# Sum of weighted records in set partitions 

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#### Abstract

The purpose of this paper is to find an explicit formula and asymptotic estimate for the total number of sum of weighted records over set partitions of $[n]$ in terms of Bell numbers. For that we study the generating function for the number of set partitions of $[n]$ according to the statistic sum of weighted records.


Keywords: Records, Sum of weighted records, Set partitions, Generating functions, Bell numbers and Asymptotic estimate.

## 1 Introduction

Let $\sigma_{i}$ be an element in the permutation $\sigma=\sigma_{1} \sigma_{2} \cdots \sigma_{\ell}$, we say that $\sigma_{i}$ is a record of position $i$ if $\sigma_{i}>\sigma_{j}$ for all $j=1,2, \cdots, i-1$. The study of records in permutations interested Rényi [7]. More recently another statistic which depends on records have been studied by Kortchemski [5] who defined the statistic srec, where $\operatorname{srec}(\sigma)$ defined as the sum of the positions of all records of $\sigma$. For example, permutation $\pi=12534$ has 3 records, $1,2,5$ and $\operatorname{srec}(\pi)=1+2+3=6$. For relevant papers about records you can see for example [2] and [3]. In this Paper we want to focus on partitions of a set. Recall that a partition $\Pi$ of set $[n]$ of size $k$ (a partition of $[n]$ with exactly $k$ blocks) is a collection $\left\{B_{1}, B_{2}, \ldots, B_{k}\right\}$, where $\emptyset \neq B_{i} \subseteq[n]$ for all $i$ and $B_{i} \bigcap B_{j}=\emptyset$ for $i \neq j$, such that $\bigcup_{i=1}^{k} B_{i}=[n]$. The elements $B_{i}$ are called blocks, and we use the assumption that $B_{1}, B_{2}, \cdots, B_{k}$ are listed in increasing order of their minimal elements, that is, $\min B_{1}<\min B_{2}<\cdots<\min B_{k}$. The set of all partitions of [ $n$ ] with exactly $k$ blocks is denoted by $P_{n, k}$ and $\left|P_{n, k}\right|=S_{n, k}$, which is known as the Stirling numbers of the second kind [8]. And the set of all partitions of $[n]$ is denoted by $P_{n}$ and $\left|P_{n}\right|=\sum_{k=1}^{n} S_{n, k}=B_{n}$, which is the $n$-th Bell number [8]. Any partition $\Pi$ can be written as $\pi_{1} \pi_{2} \cdots \pi_{n}$, where $i \in B_{\pi_{i}}$ for all $i$, and this form is called the canonical sequential form. For example $\Pi=\{\{12\},\{3\},\{4\}\}$ is a partition of [4], the canonical sequential form is $\pi=1123$. For more details about set partitions we suggest Mansour's book [6]. The important results about records, obtained by Knopfmacher, Mansour and Wagner [4] which state the asymptotic mean value and variance for the number, and for the sum of positions, of record in all partitions of $[n]$ are central to my study. In this paper, we define a new statistic swrec, where $\operatorname{swrec}(\pi)$ is the sum of the position of a record in $\pi$ multiplied by the value of the record over all the records in $P_{n}$. We will study this statistic from the point of view of canonical sequential form. For instance, if $\pi=121132$ the $\operatorname{swrec}(\pi)=1 \cdot 1+2 \cdot 2+3 \cdot 5=20$.

## 2 Main Results

### 2.1 The ordinary generating function for the number of set partitions according to the statistic swrec

Let $P_{k}(x, q)$ be the generating function for the number of partitions of [ $n$ ] with exactly $k$ blocks according to the statistic swrec, that is

$$
P_{k}(x, q)=\sum_{n \geq k} \sum_{\pi \in P_{n, k}} x^{n} q^{\mathrm{swrec}(\pi)}
$$

Theorem 1 The generating function for the number of partitions of $[n]$ with exactly $k$ blocks according to the statistic swrec is given by

$$
\begin{equation*}
P_{k}(x, q)=\prod_{i=1}^{k} \frac{x q^{i} q^{(k+1-i)(k-i)}}{1-i x \prod_{j=i+1}^{k} q^{j}} \tag{1}
\end{equation*}
$$

Proof As we know, a set partition of $[n]$ with exactly $k$ blocks can be presented as canonical sequential form:

$$
\pi=1 \pi^{(1)} 2 \pi^{(2)} \cdots k \pi^{(k)}
$$

for some $k$, where $\pi^{(j)}$ denotes an arbitrary word over an alphabet $[j]$ including the empty word. Thus, the contribution of $\pi=1 \pi^{(1)} 2 \pi^{(2)} \cdots k$ to the generating function $P_{k}(x, q)$ is $x q^{k} P_{k-1}\left(x q^{k}, q\right)$ and the contribution of $\pi^{(k)}$ to the generating function $P_{k}(x, q)$ is $\frac{1}{1-k x}$. Therefore, the corresponding generating function satisfies

$$
P_{k}(x, q)=\frac{x q^{k}}{1-k x} P_{k-1}\left(x q^{k}, q\right)
$$

By using induction on $k$ together with the initial condition $P_{1}(x, q)=\frac{x q}{1-x}$ we obtain the required result.

### 2.2 Exact and asymptotic expression for $\sum_{\pi \in P_{n}} \operatorname{swrec}(\pi)$

In this section, we aim to prove that the total number of the swrec over all partitions of $[n]$ is

$$
\frac{3}{4}\left(B_{n+3}-B_{n+2}\right)-\left(n+\frac{7}{4}\right) B_{n+1}-\frac{1}{2}(n+1) B_{n}
$$

And we want to show that asymptotically the total number of the swrec over all partitions of $[n]$ is

$$
B_{n} \frac{n^{3}}{r^{3}}\left(1+\frac{r}{n}\right)\left(1+O\left(\frac{\log n}{n}\right)\right)
$$

where $r$ is the positive root of $r e^{r}=n+1$.
For that we need to perform the following steps:

- Firstly, we find the partial derivative of $P_{k}(x, q)$ with respect to $q$ and substitute $q=1$, that is $\left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}$.
- Secondly, we pass from $\left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}$ to $\left.\frac{d}{d q} \widetilde{P}_{k}(x, u, q)\right|_{u=q=1}$, where $\widetilde{P}_{k}(x, u, q)$ is the exponential generating function for the number of partitions of $[n]$ with exactly $k$ blocks according to the statistic swrec.
- Finally, we derive the total number of swrec over all partitions of $[n]$, and the asymptotic estimate for the total number of swrec over all partitions of $[n]$.

Lemma 2 For all $k \geq 1$,

$$
\begin{equation*}
\left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}=\frac{x^{k}\left(\binom{k+1}{2}+2\binom{k+1}{3}\right)}{(1-x) \ldots(1-k x)}+\frac{x^{k+1}}{(1-x) \ldots(1-k x)} \sum_{i=1}^{k} \frac{i(i+1+k)(k-i)}{2(1-i x)} \tag{2}
\end{equation*}
$$

Proof By differentiating (11) with respect to $q$, we obtain

$$
\begin{equation*}
\left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}=P_{k}(x, 1) \sum_{i=1}^{k} \lim _{q \rightarrow 1}\left(\frac{\frac{d}{d q} L_{i}(q)}{L_{i}(q)}\right) \tag{3}
\end{equation*}
$$

where

$$
L_{i}(q)=\frac{x q^{i} q^{(k+1-i)(k-i)}}{1-i x \prod_{j=i+1}^{k} q^{j}}
$$

We have

$$
\begin{equation*}
\lim _{q \rightarrow 1} \frac{d}{d q} L_{i}(q)=\frac{x \ell A_{i, k}(x, 1)-x \lim _{q \rightarrow 1} \frac{d}{d q} A_{i, k}(x, q)}{\left(A_{i, k}(x, 1)\right)^{2}} \tag{4}
\end{equation*}
$$

Where $A_{i, k}(x, q)=1-i x \prod_{j=i+1}^{k} q^{j}$ and $\ell=(k+1-i)(k-i)+i$. By using the differentiation rules we get $\frac{d}{d q} A_{i, k}(x, q)=-i x \sum_{m=i+1}^{k} m q^{m-1} \prod_{\substack{j=i+1 \\ j \neq m}}^{k} q^{j}$. Therefore,

$$
\lim _{q \rightarrow 1} \frac{d}{d q} L_{i}(q)=\frac{x(2(i+(k+1-i)(k-i))(1-i x)+i(i+1+k)(k-i) x)}{2(1-i x)^{2}}
$$

which leads to

$$
\begin{equation*}
\lim _{q \rightarrow 1}\left(\frac{\frac{d}{d q} L_{i}(q)}{L_{i}(q)}\right)=i+(k+1-i)(k-i)+\frac{i(i+1+k)(k-i) x}{2(1-i x)} \tag{5}
\end{equation*}
$$

Hence, by substituting (5) in (3) we obtain

$$
\begin{aligned}
& \left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}=\frac{x^{k}}{(1-x) \ldots(1-k x)} \sum_{i=1}^{k}\left(i+(k+1-i)(k-i)+\frac{i(i+1+k)(k-i) x}{2(1-i x)}\right) \\
& =\frac{x^{k}\left(\binom{k+1}{2}+2\binom{k+1}{3}\right)}{(1-x) \ldots(1-k x)}+\frac{x^{k+1}}{(1-x) \ldots(1-k x)} \sum_{i=1}^{k} \frac{i(i+1+k)(k-i)}{2(1-i x)}
\end{aligned}
$$

as claimed.

Now we need to find $\left.\left[x^{n}\right] \frac{d}{d q} P_{k}(x, q)\right|_{q=1}$ to obtain the total number of swrec. We will study the exponential generating function instead of the ordinary generating function. Let $\widetilde{P}_{k}(x, u, q)$ be the exponential generating function for the number of partitions of $[n]$ with exactly $k$ blocks according to the statistic swrec, that is

$$
\widetilde{P}_{k}(x, u, q)=\sum_{n \geq k} \sum_{\pi \in P_{n, k}} \frac{x^{n} u^{k} q^{\mathrm{swrec}(\pi)}}{n!}
$$

Theorem 3 The partial derivative of $\widetilde{P}_{k}(x, u, q)$ with respect to $q$ at $u=q=1$ is given by,

$$
\begin{equation*}
\left.\frac{d}{d q} \widetilde{P}_{k}(x, u, q)\right|_{u=q=1}=e^{e^{x}-1}\left(\frac{3}{4} e^{3 x}+\frac{3}{2} e^{2 x}-\frac{7}{4} e^{x}-x e^{2 x}-\frac{3}{2} x e^{x}-\frac{1}{2}\right) \tag{6}
\end{equation*}
$$

Proof In order to prove the above result we need the following proposition:
Proposition 4 The partial derivative $\left.\frac{d}{d q} P_{k}(x, q)\right|_{\substack{x=y^{-1} \\ q=1}}$ can be decomposed as

$$
\begin{equation*}
\sum_{m=1}^{k}\left(\frac{a_{k, m}}{(y-m)^{2}}+\frac{b_{k, m}}{y-m}\right) \tag{7}
\end{equation*}
$$

where

$$
a_{k, m}=\frac{(-1)^{k-m} m(1+k+m)(k-m)}{2(m-1)!(k-m)!}
$$

and

$$
b_{k, m}=\frac{(-1)^{k-m}\left(k^{2}\left(\frac{m}{4}+1\right)+k\left(\frac{m^{2}}{2}+\frac{3 m}{4}+1\right)-\left(\frac{3 m^{2}}{2}+m\right)\right)}{(m-1)!(k-m)!}
$$

Proof We rewrite (2) as

$$
\begin{aligned}
& \left.\frac{d}{d q} P_{k}(x, q)\right|_{q=1}=x^{k} \prod_{i=1}^{k}(1-i x)^{-1}\left(\binom{k+1}{2}+2\binom{k+1}{3}+\sum_{i=1}^{k} \frac{i(1+k+i)(k-i) x}{2(1-i x)}\right) \\
= & x^{k} \prod_{i=1}^{k}(1-i x)^{-1}\left(\frac{k(k+1)(2 k+1)}{6}+\sum_{i=1}^{k} \frac{i(1+k+i)(k-i) x}{2(1-i x)}\right) .
\end{aligned}
$$

By replacing $x^{-1}=y$ in the above equation we get

$$
\begin{equation*}
\prod_{i=1}^{k}(y-i)^{-1}\left(\frac{k(k+1)(2 k+1)}{6}+\sum_{i=1}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)}\right) \tag{8}
\end{equation*}
$$

The above expression decomposed as

$$
\sum_{m=1}^{k}\left(\frac{a_{k, m}}{(y-m)^{2}}+\frac{b_{k, m}}{y-m}\right)
$$

In order to find the coefficients $a_{k, m}$ and $b_{k, m}$, we need to consider the expansion of (8) at $y=m$, as follows:

$$
\begin{aligned}
& (y-m)^{-1} \prod_{\substack{i=1 \\
i \neq m}}^{k}(y-m+m-i)^{-1}\left(\frac{k(k+1)(2 k+1)}{6}+\frac{m(1+k+m)(k-m)}{2(y-m)}+\sum_{\substack{i=1 \\
i \neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)}\right) \\
& =(y-m)^{-1} \prod_{\substack{i=1 \\
i \neq m}}^{k}\left((m-i)^{-1}\left(1+\frac{y-m}{m-i}\right)^{-1}\right) . \\
& \left(\frac{k(k+1)(2 k+1)}{6}+\frac{m(1+k+m)(k-m)}{2(y-m)}+\sum_{\substack{i=1 \\
i \neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(y-i)}\right) .
\end{aligned}
$$

Using Taylor series to expand $\left(1+\frac{y-m}{m-i}\right)^{-1}$ and $\frac{i(1+k+i)(k-i)}{2(y-i)}$ at $y=m$ we get

$$
\begin{aligned}
& (y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \prod_{\substack{i=1 \\
i \neq m}}^{k}\left(1-\frac{y-m}{m-i}+O\left((y-m)^{2}\right)\right) \\
& \left(\frac{k(k+1)(2 k+1)}{6}+\frac{m(1+k+m)(k-m)}{2(y-m)}+\sum_{\substack{i=1 \\
i \neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(m-i)}+O(y-m)\right)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& (y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!}\left(1-\sum_{\substack{i=1 \\
i \neq m}}^{k} \frac{y-m}{m-i}+O\left((y-m)^{2}\right)\right) \\
& \left(\frac{k(k+1)(2 k+1)}{6}+\frac{m(1+k+m)(k-m)}{2(y-m)}+\sum_{\substack{i=1 \\
i \neq m}}^{k} \frac{i(1+k+i)(k-i)}{2(m-i)}+O(y-m)\right)
\end{aligned}
$$

We need to simplify the product, and consider the coefficients of $(y-m)^{-1}$ and $(y-m)^{-2}$ as follows:
$(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!}$
$\cdot\left(\frac{k(k+1)(2 k+1)}{6}+\frac{m(1+k+m)(k-m)}{2(y-m)}+\sum_{\substack{i=1 \\ i \neq m}}^{k} \frac{i(1+k+i)(k-i)-m(1+k+m)(k-m)}{2(m-i)}+O(y-m)\right)$.

By using Maple we compute the term in the summation, which hints

$$
\begin{aligned}
& (y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \\
& \cdot\left(\frac{m(1+k+m)(k-m)}{2(y-m)}+\frac{k(k+1)(2 k+1)}{6}+\frac{1}{2} \sum_{\substack{i=1 \\
i \neq m}}^{k}\left(i^{2}+i+i m-k^{2}-k+m+m^{2}\right)+O(y-m)\right) \\
& =(y-m)^{-1} \frac{(-1)^{k-m}}{(m-1)!(k-m)!} \\
& \cdot\left(\frac{m(1+k+m)(k-m)}{2(y-m)}+k^{2}\left(\frac{m}{4}+1\right)+k\left(\frac{m^{2}}{2}+\frac{3 m}{4}+1\right)-\left(\frac{3 m^{2}}{2}+m\right)+O(y-m)\right)
\end{aligned}
$$

Hence, by finding the coefficients of of $(y-m)^{-1}$ and $(y-m)^{-2}$ we complete the proof.
Now we return to the proof of the theorem. We use (7) for passing to exponential generating function, by substituting

$$
\frac{e^{m x}-1}{m}=\sum_{\ell \geq 0} \frac{m^{\ell} x^{\ell+1}}{(\ell+1)!}
$$

in

$$
\frac{1}{y-m}=\frac{x}{1-m x}=\sum_{\ell \geq 0} m^{\ell} x^{\ell+1}
$$

and $\frac{e^{m x}(m x-1)+1}{m^{2}}$ in $\frac{1}{(y-m)^{2}}$. Moreover, by summing over all $k$ we obtain the generating function

$$
\begin{aligned}
& \sum_{k \geq 1} u^{k} \sum_{m=1}^{k} \frac{(-1)^{k-m} m(1+k+m)(k-m)}{2(m-1)!(k-m)!} \cdot \frac{e^{m x}(m x-1)+1}{m^{2}} \\
& +\sum_{k \geq 1} u^{k} \sum_{m=1}^{k} \frac{(-1)^{k-m}\left(k^{2}\left(\frac{m}{4}+1\right)+k\left(\frac{m^{2}}{2}+\frac{3 m}{4}+1\right)-\left(\frac{3 m^{2}}{2}+m\right)\right)}{(m-1)!(k-m)!} \cdot \frac{e^{m x}-1}{m} .
\end{aligned}
$$

We need to change the order of the summation as follows:

$$
\begin{aligned}
& \sum_{m \geq 1} \frac{e^{m x}(m x-1)+1}{m!} \sum_{k \geq m} \frac{(-1)^{k-m}(1+k+m)(k-m) u^{k}}{2(k-m)!} \\
& +\sum_{m \geq 1} \frac{e^{m x}-1}{m!} \sum_{k \geq m} \frac{(-1)^{k-m}\left(k^{2}\left(\frac{m}{4}+1\right)+k\left(\frac{m^{2}}{2}+\frac{3 m}{4}+1\right)-\left(\frac{3 m^{2}}{2}+m\right)\right)}{(k-m)!} u^{k} .
\end{aligned}
$$

By substituting $\ell=k-m$ and rewriting the above result we obtain the following form:

$$
\begin{aligned}
& \sum_{m \geq 1} \frac{e^{m x}(m x-1)+1}{m!} \sum_{\ell \geq 0} \frac{(-1)^{\ell}(2 m+\ell+1) \ell u^{m+\ell}}{2 \ell!} \\
& +\sum_{m \geq 1} \frac{e^{m x}-1}{m!} \sum_{\ell \geq 0} \frac{(-1)^{\ell}\left((m+\ell)^{2}\left(\frac{m}{4}+1\right)+(m+\ell)\left(\frac{m^{2}}{2}+\frac{3 m}{4}+1\right)-\left(\frac{3 m^{2}}{2}+m\right)\right)}{\ell!} u^{m+\ell}
\end{aligned}
$$

By evaluating the previous terms in $u=1$ we complete the proof.

Theorem 5 The total number of swrec taken over all set partitions of [ $n$ ], is given by

$$
\frac{3}{4}\left(B_{n+3}-B_{n+2}\right)-\left(n+\frac{7}{4}\right) B_{n+1}-\frac{1}{2}(n+1) B_{n} .
$$

Proof In order to find the total number of swrec, we need to find an explicit formula for the coefficient of $x^{n}$ in the generating function $\left.\frac{d}{d q} \widetilde{P}_{k}(x, u, q)\right|_{u=q=1}$. By Theorem 3

$$
\left.\frac{d}{d q} \widetilde{P}_{k}(x, u, q)\right|_{u=q=1}=e^{e^{x}-1}\left(\frac{3}{4} e^{3 x}+\frac{3}{2} e^{2 x}-\frac{7}{4} e^{x}-x e^{2 x}-\frac{3}{2} x e^{x}-\frac{1}{2}\right) .
$$

By differentiating the well known generating function $e^{e^{x}-1}=\sum_{n \geq 0} B_{n} \frac{x^{n}}{n!}$ three times we obtain

$$
\begin{gathered}
e^{x} e^{e^{x}-1}=\sum_{n \geq 0} B_{n+1} \frac{x^{n}}{n!} \\
e^{2 x} e^{e^{x}-1}=\sum_{n \geq 0} B_{n+2} \frac{x^{n}}{n!}-\sum_{n \geq 0} B_{n+1} \frac{x^{n}}{n!}
\end{gathered}
$$

and

$$
e^{3 x} e^{e^{x}-1}=\sum_{n \geq 0} B_{n+3} \frac{x^{n}}{n!}-3 \sum_{n \geq 0} B_{n+2} \frac{x^{n}}{n!}+2 \sum_{n \geq 0} B_{n+1} \frac{x^{n}}{n!}
$$

From the above equations, we can derive that

$$
x e^{x} e^{e^{x}-1}=\sum_{n \geq 0} n B_{n} \frac{x^{n}}{n!}
$$

and

$$
x e^{2 x} e^{e^{x}-1}=\sum_{n \geq 0} n B_{n+1} \frac{x^{n}}{n!}-\sum_{n \geq 0} n B_{n} \frac{x^{n}}{n!}
$$

Using all these facts together leads to

$$
\left.\frac{d}{d q} \widetilde{P}_{k}(x, u, q)\right|_{u=q=1}=\sum_{n \geq 0}\left(\frac{3}{4} B_{n+3}-\frac{3}{4} B_{n+2}-\left(n+\frac{7}{4}\right) B_{n+1}-\frac{1}{2}(n+1) B_{n}\right) \frac{x^{n}}{n!}
$$

Hence the total number of swrec is given by

$$
\frac{3}{4}\left(B_{n+3}-B_{n+2}\right)-\left(n+\frac{7}{4}\right) B_{n+1}-\frac{1}{2}(n+1) B_{n} .
$$

In order to obtain asymptotic estimate for the moment as well as limiting distribution, we need the fact

$$
B_{n+h}=B_{n} \frac{(n+h)!}{n!r^{h}}\left(1+O\left(\frac{\log n}{n}\right)\right)
$$

uniformly for $h=O(\log n)$, where $r$ is the positive root of $r e^{r}=n+1$. For more details about the asymptotic expansion of Bell numbers see [1]. Therefore, Theorem 5] gives the following corollary.

Corollary 6 Asymptotically, the total number of swrec taken over all set partitions of $[n]$, is given by

$$
B_{n} \frac{n^{3}}{r^{3}}\left(1+\frac{r}{n}\right)\left(1+O\left(\frac{\log n}{n}\right)\right) .
$$

Acknowledgement. The research of the author was supported by the Ministry of Science and Technology, Israel.

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