Sum of weighted records in set partitions

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Abstract

The purpose of this paper is to find an explicit formula and asymptotic estimate for the total number of sum of weighted records over set partitions of [n] in terms of Bell numbers. For that we study the generating function for the number of set partitions of [n] according to the statistic sum of weighted records.

Keywords: Records, Sum of weighted records, Set partitions, Generating functions, Bell numbers and Asymptotic estimate.

1 Introduction

Let σ_i be an element in the permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_\ell$, we say that σ_i is a *record* of *position i* if $\sigma_i > \sigma_i$ for all $j = 1, 2, \dots, i-1$. The study of records in permutations interested Rényi [7]. More recently another statistic which depends on records have been studied by Kortchemski [5] who defined the statistic *srec*, where $\operatorname{srec}(\sigma)$ defined as the sum of the positions of all records of σ . For example, permutation $\pi = 12534$ has 3 records, 1, 2, 5 and srec $(\pi) = 1 + 2 + 3 = 6$. For relevant papers about records you can see for example [2] and [3]. In this Paper we want to focus on partitions of a set. Recall that a partition Π of set [n] of size k (a partition of [n] with exactly k blocks) is a collection $\{B_1, B_2, \ldots, B_k\}$, where $\emptyset \neq B_i \subseteq [n]$ for all i and $B_i \cap B_j = \emptyset$ for $i \neq j$, such that $\bigcup_{i=1}^k B_i = [n]$. The elements B_i are called *blocks*, and we use the assumption that B_1, B_2, \dots, B_k are listed in increasing order of their minimal elements, that is, $minB_1 < minB_2 < \cdots < minB_k$. The set of all partitions of [n] with exactly k blocks is denoted by $P_{n,k}$ and $|P_{n,k}| = S_{n,k}$, which is known as the Stirling numbers of the second kind [8]. And the set of all partitions of [n] is denoted by P_n and $|P_n| = \sum_{k=1}^n S_{n,k} = B_n$, which is the *n*-th Bell number [8]. Any partition Π can be written as $\pi_1 \pi_2 \cdots \pi_n$, where $i \in B_{\pi_i}$ for all i, and this form is called the *canonical sequential form*. For example $\Pi = \{\{12\}, \{3\}, \{4\}\}$ is a partition of [4], the canonical sequential form is $\pi = 1123$. For more details about set partitions we suggest Mansour's book [6]. The important results about records, obtained by Knopfmacher, Mansour and Wagner [4] which state the asymptotic mean value and variance for the number, and for the sum of positions, of record in all partitions of [n] are central to my study. In this paper, we define a new statistic swree, where swree(π) is the sum of the position of a record in π multiplied by the value of the record over all the records in P_n . We will study this statistic from the point of view of canonical sequential form. For instance, if $\pi = 121132$ the swrec $(\pi) = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 5 = 20$.

2 Main Results

2.1 The ordinary generating function for the number of set partitions according to the statistic swrec

Let $P_k(x,q)$ be the generating function for the number of partitions of [n] with exactly k blocks according to the statistic swree, that is

$$P_k(x,q) = \sum_{n \ge k} \sum_{\pi \in P_{n,k}} x^n q^{\operatorname{swrec}(\pi)}.$$

Theorem 1 The generating function for the number of partitions of [n] with exactly k blocks according to the statistic swree is given by

$$P_k(x,q) = \prod_{i=1}^k \frac{xq^i q^{(k+1-i)(k-i)}}{1 - ix \prod_{j=i+1}^k q^j}.$$
(1)

Proof As we know, a set partition of [n] with exactly k blocks can be presented as canonical sequential form:

$$\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$$

for some k, where $\pi^{(j)}$ denotes an arbitrary word over an alphabet [j] including the empty word. Thus, the contribution of $\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k$ to the generating function $P_k(x,q)$ is $xq^k P_{k-1}(xq^k,q)$ and the contribution of $\pi^{(k)}$ to the generating function $P_k(x,q)$ is $\frac{1}{1-kx}$. Therefore, the corresponding generating function satisfies

$$P_k(x,q) = \frac{xq^k}{1-kx} P_{k-1}(xq^k,q)$$

By using induction on k together with the initial condition $P_1(x,q) = \frac{xq}{1-x}$ we obtain the required result.

2.2 Exact and asymptotic expression for $\sum_{\pi \in P_n} \operatorname{swrec}(\pi)$

In this section, we aim to prove that the total number of the swree over all partitions of [n] is

$$\frac{3}{4}(B_{n+3} - B_{n+2}) - (n + \frac{7}{4})B_{n+1} - \frac{1}{2}(n+1)B_n.$$

And we want to show that asymptotically the total number of the swree over all partitions of [n] is

$$B_n \frac{n^3}{r^3} \left(1 + \frac{r}{n} \right) \left(1 + O(\frac{\log n}{n}) \right),$$

where r is the positive root of $re^r = n + 1$.

For that we need to perform the following steps:

• Firstly, we find the partial derivative of $P_k(x,q)$ with respect to q and substitute q = 1, that is $\frac{d}{dq}P_k(x,q)|_{q=1}$.

• Secondly, we pass from $\frac{d}{dq}P_k(x,q)|_{q=1}$ to $\frac{d}{dq}\widetilde{P}_k(x,u,q)|_{u=q=1}$, where $\widetilde{P}_k(x,u,q)$ is the exponential generating function for the number of partitions of [n] with exactly k blocks according to the statistic swree.

• Finally, we derive the total number of swrec over all partitions of [n], and the asymptotic estimate for the total number of swrec over all partitions of [n].

Lemma 2 For all $k \geq 1$,

$$\frac{d}{dq}P_k(x,q)\mid_{q=1} = \frac{x^k(\binom{k+1}{2} + 2\binom{k+1}{3})}{(1-x)\dots(1-kx)} + \frac{x^{k+1}}{(1-x)\dots(1-kx)}\sum_{i=1}^k \frac{i(i+1+k)(k-i)}{2(1-ix)}.$$
(2)

Proof By differentiating (1) with respect to q, we obtain

$$\frac{d}{dq}P_k(x,q)\mid_{q=1} = P_k(x,1)\sum_{i=1}^k \lim_{q \to 1} \left(\frac{\frac{d}{dq}L_i(q)}{L_i(q)}\right),\tag{3}$$

where

$$L_i(q) = \frac{xq^i q^{(k+1-i)(k-i)}}{1 - ix \prod_{j=i+1}^k q^j}.$$

We have

$$\lim_{q \to 1} \frac{d}{dq} L_i(q) = \frac{x \ell A_{i,k}(x,1) - x \lim_{q \to 1} \frac{d}{dq} A_{i,k}(x,q)}{(A_{i,k}(x,1))^2}.$$
(4)

Where $A_{i,k}(x,q) = 1 - ix \prod_{j=i+1}^{k} q^j$ and $\ell = (k+1-i)(k-i) + i$. By using the differentiation rules we get $\frac{d}{dq}A_{i,k}(x,q) = -ix \sum_{m=i+1}^{k} mq^{m-1} \prod_{\substack{j=i+1 \ j\neq m}}^{k} q^j$. Therefore,

$$\lim_{q \to 1} \frac{d}{dq} L_i(q) = \frac{x \left(2(i + (k+1-i)(k-i))(1-ix) + i(i+1+k)(k-i)x\right)}{2(1-ix)^2}$$

which leads to

$$\lim_{q \to 1} \left(\frac{\frac{d}{dq} L_i(q)}{L_i(q)} \right) = i + (k+1-i)(k-i) + \frac{i(i+1+k)(k-i)x}{2(1-ix)}.$$
(5)

Hence, by substituting (5) in (3) we obtain

$$\frac{d}{dq}P_k(x,q)|_{q=1} = \frac{x^k}{(1-x)\dots(1-kx)} \sum_{i=1}^k \left(i + (k+1-i)(k-i) + \frac{i(i+1+k)(k-i)x}{2(1-ix)}\right)$$
$$= \frac{x^k(\binom{k+1}{2} + 2\binom{k+1}{3})}{(1-x)\dots(1-kx)} + \frac{x^{k+1}}{(1-x)\dots(1-kx)} \sum_{i=1}^k \frac{i(i+1+k)(k-i)}{2(1-ix)},$$

as claimed.

Now we need to find $[x^n] \frac{d}{dq} P_k(x,q) |_{q=1}$ to obtain the total number of swree. We will study the exponential generating function instead of the ordinary generating function. Let $\tilde{P}_k(x,u,q)$ be the exponential generating function for the number of partitions of [n] with exactly k blocks according to the statistic swree, that is

$$\widetilde{P}_k(x, u, q) = \sum_{n \ge k} \sum_{\pi \in P_{n,k}} \frac{x^n u^k q^{\operatorname{swrec}(\pi)}}{n!}.$$

Theorem 3 The partial derivative of $\widetilde{P}_k(x, u, q)$ with respect to q at u = q = 1 is given by,

$$\frac{d}{dq}\widetilde{P}_k(x,u,q)\mid_{u=q=1} = e^{e^x-1}(\frac{3}{4}e^{3x} + \frac{3}{2}e^{2x} - \frac{7}{4}e^x - xe^{2x} - \frac{3}{2}xe^x - \frac{1}{2}).$$
(6)

Proof In order to prove the above result we need the following proposition:

Proposition 4 The partial derivative $\frac{d}{dq}P_k(x,q) \mid_{\substack{x=y^{-1}\\q=1}}$ can be decomposed as

$$\sum_{m=1}^{k} \left(\frac{a_{k,m}}{(y-m)^2} + \frac{b_{k,m}}{y-m} \right),\tag{7}$$

where

$$a_{k,m} = \frac{(-1)^{k-m}m(1+k+m)(k-m)}{2(m-1)!(k-m)!}$$

and

$$b_{k,m} = \frac{(-1)^{k-m} \left(k^2 \left(\frac{m}{4}+1\right)+k \left(\frac{m^2}{2}+\frac{3m}{4}+1\right)-\left(\frac{3m^2}{2}+m\right)\right)}{(m-1)! (k-m)!}$$

Proof We rewrite (2) as

$$\frac{d}{dq}P_k(x,q)\mid_{q=1} = x^k \prod_{i=1}^k (1-ix)^{-1} \left(\binom{k+1}{2} + 2\binom{k+1}{3} + \sum_{i=1}^k \frac{i(1+k+i)(k-i)x}{2(1-ix)} \right)$$
$$= x^k \prod_{i=1}^k (1-ix)^{-1} \left(\frac{k(k+1)(2k+1)}{6} + \sum_{i=1}^k \frac{i(1+k+i)(k-i)x}{2(1-ix)} \right).$$

By replacing $x^{-1} = y$ in the above equation we get

$$\prod_{i=1}^{k} (y-i)^{-1} \left(\frac{k(k+1)(2k+1)}{6} + \sum_{i=1}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)} \right).$$
(8)

The above expression decomposed as

$$\sum_{m=1}^{k} \left(\frac{a_{k,m}}{(y-m)^2} + \frac{b_{k,m}}{y-m} \right).$$

In order to find the coefficients $a_{k,m}$ and $b_{k,m}$, we need to consider the expansion of (8) at y = m, as follows:

$$\begin{aligned} (y-m)^{-1} \prod_{\substack{i=1\\i\neq m}}^{k} (y-m+m-i)^{-1} \left(\frac{k(k+1)(2k+1)}{6} + \frac{m(1+k+m)(k-m)}{2(y-m)} + \sum_{\substack{i=1\\i\neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)} \right) \\ &= (y-m)^{-1} \prod_{\substack{i=1\\i\neq m}}^{k} \left((m-i)^{-1}(1+\frac{y-m}{m-i})^{-1} \right) \cdot \\ &\left(\frac{k(k+1)(2k+1)}{6} + \frac{m(1+k+m)(k-m)}{2(y-m)} + \sum_{\substack{i=1\\i\neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)} \right) . \end{aligned}$$

Using Taylor series to expand $(1 + \frac{y-m}{m-i})^{-1}$ and $\frac{i(1+k+i)(k-i)}{2(y-i)}$ at y = m we get

$$(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \prod_{\substack{i=1\\i\neq m}}^{k} \left(1 - \frac{y-m}{m-i} + O((y-m)^2)\right)$$
$$\cdot \left(\frac{k(k+1)(2k+1)}{6} + \frac{m(1+k+m)(k-m)}{2(y-m)} + \sum_{\substack{i=1\\i\neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(m-i)} + O(y-m)\right),$$

which is equivalent to

$$(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \left(1 - \sum_{\substack{i=1\\i \neq m}}^{k} \frac{y-m}{m-i} + O((y-m)^2) \right) \\ \cdot \left(\frac{k(k+1)(2k+1)}{6} + \frac{m(1+k+m)(k-m)}{2(y-m)} + \sum_{\substack{i=1\\i \neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(m-i)} + O(y-m) \right).$$

We need to simplify the product, and consider the coefficients of $(y-m)^{-1}$ and $(y-m)^{-2}$ as follows:

$$(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \cdot \left(\frac{k(k+1)(2k+1)}{6} + \frac{m(1+k+m)(k-m)}{2(y-m)} + \sum_{\substack{i=1\\i\neq m}}^{k} \frac{i(1+k+i)(k-i) - m(1+k+m)(k-m)}{2(m-i)} + O(y-m)\right).$$

By using Maple we compute the term in the summation, which hints

$$(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \cdot \left(\frac{m(1+k+m)(k-m)}{2(y-m)} + \frac{k(k+1)(2k+1)}{6} + \frac{1}{2} \sum_{\substack{i=1\\i \neq m}}^{k} (i^2+i+im-k^2-k+m+m^2) + O(y-m)\right)$$
$$= (y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!}$$

$$\left(\frac{m(1+k+m)(k-m)}{2(y-m)} + k^2(\frac{m}{4}+1) + k(\frac{m^2}{2} + \frac{3m}{4}+1) - (\frac{3m^2}{2}+m) + O(y-m)\right)$$

Hence, by finding the coefficients of of $(y - m)^{-1}$ and $(y - m)^{-2}$ we complete the proof.

Now we return to the proof of the theorem. We use (7) for passing to exponential generating function, by substituting

$$\frac{e^{mx} - 1}{m} = \sum_{\ell \ge 0} \frac{m^{\ell} x^{\ell+1}}{(\ell+1)!}$$

 in

$$\frac{1}{y-m} = \frac{x}{1-mx} = \sum_{\ell \ge 0} m^{\ell} x^{\ell+1}$$

and $\frac{e^{mx}(mx-1)+1}{m^2}$ in $\frac{1}{(y-m)^2}$. Moreover, by summing over all k we obtain the generating function

$$\sum_{k\geq 1} u^k \sum_{m=1}^k \frac{(-1)^{k-m}m(1+k+m)(k-m)}{2(m-1)!(k-m)!} \cdot \frac{e^{mx}(mx-1)+1}{m^2} + \sum_{k\geq 1} u^k \sum_{m=1}^k \frac{(-1)^{k-m}\left(k^2(\frac{m}{4}+1)+k(\frac{m^2}{2}+\frac{3m}{4}+1)-(\frac{3m^2}{2}+m)\right)}{(m-1)!(k-m)!} \cdot \frac{e^{mx}-1}{m}$$

We need to change the order of the summation as follows:

$$\sum_{m\geq 1} \frac{e^{mx}(mx-1)+1}{m!} \sum_{k\geq m} \frac{(-1)^{k-m}(1+k+m)(k-m)u^k}{2(k-m)!} + \sum_{m\geq 1} \frac{e^{mx}-1}{m!} \sum_{k\geq m} \frac{(-1)^{k-m} \left(k^2(\frac{m}{4}+1)+k(\frac{m^2}{2}+\frac{3m}{4}+1)-(\frac{3m^2}{2}+m)\right)}{(k-m)!} u^k.$$

By substituting $\ell = k - m$ and rewriting the above result we obtain the following form:

$$\sum_{m\geq 1} \frac{e^{mx}(mx-1)+1}{m!} \sum_{\ell\geq 0} \frac{(-1)^{\ell}(2m+\ell+1)\ell u^{m+\ell}}{2\ell!} + \sum_{m\geq 1} \frac{e^{mx}-1}{m!} \sum_{\ell\geq 0} \frac{(-1)^{\ell} \left((m+\ell)^2(\frac{m}{4}+1)+(m+\ell)(\frac{m^2}{2}+\frac{3m}{4}+1)-(\frac{3m^2}{2}+m)\right)}{\ell!} u^{m+\ell}.$$

By evaluating the previous terms in u = 1 we complete the proof.

Theorem 5 The total number of swrec taken over all set partitions of [n], is given by

$$\frac{3}{4}(B_{n+3} - B_{n+2}) - (n + \frac{7}{4})B_{n+1} - \frac{1}{2}(n+1)B_n.$$

Proof In order to find the total number of swree, we need to find an explicit formula for the coefficient of x^n in the generating function $\frac{d}{dq} \tilde{P}_k(x, u, q) \mid_{u=q=1}$. By Theorem 3

$$\frac{d}{dq}\widetilde{P}_k(x,u,q)\mid_{u=q=1} = e^{e^x-1}(\frac{3}{4}e^{3x} + \frac{3}{2}e^{2x} - \frac{7}{4}e^x - xe^{2x} - \frac{3}{2}xe^x - \frac{1}{2})$$

By differentiating the well known generating function $e^{e^x-1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ three times we obtain

$$e^{x}e^{e^{x}-1} = \sum_{n\geq 0} B_{n+1}\frac{x^{n}}{n!},$$
$$e^{2x}e^{e^{x}-1} = \sum_{n\geq 0} B_{n+2}\frac{x^{n}}{n!} - \sum_{n\geq 0} B_{n+1}\frac{x^{n}}{n!}$$

and

$$e^{3x}e^{e^x-1} = \sum_{n\geq 0} B_{n+3}\frac{x^n}{n!} - 3\sum_{n\geq 0} B_{n+2}\frac{x^n}{n!} + 2\sum_{n\geq 0} B_{n+1}\frac{x^n}{n!}$$

From the above equations, we can derive that

$$xe^{x}e^{e^{x}-1} = \sum_{n\geq 0} nB_n \frac{x^n}{n!}$$

and

$$xe^{2x}e^{e^x-1} = \sum_{n\geq 0} nB_{n+1}\frac{x^n}{n!} - \sum_{n\geq 0} nB_n\frac{x^n}{n!}$$

Using all these facts together leads to

$$\frac{d}{dq}\widetilde{P}_k(x,u,q)\mid_{u=q=1} = \sum_{n\geq 0} (\frac{3}{4}B_{n+3} - \frac{3}{4}B_{n+2} - (n+\frac{7}{4})B_{n+1} - \frac{1}{2}(n+1)B_n)\frac{x^n}{n!}$$

Hence the total number of swrec is given by

$$\frac{3}{4}(B_{n+3} - B_{n+2}) - (n + \frac{7}{4})B_{n+1} - \frac{1}{2}(n+1)B_n.$$

In order to obtain asymptotic estimate for the moment as well as limiting distribution, we need the fact

$$B_{n+h} = B_n \frac{(n+h)!}{n!r^h} \left(1 + O(\frac{\log n}{n}) \right)$$

uniformly for $h = O(\log n)$, where r is the positive root of $re^r = n + 1$. For more details about the asymptotic expansion of Bell numbers see [1]. Therefore, Theorem 5 gives the following corollary.

Corollary 6 Asymptotically, the total number of swrec taken over all set partitions of [n], is given by

$$B_n \frac{n^3}{r^3} \left(1 + \frac{r}{n} \right) \left(1 + O(\frac{\log n}{n}) \right).$$

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