that are in W , so the total number of requested edges in W is not greater than γd^2 . Similarly, the total number of conflict edges in each 27-neighborhood W is not greater than γd^2 .

We shall construct the coloring φ' by assigning a color to every conflict edge; this is done by iteratively constructing a d -edge precoloring ϕ of the conflict edges of Q_d ; in each step we color a hitherto uncolored conflict edge, thereby transforming a conflict edge to a prescribed edge. At each step of transforming a conflict edge uv into prescribed edge, the number of requested edges will increase by 2. Hence, after constructing the proper d -edge precoloring φ' , the total number of requested edges of each 26-neighborhood is at most $\gamma d^2 + 2\gamma d^2 = 3\gamma d^2$.

Suppose now that we have constructed the precoloring φ' . A vertex u in Q_d is φ' -overloaded if E_u contains at least $\epsilon_0 d$ requested edges; note that no more than $\frac{3\gamma d^2}{\epsilon_0 d} = \frac{3\gamma}{\epsilon_0}d$ vertices of each 25-neighborhood are φ' -overloaded.

A color c is φ' -overloaded in a t -neighborhood W if c appears on at least $\epsilon_0 d$ edges in W under φ' ; note that at most

$$\frac{\gamma d^2}{\epsilon_0 d} + \frac{\alpha d^2}{\epsilon_0 d} = \frac{\gamma + \alpha}{\epsilon_0} d$$

colors are φ' -overloaded in each 25-neighborhood W . These upper bounds hold for any choice of the precoloring φ' obtained from φ by coloring the conflict edges of Q_d .

Let G be the subgraph of the hypercube Q_d induced by all conflict edges of Q_d . Let us now construct the d -edge coloring ϕ of G . We color the edges of G by steps, and in each step we define a list $\mathcal{L}(e)$ of allowed colors for a hitherto uncolored edge $e = uv$ of G by for every color $c \in \{1, \dots, d\}$ including c in $\mathcal{L}(e)$ if

- $c \notin L(uv)$,
- c does not appear in $\varphi(u)$ or $\varphi(v)$, or on any previously colored edge of G that is adjacent to e .
- c is distinct from the color of the edge uu' (or vv') under h' if u' (or v') is φ' -overloaded.
- c is not φ' -overloaded in the 25-neighborhood of e .

Our goal is then to pick a color $\phi(e)$ from $\mathcal{L}(e)$ for e . Given that this is possible for each edge of G , this procedure clearly produces a d -edge-coloring ϕ of G , so that ϕ and φ taken together form a proper d -edge precoloring of Q_d .

Using the estimates above and the facts that G has maximum degree γd , and $|\varphi(v)| \leq \alpha d$ for any vertex v of Q_d , we have

$$|\mathcal{L}(e)| \geq d - \beta d - 2\alpha d - 2\gamma d - \frac{3\gamma}{\epsilon_0} d - \frac{\gamma + \alpha}{\epsilon_0} d,$$

for every edge e of G in the process of constructing ϕ , and by assumption $|\mathcal{L}(e)| \geq 1$. Thus, we conclude that we can choose an allowed color for each conflict edge so that the coloring ϕ satisfies the above conditions. This implies that taking ϕ and φ together we obtain a proper d -precoloring φ' of the edges of Q_d . Let us now prove that the precoloring φ' satisfy the conditions in the lemma.

Let $\alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$. Then φ' satisfies the following:

- If uv is precolored under φ , then $\varphi'(uv) = \varphi(uv)$. For every conflict edge uv , there is a precolor $\varphi'(uv)$ such that $\varphi'(uv) \notin L(uv)$.
- There are at most $\alpha' d$ precolored edges at each vertex.

Let us next prove that the precoloring φ' satisfies conditions (d) and (e) of the lemma. Suppose that some 12-neighborhood W in Q_d contains more than $\alpha' d$ precolored edges with color i , for some $i \in \{1, \dots, d\}$. Consider an edge e in W with $\phi(e) = i$. By the construction of ϕ , in the 25-neighborhood W' of e no color is φ' -overloaded. Note further that every 12-neighborhood in Q_d that e lies in is contained in W' ; thus W is contained in W' , so the color i is φ' -overloaded in W' , a contradiction. We conclude that condition (d) holds. A similar argument shows that condition (e) holds as well.

Let us now turn to conditions (f) and (g). There are at most $\alpha'd$ precolored edges with color i , $i = 1, \dots, d$, in every 12-neighborhood in Q_d , and all edges that are in the same dimensional matching have the same color under h' . This implies that for each 11-neighborhood W and every dimensional matching M , the maximum number of requested edges in M that are in W is $\alpha'd$. Since

$$\kappa \geq \max\{\alpha + \gamma, \alpha + \epsilon_0, \gamma + \epsilon_0\},$$

condition (f) holds. Similarly, at each step of transforming a conflict edge into a prescribed edge under ϕ , we create 2 new requested edges, 1 at each vertex which is incident with the conflict edge. Since the maximum degree in G is γd , and no vertex is φ' -overloaded, no vertex x in Q_d satisfies that E_x contains more than $\epsilon_0 d + \gamma d$ requested edges. Thus every vertex x in Q_d satisfies that E_x contains at most κd requested edges. \square

Step III: Let φ' be the proper d -precoloring of Q_d obtained in the previous step and h' the d -edge coloring of Q_d obtained in Step I. By a *clash edge (of h')* in Q_d we mean an edge which is both prescribed and requested (under φ'). We use the following lemma for constructing, from h' , a proper d -edge coloring h'' of Q_d with no clash edge. The coloring h'' will also have the property that every requested edge e of h'' is adjacent to at most one prescribed edge e' such that $h''(e) = \varphi'(e')$.

Lemma 3.3. *Let $\kappa, \epsilon, \mu, \tau, \alpha' = \max(\alpha + \gamma, \alpha + \epsilon_0)$ be constants such that $\mu = 3\kappa + \epsilon + 1$ and*

$$d - \tau d - 9\kappa d - 3\alpha' d - 3\epsilon d - \frac{12\kappa}{\epsilon} d - 3 > 0.$$

By performing a sequence of swaps on disjoint allowed 2-colored 4-cycles in h' , we obtain a proper d -edge coloring h'' of Q_d satisfying the following:

- (a) *There is no clash edge in h'' .*
- (b) *For each requested edge e of h'' , e is adjacent to at most one edge e' satisfying that $h''(e) = \varphi'(e')$.*
- (c) *For each vertex $u \in V(Q_d)$, at most $2\kappa d + \epsilon d + 1$ edges incident with u appears in swaps for constructing h'' from h' .*
- (d) *For every 3-neighborhood W of Q_d , and every dimensional matching M , at most $2\kappa d + \epsilon d + 1$ edges of $E(W) \cap M$ appears in swaps for constructing h'' from h' .*
- (e) *For every 3-neighborhood W in Q_d , and every dimensional matching M , there are at most μd requested edges in $M \cap E(W)$.*
- (f) *No vertex in Q_d is incident with more than μd requested edges.*

Proof. An *unexpected edge* of h' is a clash edge or a requested edge e of h' that is adjacent to more than one edge e' satisfying that $h'(e) = \varphi'(e')$. For constructing h'' from h' , we will perform a number of swaps on 2-colored 4-cycles, and we shall refer to this procedure as *S-swap*. In more detail, we are going to construct a set S of disjoint allowed 4-cycles, each such cycle containing exactly one unexpected edge in h' . An edge that belongs to a cycle in S is called *used* in *S-swap*.

Let us first deduce some properties that our set S , which is yet to be constructed, will satisfy.

By Lemma 3.2, for every 11-neighborhood W in Q_d , and every dimensional matching M , the number of unexpected edges in $E(W) \cap M$ is not greater than κd . Suppose we have included a 4-cycle C in S . Every edge in C is at distance at most 1 from the unexpected edge contained in C ; this implies that for every 10-neighborhood W in Q_d , the total number of edges in W that are used in *S-swap* is at most $4\kappa d^2$.

A vertex u in Q_d is *S-overloaded* if E_u contains at least ϵd edges that are used in *S-swap*; note that no more than $\frac{4\kappa d^2}{\epsilon d} = \frac{4\kappa}{\epsilon}d$ vertices of each 9-neighborhood are *S-overloaded*. A dimensional matching M in Q_d is *S-overloaded* in a t -neighborhood W if $M \cap E(W)$ contains at least ϵd edges that are used in *S-swap*; note that for each 10-neighborhood W , no more than $\frac{4\kappa}{\epsilon}d$ dimensional matchings of Q_d are *S-overloaded* in W .

Using these facts, let us now construct our set S by steps; at each step we consider an unexpected edge e and include an allowed 2-colored 4-cycle containing e in S . Initially, the set S is empty. Next, for each unexpected edge $e = uv$ in Q_d , there are at least $d - \tau d$ allowed cycles containing e . We choose an allowed cycle $uvztu$ which contains e and satisfies the following:

- (1) z and t and the dimensional matching that contains vz and ut are not *S-overloaded* in the 9-neighborhood W_e of e ; this eliminates at most $\frac{12\kappa}{\epsilon}d$ choices.

Note that with this strategy for including 4-cycles in S , after completing the construction of S , every vertex is incident with at most $2\kappa d + \epsilon d + 1$ edges that are used in *S-swap*; that is, condition (c) holds.

Furthermore, after we have constructed the set S , no dimensional matching is *S-overloaded* in a 3-neighborhood of Q_d ; this follows from the fact that every 3-neighborhood W' in Q_d that ut , vz or zt belongs to is contained in W_e . Moreover, this implies that condition (d) holds.

- (2) None of the edges vz , zt , ut are prescribed, or requested, or used before in *S-swap*.

All possible choices for these edges are in the 3-neighborhood W_e of e in Q_d . By Lemma 3.2, no vertex in W_e , or subset of a dimensional matching that is in W_e , contains more than κd requested edges or $\alpha'd$ prescribed edges. Moreover, *S-swap* uses at most $2\kappa d + \epsilon d + 1$ edges at each vertex and in each subset of a dimensional matching contained in W_e . Hence, these restrictions eliminate at most $3(\kappa d + \alpha'd) + 3(2\kappa d + \epsilon d + 1)$ or $9\kappa d + 3\alpha'd + 3\epsilon d + 3$ choices.

It follows that we have at least

$$d - \tau d - 9\kappa d - 3\alpha'd - 3\epsilon d - \frac{12\kappa}{\epsilon}d - 3$$

choices for an allowed cycle $uvztu$ which contains uv . By assumption, this expression is greater than zero, so we conclude that there is a cycle satisfying these conditions, and thus we may construct the set S by iteratively adding disjoint allowed 2-colored 4-cycles such that each cycle contains a unexpected edge.

After this process terminates we have a set S of disjoint allowed cycles; we swap on all the cycles in S to obtain the coloring h'' . Note that for the cycle $uvztu$ constructed above, since none of the edges vz, zt, ut are prescribed or requested, $\{\varphi'(z) \cup \varphi'(t)\}$ does not contain the color $h'(uv)$; so after swapping colors on the cycle $uvztu$, none of the edges uv, vz, zt, ut are unexpected edges in the obtained coloring; that is, condition (a) and (b) hold.

Let us finally verify that conditions (e) and (f) hold. As noted above, for every dimensional matching M and every 3-neighborhood W , S -swap uses at most $2\kappa d + \epsilon d + 1$ from $E(W) \cap M$. Moreover, by Lemma 3.2, $E(W) \cap M$ contains at most κd requested edges under h' with respect to φ' . Thus the proper coloring h'' satisfies that for every dimensional matching M and for every 3-neighborhood W in Q_d , at most μd requested edges are contained in $E(W) \cap M$. Similarly, no vertex x in Q_d satisfies that E_x contains more than μd requested edges. \square

Step IV: Let h'' be the proper d -edge coloring of Q_d obtained in the previous step and let φ' be the precoloring of Q_d obtained in Step II. Then h'' and φ' satisfies (a)-(f) of Lemma 3.3, and also the following:

- each vertex of Q_d is incident with at most $\alpha'd$ edges that are precolored under φ' ;
- for every 12-neighborhood W and every dimensional matching M in Q_d , at most $\alpha'd$ edges of M are precolored under φ' in W ;
- for every 12-neighborhood W , there are at most $\alpha'd$ edges that are precolored i under φ' in W , $i = 1, \dots, d$.

As in the proof of Lemma 3.3, we say that an edge e in Q_d with $h'(e) \neq h''(e)$ is used in S -swap. Note that since in every 12-neighborhood, the number of edges that are precolored i ($i \in \{1, \dots, d\}$) is at most $\alpha'd$, and since the number of precolored edges in a subset of a dimensional matching of Q_d that is contained in a 12-neighborhood is also bounded, there is a bounded number of edges colored i under h'' that have been used in S -swap in each 3-neighborhood. Moreover, since S -swap uses a bounded number of edges at each vertex, and in the intersection of every dimensional matching and 3-neighborhood (by condition (c) and (d) in Lemma 3.3), most edges in Q_d are in a large number of allowed 2-colored 4-cycles under h'' . Those two properties are central for completing the proof of Theorem 2.2 in Step IV; this is done by proving the following lemma.

Lemma 3.4. *Let $\kappa, \epsilon, \tau, \mu = 3\kappa + \epsilon + 1, \alpha' = (\alpha + \gamma, \alpha + \epsilon_0)$ be constants such that*

$$d - 64\mu d - 64\alpha'd - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 86 > 0.$$

There is a proper d -edge coloring of Q_d that is an extension of φ' and which avoids L .

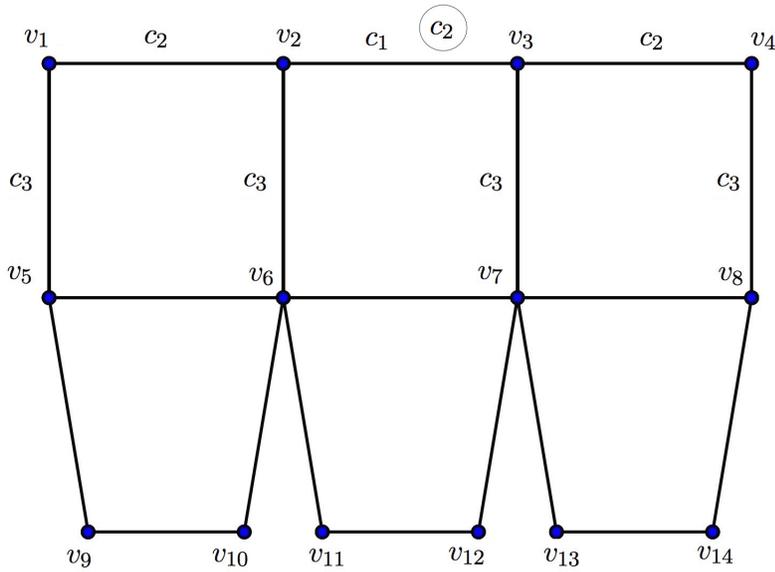


Figure 1: An example of a configuration T_e .

Proof. If $h''(e) = \varphi'(e)$ for all precolored edges e then we do nothing; h'' is the required proper edge coloring. Else, we construct a set $T \subseteq E(Q_d)$, such that performing a sequence of swaps on allowed 2-colored 4-cycles of the subgraph of Q_d induced by T , we obtain the required extension of φ' . We refer to this construction as T -swap. For each φ' -precolored edge e , the set T will contain a subset T_e of edges associated with e ; if e and e' are distinct φ' -precolored edges of Q_d , then we will have $T_e \cap T_{e'} = \emptyset$. An example of a subset T_e can be seen in Figure 1, where v_2v_3 is a prescribed edge, and v_1v_2 and v_3v_4 are requested. Since distinct sets T_e and $T_{e'}$ are disjoint, every requested edge is in at most one set T_e ; this property is ensured by Lemma 3.3 (b).

An edge that belongs to T is called *used* in T -swap. By Lemma 3.2, for every 12-neighborhood W in Q_d , and every dimensional matching M , at most $\alpha'd$ edges of $M \cap E(W)$ are precolored under φ' . For each configuration T_e in T , every edge of T_e is at distance at most 2 from the prescribed edge; this implies that for every 10-neighborhood W in Q_d , the total number of edges in W that are used in T -swap is at most $19\alpha'd^2$.

A vertex u in Q_d is T -overloaded if at least ϵd edges from E_u are used in T -swap; note that no more than $\frac{19\alpha'd^2}{\epsilon d} = \frac{19\alpha'}{\epsilon}d$ vertices of each 9-neighborhood are T -overloaded. A dimensional matching M in Q_d is T -overloaded in a t -neighborhood W if $M \cap E(W)$ contains at least ϵd edges that are used in T -swap; note that for each 10-neighborhood W no more than $\frac{19\alpha'}{\epsilon}d$ dimensional matchings are T -overloaded in W .

Consider the setup in Figure 1. We now describe how to construct the set T_e for the prescribed edge $e = v_2v_3$. Suppose that $\varphi'(v_2v_3) = c_2 \neq h''(v_2v_3) = c_1$. Since every vertex in Q_d has degree d we initially have at least $d - 3$ choices for a subgraph as in Figure 1.

Let v_1v_2 and v_3v_4 be the edges adjacent to v_2v_3 that are colored c_2 . The set $T_{v_2v_3}$ will consist of edges incident with 14 vertices v_1, \dots, v_{14} . We shall choose the vertices v_5, \dots, v_{14} such that they satisfy a number of properties:

- (1) v_5, \dots, v_{14} and the dimensional matchings that contain $v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_5v_9, v_6v_{10}, v_7v_{11}, v_8v_{12}, v_9v_{13}, v_{10}v_{14}$ are not T -overloaded in the 9-neighborhood W_e of e .

These edges are in at most four dimensional matchings and we select 10 new vertices, so this eliminates at most $\frac{19 \times 14\alpha'}{\epsilon}d$ or $\frac{266\alpha'}{\epsilon}d$ choices.

Moreover, every 2-neighborhood W in Q_d that one of these edges lie in is contained in W_e , so with this strategy and with the bounds on the number of requested and prescribed edges under h'' , in the process of choosing the set T , for every dimensional matching M and every 2-neighborhood W in Q_d , $E(W) \cap M$ contains at most

$$3\mu d + 3\alpha'd + (\epsilon d - 1) + 4 = 3\mu d + 3\alpha'd + \epsilon d + 3$$

edges that are used in T -swap. Similarly, every vertex is incident with at most

$$3\mu d + 3\alpha'd + (\epsilon d - 1) + 5 = 3\mu d + 3\alpha'd + \epsilon d + 4$$

edges that are used in T -swap; these upper bounds follow from the facts that the maximum number of edges of T_e incident with one vertex is 5, and the maximum number of edges from a given dimensional matching in T_e is 4.

- (2) None of the edges $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ are prescribed or requested or used before in T -swap.

Since all the possible choices for these edges are in the 2-neighborhood W_e of e in Q_d , and under h'' no vertex contains more than $\alpha'd$ prescribed edges and μd requested edges, T -swap uses at most $3\mu d + 3\alpha'd + \epsilon d + 4$ edges at each vertex, this condition eliminates at most

$$4 \times (\mu d + \alpha'd) + 4 \times (3\mu d + 3\alpha'd + \epsilon d + 4) = 16\mu d + 16\alpha'd + 4\epsilon d + 16$$

choices.

- (3) $v_1v_5, v_2v_6, v_3v_7, v_4v_8$ are not used before in S -swap.

Note that this condition ensures that $h''(v_1v_5) = h''(v_2v_6) = h''(v_3v_7) = h''(v_4v_8) = c_3$, where c_3 is the most common color of the dimensional matching that contains v_1v_5 . This eliminates at most $8\kappa d + 4\epsilon d + 4$ choices based on the conditions (c) and (d) of Lemma 3.3.

- (4) The three cycles $v_1v_2v_6v_5v_1, v_2v_3v_7v_6v_2, v_3v_4v_8v_7v_3$ are allowed before S -swap.

Each of the edges v_1v_2, v_2v_3, v_3v_4 belongs to at most τd non-allowed cycles, so this eliminates at most $3\tau d$ choices.

- (5) $c_2 \notin \{L(v_1v_5) \cup L(v_4v_8)\}$.

Color c_2 appears at most βd times in lists of the set of edges incident to v_1 (or v_4), so this eliminates at most $2\beta d$ choices.

(6) $c_1 \notin \{L(v_2v_6) \cup L(v_3v_7)\}$.

Similarly, this eliminates at most $2\beta d$ choices.

(7) If v_1v_2 is in a dimensional matching of color c_2 of h'' , then we choose v_5, v_6 such that v_5v_6 is not used in S -swap or T -swap, and v_5v_6 is not prescribed or requested. Note that the conditions imply that $h''(v_5v_6) = c_2$; in this case we choose v_9, v_{10} arbitrarily. Based on the restriction of edges used in each dimensional matching and $v_5v_6 \neq v_1v_2$, these restrictions eliminate at most

$$(2\kappa d + \epsilon d + 1) + (3\mu d + 3\alpha' d + \epsilon d + 4) + 1 = 2\kappa d + 3\alpha' d + 2\epsilon d + 3\mu d + 6$$

choices.

Else, we choose v_5, v_6, v_9, v_{10} such that

- (a) v_5v_6 is not used in S -swap (and also $v_5v_6 \neq v_1v_2$); this eliminates at most $2\kappa d + \epsilon d + 1$ choices. We can assume that v_5v_6 is in a dimensional matching of color c_4 .
- (b) v_5v_6 is in an allowed cycle $v_5v_6v_{10}v_9v_5$ with color c_2 (which means $h''(v_5v_9) = h''(v_6v_{10}) = c_2$, $h''(v_9v_{10}) = h''(v_5v_6) = c_4$), $c_4 \notin \{L(v_5v_9) \cup L(v_6v_{10})\}$ and $c_2 \notin \{L(v_5v_6) \cup L(v_9v_{10})\}$.

In the 3-neighborhood W of e in Q_d , S -swap uses at most $2\kappa d + \epsilon d + 1$ edges in the dimensional matching of color c_2 , so this eliminates at most $4\kappa d + 2\epsilon d + 2$ choices for v_5v_9 and v_6v_{10} . We also require that $v_9v_{10} \neq v_5v_6$ and that v_9v_{10} is not used in S -swap to make sure $h''(v_9v_{10}) = h''(v_5v_6) = c_4$; this eliminates at most $2\kappa d + \epsilon d + 2$ choices.

Since c_2 occurs βd times in the subset of the dimensional matching of color c_4 contained in the 27-neighborhood W of e in Q_d , and c_4 occurs βd times in the subset of the dimensional matching of color c_2 contained in W , the two conditions $c_4 \notin \{L(v_5v_9) \cup L(v_6v_{10})\}$ and $c_2 \notin \{L(v_5v_6) \cup L(v_9v_{10})\}$ eliminate at most $2\beta d$ choices.

- (c) $v_5v_6, v_5v_9, v_9v_{10}, v_6v_{10}$ are not prescribed or requested or used before in T -swap. Since all the possible choices for these edges are in the 2-neighborhood W of e in Q_d , this eliminates at most $4 \times (\mu d + \alpha' d) + 4 \times (3\mu d + 3\alpha' d + \epsilon d + 4)$ choices.

So in both cases, the choosing process eliminates at most

$$16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21$$

choices.

(8) v_7, v_8, v_{14}, v_{13} is chosen with same strategy as v_5, v_6, v_9, v_{10} .

Similarly, this eliminates at most $16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21$ choices.

(9) v_6, v_7, v_{11}, v_{12} is chosen with same strategy with v_5, v_6, v_9, v_{10} but the color c_2 is replaced by c_1 .

Again, this eliminates at most $16\mu d + 16\alpha' d + 8\kappa d + 8\epsilon d + 2\beta d + 21$ choices.

Summing up, we conclude that in total, there are at most

$$64\mu d + 64\alpha'd + 32\kappa d + 32\epsilon d + 10\beta d + 3\tau d + \frac{266\alpha'}{\epsilon}d + 83$$

forbidden choices for the configuration T_e .

This implies that we have

$$d - 3 - 64\mu d - 64\alpha'd - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 83$$

or

$$Z = d - 64\mu d - 64\alpha'd - 32\kappa d - 32\epsilon d - 10\beta d - 3\tau d - \frac{266\alpha'}{\epsilon}d - 86$$

choices for a configuration $T_{e'}$ in the process of constructing T , whenever e' is a prescribed edge.

By assumption, $Z > 0$, so there is a set $T_{v_2v_3}$ that satisfies all the above conditions. We add this set to T and apply this procedure for all prescribed edges uv with $h''(uv) \neq \varphi'(uv)$. Since the resulting subsets of T are disjoint, we can do the following transformation for each subset $T_{v_2v_3}$ as above.

- If $h''(v_5v_6) \neq c_2$, then interchange colors of the cycle $v_5v_6v_{10}v_9v_5$.
- If $h''(v_6v_7) \neq c_1$, then interchange colors of the cycle $v_6v_7v_{12}v_{11}v_6$.
- If $h''(v_7v_8) \neq c_2$, then interchange colors of the cycle $v_7v_8v_{14}v_{13}v_7$.
- Next, interchange colors of the cycles $v_1v_2v_6v_5v_1$ and $v_3v_4v_8v_7v_3$.
- Finally, interchange colors of the cycle $v_2v_3v_7v_6v_2$.

In the resulting edge coloring obtained from h'' , v_2v_3 is colored c_2 . Moreover, it follows from conditions (3), (4), (5), (7) that we do not create any new conflict edges by performing these swaps. We thus conclude that by repeating this swapping procedure for every prescribed edge, we obtain a new proper d -edge coloring which agrees with the precoloring φ' . \square

We have proved that it is possible to complete all the steps I-IV outlined in Section 2, thereby obtaining an extension of φ that avoids L ; this completes the proof of Theorem 2.2. \square

Let us now turn to the proof of Theorem 1.3. As we shall see, Property 2.1 of the standard edge coloring h of Q_d trivially yields the result.

Proof of Theorem 1.3 (sketch). Let φ be a d -edge precoloring of Q_d , and L a β -sparse list assignment for the non-precolored edges of Q_d , such that any edge e which is either precolored or satisfies $L(e) \neq \emptyset$ belongs to a distance-3 matching M in Q_d . Let h be the standard d -edge coloring of Q_d defined above.

Now, by arbitrarily picking a color from the set $\{1, \dots, d\} \setminus L(e')$ for each conflict edge e' , we can construct a precoloring φ' from φ such that an edge e of Q_d is precolored under φ' if and only if e is precolored under φ or e is a conflict of h with L . Furthermore, any 2-colored

4-cycle C with colors c_1 and c_2 under h , and satisfying that there is an edge $e \in E(C)$ with $\varphi'(e) = c_1$ and $h(e) = c_2$ is allowed. Moreover, since edges that are precolored under φ' are at distance at least 3 from each other, two 4-cycles containing distinct precolored edges are disjoint. Now, by Property 2.1, every edge in Q_d is contained in $d - 1$ 2-colored 4-cycles under h ; thus, we may complete the proof by simply swapping on a suitable set of disjoint 2-colored 4-cycles. \square

4 Upper bounds and further problems

We have proved that there are constants α and β such that every α -dense d -edge precoloring of Q_d can be extended to a proper d -edge coloring avoiding any given β -sparse list assignment for Q_d . The values we have found for α and β are quite small, to a large extent due to the calculations in Lemma 3.1.

Let us briefly compare our results obtained in this paper with corresponding results for complete bipartite graphs. Recall that a list assignment L for $K_{n,n}$ is β -sparse if each edge e of $K_{n,n}$ is assigned a list $L(e)$ of at most βn forbidden colors from $\{1, \dots, n\}$, and at every vertex v each color appears in lists of at most βn edges adjacent to v ; similarly an n -edge precoloring of $K_{n,n}$ is α -dense if every color is used at most αn times in the precoloring and at every vertex v at most αn edges incident to v are precolored. For $K_{n,n}$ Daykin and Häggkvist [DH84] conjectured that $\alpha = 1/4$ is the optimal value, and Häggkvist conjectured that $\beta = 1/3$ is optimal. The currently best value is $\alpha = 1/25$, as proven in [BKL⁺16]. The best known value for β is given in [ACÖ13] is far smaller, due to probabilistic tools. That one can simultaneously take α and β to be positive was proven in [ACM16].

For the hypercube Q_d , the following general proposition yields an upper bound on the values of α and β in Theorem 1.1.

Proposition 4.1. *Let G be a d -regular d -edge-colorable graph.*

- (i) *If every d -edge precoloring of G , satisfying that each vertex of G is incident to at most αd precolored edges, is extendable, then $\alpha < \frac{1}{2}$.*
- (ii) *If every list assignment L , such that $|L(e)| \leq \beta d$ for each edge $e \in E(G)$, and for each vertex v each color appears in at most βd lists of edges incident with v , is avoidable, then $\beta < \frac{1}{2}$.*
- (iii) *If every precoloring as in (i) is extendable to a coloring avoiding any list assignment as in (ii), then $\alpha + \beta < \frac{1}{2}$.*

Proof. (i) Let $u_1 u_2$ be an edge of G . We define an edge precoloring φ of G by coloring $\lceil d/2 \rceil$ edges incident with u_1 and distinct from $u_1 u_2$ by colors $1, \dots, \lceil d/2 \rceil$; next, color $\lceil d/2 \rceil$ edges incident to u_2 and distinct from $u_1 u_2$ by colors $\lceil d/2 \rceil + 1, \dots, d$. This yields an edge d -precoloring which is not extendable to a proper d -edge coloring, so necessarily $\alpha < 1/2$.

- (ii) Let $u_1 u_2$ be an edge of G . Next, to $\lceil d/2 \rceil$ edges incident with u_1 , but not u_2 , assign identical color lists containing all the colors $1, \dots, \lceil d/2 \rceil$. Similarly assign to $\lceil d/2 \rceil$ edges

incident with u_2 , but not u_1 , identical color lists containing all the colors $\lceil d/2 \rceil + 1, \dots, d$. Now, since apart from u_1u_2 , there are at most $\lceil d/2 \rceil - 1$ edges incident with u_1 , where colors $1, \dots, \lceil d/2 \rceil$ are not forbidden, we must have that u_1u_2 is colored with a color from $1, \dots, \lceil d/2 \rceil$ in any proper d -edge coloring of G avoiding the list assignment; similarly by the restrictions at u_2 , u_1u_2 must be colored with a color from $\lceil d/2 \rceil + 1, \dots, d$ in any coloring of G avoiding the list assignments at u_2 . This is clearly not possible, so the list assignment is unavoidable, and thus $\beta < 1/2$.

- (iii) The precoloring and list-assignments defined above can be combined in the following way (we assume that $1/2 > \beta > \alpha$ and that αd and βd are integers):

Let u_1u_2 be an edge as above, let H_1 be the star induced by u_1 and its neighbors except for u_2 , and H_2 the corresponding star for u_2 .

We now consider the assignment where in H_1 there are αd precolored edges incident with u_1 using colors $d - \alpha d + 1, \dots, d$; moreover, exactly βd edges in H_1 incident with u_1 , distinct from the precolored ones, are assigned identical lists with colors $1, \dots, \beta d$. Similarly, in H_2 there are αd precolored edges incident with u_2 using colors $1, \dots, \alpha d$; moreover, there are precisely βd edges in H_2 incident with u_2 , distinct from the precolored edges, all of which are assigned identical color lists containing colors $d - \beta d + 1, \dots, d$.

Now, for any proper d -edge coloring f of G which is an extension of the precoloring and which avoids the the list assignment, the colors $1, \dots, \beta d$ must appear on edges incident with u_1 which are neither precolored nor are assigned a non-empty list of forbidden colors. By a similar argument for H_2 , we must have that for any coloring f which is an extension of the precoloring and also avoids the list assignment, colors $d - \beta d + 1, \dots, d$ must appear on edges incident with u_2 which are neither precolored nor are assigned a non-empty list of forbidden colors. Note that both u_1 and u_2 are incident with exactly $d - \beta d - \alpha d$ edges which are neither precolored nor contain a non-empty list of forbidden colors. Thus if $d - \alpha d - \beta d \leq \beta d$, then the edge u_1u_2 must receive a color both from the set $\{1, \dots, \beta d\}$ and from the set $\{d - \beta d + 1, \dots, d\}$ under f . Moreover, if $\beta d \leq d - \beta d + 1$, then these sets are disjoint, implying that there is no extension of the precoloring which avoids the given list assignment. Here, by choosing β close to $1/2$ and α small, we can make the sum $\alpha + \beta$ arbitrarily close to $1/2$. □

Returning to the setup of Theorem 1.1, we have attempted to find constructions which yield better upper bounds for α and β for the hypercubes, but have not been able to do so. Moreover, the conditions (ii) and (iii) for a precoloring of Q_d to be α -dense are not probably not best possible in terms of size of the neighborhoods. Those conditions are required in our proof, but might be far stronger than what is actually needed in order for a compatible edge coloring to exist. Nonetheless it would be interesting to see how far Theorem 1.1 can be improved in its current form (possibly with decreased size of the neighborhoods).

Problem 4.2. *What are the optimal values for α and β in Theorem 1.1?*

Our focus here has been the family of hypercubes but of course the type of problem we have considered is interesting for more general graphs as well. The examples in [EGv⁺14]

show that in order to get results similar to those for $K_{n,n}$, and those given in this paper, one must impose some structural conditions on the considered family of graphs. Both $K_{n,n}$, and Q_d are well connected bipartite graphs and it would be interesting to see how far Proposition 4.1 can be improved for this general class of graphs.

Problem 4.3. *Given a precoloring and a list assignment as in Proposition 4.1, what are the optimal values for α and β for the family of d -regular, d -edge connected, bipartite graphs?*

Here the cases closest to our results are of course those where d is a function of the number of vertices in the graph.

Finally, as mentioned in the introduction, our proof method easily give us Theorem 1.3 where the edges which are precolored or have non-empty lists of forbidden colors on them are forced to lie in a distance-3 matching. Here it is natural to ask if this result holds for distance-2 matchings as well.

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