Cubic graphs with equal independence number and matching number

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Abstract

Caro, Davila, and Pepper recently proved $\delta(G)\alpha(G) \leq \Delta(G)\mu(G)$ for every graph G with minimum degree $\delta(G)$, maximum degree $\Delta(G)$, independence number $\alpha(G)$, and matching number $\mu(G)$. Answering some problems they posed, we characterize the extremal graphs for $\delta(G) < \Delta(G)$ as well as for $\delta(G) = \Delta(G) = 3$.

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1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology. Recently, Caro, Davila, and Pepper [1] proved the inequality

$$\delta(G)\alpha(G) \leq \Delta(G)\mu(G)$$

for every graph G with minimum degree $\delta(G)$, maximum degree $\Delta(G)$, independence number $\alpha(G)$, and matching number $\mu(G)$. As an open problem they asked for the characterization of the extremal graphs, that is, those graphs that satisfy this inequality with equality. In particular, they asked for the characterization of the cubic graphs G with $\alpha(G) = \mu(G)$. In the present note, we give a simple proof of the above inequality, which allows to characterize the non-regular extremal graphs. Furthermore, we characterize the cubic extremal graphs.

2 Results

For positive integers δ and Δ with $\delta < \Delta$, a bipartite graph is (δ, Δ)-*regular* if it has a bipartition with partite sets A and B such that every vertex in A has degree δ and every vertex in B has degree Δ.

Theorem 1. *If G is a graph with minimum degree δ and maximum degree Δ, then*

$$\delta\alpha(G) \leq \Delta\mu(G). \tag{1}$$

Furthermore, if $\delta < \Delta$, then equality holds in (1) if and only if G is bipartite and (δ, Δ)-regular.

Proof. Let I be a maximum independent set in G. Let $R = V(G) \setminus I$, and let H be the bipartite spanning subgraph of G that contains all edges of G between I and R. Let M be a maximum matching in H, and let U be a minimum vertex cover in H. See Figure 1 for an illustration.

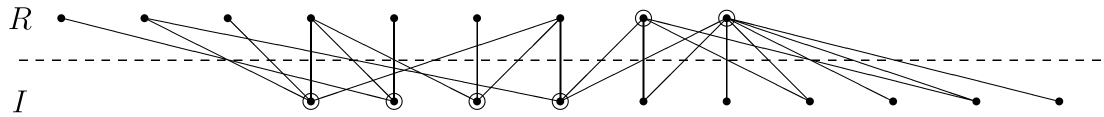

Figure 1: The partition into I and R. The vertical edges form the matching M, and the encircled vertices form the vertex cover U.

Since G is bipartite, König's theorem [3] implies that $\mu(H) = |M| = |U|$, and that U intersects each edge in M in exactly one vertex. Let m be the number of edges of H between $I \setminus U$ and $U \cap R$. Since each edge leaving $I \setminus U$ enters $U \cap R$, we obtain, for $k = |I \cap U|$, that

$$\delta(\alpha(G) - k) = \delta|I \setminus U| \leq m \leq \Delta|U \cap R| = \Delta(\mu(H) - k) \leq \Delta(\mu(G) - k). \tag{2}$$

This implies

$$\delta\alpha(G) \leq \delta\alpha(G) + (\Delta - \delta)k \leq \Delta\mu(H) \leq \Delta\mu(G), \tag{3}$$

that is, the inequality (1) follows.

We proceed to the characterization of the graphs G that satisfy $\delta\alpha(G) = \Delta\mu(G)$ for $\delta < \Delta$. Since $\alpha(G)$ and $\mu(G)$ are additive with respect to the components, it suffices to characterize the connected graphs. If G is bipartite and (δ, Δ)-regular, and has partite sets A and B as above, then Hall's theorem [2] and König's theorem imply that $\mu(G) = |B|$, and that $\alpha(G) = n - \mu(G) = |A|$. Furthermore, the number of edges of G equals $\delta|A| = \delta\alpha(G)$ and $\Delta|B| = \Delta\mu(G)$, that is, $\delta\alpha(G) = \Delta\mu(G)$. Now, let G be a connected graph with $\delta\alpha(G) = \Delta\mu(G)$. If H, M, U, m, and k are as above, then equality holds throughout (2) and (3). This implies that $k = 0$ and $\delta|I \setminus U| = m = \Delta|U \cap R|$, which implies that $G[(I \setminus U) \cup (U \cap R)]$ is a non-empty bipartite graph, where every vertex in $I \setminus U$ has degree δ and every vertex in $U \cap R$ has degree Δ. Since all edges in G leaving $I \setminus U$ enter $U \cap R$, this implies that no edge of G has only one endpoint in $(I \setminus U) \cup (U \cap R)$. Since G is connected, this implies that G equals $G[(I \setminus U) \cup (U \cap R)]$, that is bipartite and (δ, Δ)-regular. This completes the proof. $\square$

Note that all cycles satisfy (1) with equality, in particular, there are non-bipartite extremal graphs. For higher degrees of regularity, that is, for $\delta = \Delta \geq 3$, the extremal graphs have a richer structure, which we elucidate for $\delta = \Delta = 3$. A graph G is a *bubble with contact vertex z and partition* (I, R) if the vertex set of G can be partitioned into two sets I and R such that

- every vertex in $V(G) \setminus \{z\}$ has degree 3 and z has degree 2,
- I is independent, and
- z lies in R and $G[R]$ contains exactly one edge.

Since a bubble G has degree sequence $3, \ldots, 3, 2$, it is not bipartite. Counting the edges of G implies that $|R| = |I| + 1$, and, hence, $|I| = (n(G) - 1)/2$. Figure 2 illustrates some connected bubbles.

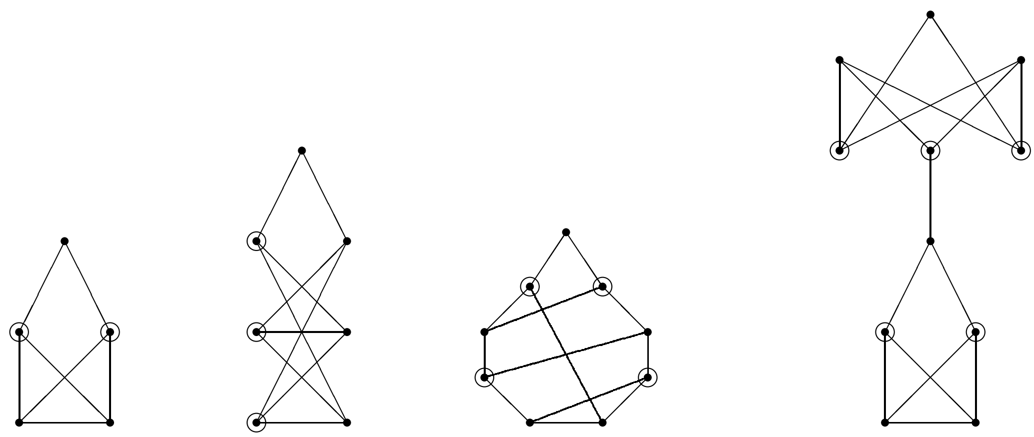

Figure 2: Four connected bubbles; the contact vertices are the topmost vertices, the encircled vertices form the sets I, and the rightmost bubble illustrates that a bubble may properly contain a smaller bubble.

Lemma 1. *If G is a bubble with contact vertex z and partition (I, R), then*

$$\alpha(G) = \alpha(G - z) = \mu(G) = \mu(G - z) = (n(G) - 1)/2.$$

Furthermore, if G is not 2-connected, then some proper induced subgraph G' of G is also a bubble with partition (I', R') such that $I' \subseteq I$ and $R' \subseteq R$.

Proof. Let $p = (n(G) - 1)/2$, and let xy be the unique edge of $G[R]$. Note that z may coincide with x or y.

Since every matching M of G either contains xy and at most $|R| - 2$ further edges incident with the vertices in $R \setminus \{x, y\}$ or does not contain xy and at most $|I|$ edges incident with the vertices in I, we have $\mu(G) \leq |R| - 1 = |I| = p$.

Let u be any vertex from $\{x, y, z\}$. For some set $S \subseteq I$, let $T = N_G(S)$. Since I is independent, we obtain $T \subseteq R$, and the vertex degrees imply $|T| \geq |S|$. Furthermore, if T contains u, then the edge xy and the degree of z imply that $|T| \geq |S| + 1$. Altogether, $|N_{G-u}(S)| = |T \setminus \{u\}| \geq |S|$ for every set $S \subseteq I$, and Hall's Theorem implies the existence of a matching M_u in $G-u$ that saturates each vertex in I, in particular, $\mu(G) = \mu(G-z) = p$.

Now, let J be a maximum independent set in G. If J does not contain x, then M_x and the edges xy imply that $|J| \leq p$. Similarly, if J does not contain y, then $|J| \leq p$, which implies $\alpha(G) \leq p$. Since I is an independent set of order p, we obtain that $\alpha(G) = \alpha(G - z) = p$.

Finally, suppose that G is not 2-connected. If G is not connected, then the vertex degrees easily imply that the component of G containing the edge xy is also a bubble with a partition (I', R') such that $I' \subseteq I$ and $R' \subseteq R$. Hence, we may assume that G is connected but not 2-connected. Since G is subcubic, this implies that G has a bridge uv. Let G_u and G_v be the components of $G - uv$ containing u and v, respectively. Let $I_u = V(G_u) \cap I$, $d_1 = \sum_{w \in I_u} d_{G_u}(w)$, $R_u = V(G_u) \cap R$, and $d_2 = \sum_{w \in R_u} d_{G_u}(w)$. If $u, v \in R$, then uv is the unique edge xy in $G[R]$, the graph G_u is a bipartite graph with partite sets I_u and R_u, but d_1 and d_2 have different parities modulo 3, which is a contradiction. Hence, by symmetry, we may assume that $u \in R$ and $v \in I$. Since d_1 is a multiple of 3, it follows that the unique edge xy of $G[R]$ lies within R_u, and that the contact vertex of G does not lie R_u. Hence, G_u is a bubble with partition (I_u, R_u), which completes the proof. $\square$

A graph G is *special* if it is connected, cubic, and the vertex set of G can be partitioned into sets $V_0, V_1, \dots, V_\ell$ such that

- the graph $G[V_0]$ is a non-empty bipartite graph with partite sets I_0 and R_0 such that every vertex in R_0 has degree 3 in $G[V_0]$, and
- for every i in $[\ell]$, the graph $G[V_i]$ is a 2-connected bubble with contact vertex z_i.

Note that, since G is connected and V_0 is non-empty, it follows that $G[V_0]$ is connected, and that, for every i in $[\ell]$, the graph G contains a bridge between z_i and some vertex in I_0. Since G is cubic, this implies $\ell = \sum_{u \in I_0}(3 - d_{G[V_0]}(u))$. In particular, if $\ell = 0$, then G is bipartite. See Figure 3 for an illustration.

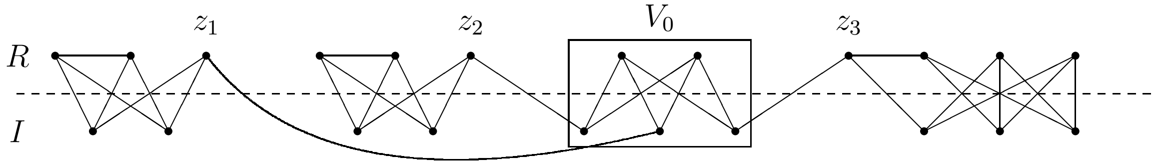

Figure 3: A cubic graph G with $\alpha(G) = \mu(G) = 10$.

Theorem 2. *A connected cubic graph G satisfies $\alpha(G) = \mu(G)$ if and only if it is special.*

Proof. First, we assume that G is special. For every i in $\{0\} \cup [\ell]$, let $G_i = G[V_i]$. Let the partite sets I_0 and R_0 of G_0 be as above. For every i in $[\ell]$, let the bubble G_i have partition (I_i, R_i). Let $I = I_0 \cup I_1 \cup \dots \cup I_\ell$ and $R = R_0 \cup R_1 \cup \dots \cup R_\ell$. Since $\alpha(G_i) = \alpha(G_i - z_i)$ for every i in $[\ell]$, we have $\alpha(G) = \alpha(G_0) + \sum_{i=1}^{\ell} \alpha(G_i)$. By the vertex degrees, we have $|N_{G_0}(S)| \geq |S|$ for every set $S \subseteq R_0$, which implies $\alpha(G_0) = |I_0|$. Together with Lemma 1, we obtain $\alpha(G) = |I_0| + \sum_{i=1}^{\ell}(n(G_i) - 1)/2$. Now, let M be a maximum matching in G. By Lemma 1, the set M contains at most $(n(G_i) - 1)/2$ edges of G_i for every i in $[\ell]$, which implies $\mu(G) \leq \mu(G') + \sum_{i=1}^{\ell}(n(G_i) - 1)/2$, where $G' = G[V_0 \cup \{z_1, \dots, z_\ell\}]$. Since G' is bipartite with partite sets I_0 and $R_0 \cup \{z_1, \dots, z_\ell\}$, we obtain

$$|I_0| + \sum_{i=1}^{\ell} \frac{n(G_i) - 1}{2} = \alpha(G) \overset{(1)}{\leq} \mu(G) \leq |I_0| + \sum_{i=1}^{\ell} \frac{n(G_i) - 1}{2},$$

in particular, $\alpha(G) = \mu(G)$.

Now, let G be a connected cubic graph with $\alpha(G) = \mu(G)$. Let I be a maximum independent set in G, and let $R = V(G) \setminus I$. Let $V_1, \dots, V_\ell$ be a maximal collection of disjoint sets of vertices of G such that, for every i in $[\ell]$, the graph $G[V_i]$ is a 2-connected bubble with contact vertex z_i and partition (I_i, R_i), where $I_i \subseteq I$ and $R_i \subseteq R$, and the unique neighbor of z_i outside of V_i belongs to I. Let $I_0 = I \setminus (I_1 \cup \dots \cup I_\ell)$ and $R_0 = R \setminus (R_1 \cup \dots \cup R_\ell)$. If R_0 is an independent set, then G is special. Therefore, for a contradiction, we may assume that xy is an edge of G with $x, y \in R_0$. Let $G' = G - \bigcup_{i \in [\ell]}(V_i \setminus \{z_i\})$, and let $G'' = G' - \{x, y\}$. See Figure 4 for an illustration.

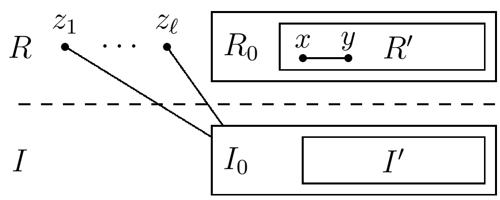

Figure 4: The graph G'; removing from G' the vertices x and y yields G'' while removing from G' all vertices from the bubble B except for its contact vertex yields G'''.

If G'' contains a matching M_0 that saturates I_0, then, by Lemma 1, the union of $\{xy\}$, the matching M_0, and maximum matchings in each $G[V_i] - z_i$ yields a matching of size more than $|I| = \alpha(G) = \mu(G)$, which is a contradiction. Hence, by Hall's theorem, there is a set $I' \subseteq I_0$ with $|N_{G''}(I')| < |I'|$. Let $R' = N_{G'}(I')$, and let $B = G[I' \cup R']$. By the vertex degrees, the independence of I, and the edge xy, we obtain $|R'| > |I'|$, and, hence, $R' = N_{G''}(I') \cup \{x, y\}$ and $|R'| = |I'| + 1$, see Figure 4. Since $\sum_{u \in I'} d_{G'}(u)$ is a multiple of 3, this implies that xy is the only edge of G within R', that exactly one vertex in R' has degree 2 in B, and that all remaining vertices of B have degree 3 in B, that is, the graph B is a bubble with some contact vertex z and partition (I', R'). Since each z_i has degree 1 in G', and B contains no vertex of degree less than 2, we have $I' \subseteq I_0$ and $R' \subseteq R_0$, see Figure 4. By Lemma 1, we may assume that B is 2-connected. Let $G''' = G' - (V(B) \setminus \{z\})$. Let z' be the neighbor of z outside of $V(B)$. Suppose, for a contradiction, that z' lies in R_0. By the vertex degrees and the independence of I, for every $S \subseteq I_0 \setminus I'$, we have $|N_{G'''}(S)| \geq |S|$, and, in view of the edge zz', if $N_{G'''}(S)$ contains z', then $|N_{G'''}(S)| > |S|$, which implies $|N_{G'''-z'}(S)| \geq |S|$. By Hall's theorem, the graph $G''' - z'$ has a matching saturating I_0, which, by Lemma 1, together with the edge zz', maximum matchings in each $G[V_i] - z_i$, and a maximum matching in $B - z$ yields a matching of size more than $|I| = \alpha(G) = \mu(G)$, which is a contradiction. Hence, $z' \in I$. Now, the set $V(B)$ can be added to the collection $V_1, \ldots, V_\ell$, contradicting its maximality, which completes the proof. $\square$

For a given connected cubic graph G, the constructive proofs of Lemma 1 and Theorem 2 easily allow to design an efficient algorithm that decides whether $\alpha(G) = \mu(G)$, and, that returns the partition of $V(G)$ into the sets $V_0, V_1, \ldots, V_\ell$ in this case. It remains an open problem to characterize the extremal k-regular graphs for every k at least 4; it might even be the case that these graphs are NP-hard to recognize.

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