Eigenfunctions and minimum 1-perfect bitrades in the Hamming graph $\stackrel{\approx}{\Rightarrow}$

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Abstract

The Hamming graph H(n,q) is the graph whose vertices are the words of length n over the alphabet $\{0, 1, \ldots, q-1\}$, where two vertices are adjacent if they differ in exactly one coordinate. The adjacency matrix of H(n,q) has n + 1 distinct eigenvalues $n(q-1) - q \cdot i$ with corresponding eigenspaces $U_i(n,q)$ for $0 \leq i \leq n$. In this work we study functions belonging to a direct sum $U_i(n,q) \oplus U_{i+1}(n,q) \oplus \ldots \oplus U_j(n,q)$ for $0 \leq i \leq j \leq n$. We find the minimum cardinality of the support of such functions for q = 2 and for q = 3, i + j > n. In particular, we find the minimum cardinality of the support of eigenfunctions from the eigenspace $U_i(n,3)$ for $i > \frac{n}{2}$. Using the correspondence between 1-perfect bitrades and eigenfunctions with eigenvalue -1, we find the minimum size of a 1-perfect bitrade in the Hamming graph H(n,3).

Keywords: Hamming graph, eigenfunction, eigenfunctions of graphs, eigenspace, minimum support, trade, bitrade, 1-perfect bitrade

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1. Introduction

In this work we consider the following extremal problem for eigenfunctions of graphs.

Problem 1. Let G be a graph and let λ be an eigenvalue of the adjacency matrix of G. Find the minimum cardinality of the support of a λ -eigenfunction of G.

Problem 1 is directly related to the intersection problem of two combinatorial objects and to the problem of finding the minimum cardinality of bitrades. Often such problems can be considered as Problem 1 for the corresponding graph and some eigenvalue with some additional discrete restrictions on the functions. In this context we would like to mention the papers by Graham et al. [10], Deza and Frankl [7], Frankl and Pach [9] and Cho [4, 5]

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on null designs and the paper by Hwang [13] on combinatorial trades. In more details, connections between eigenfunctions and bitrades are described in [15, 16, 18, 27].

Problem 1 was studied for the bilinear forms graphs in [24], for the cubical distanceregular graphs in [23], for the Doob graphs in [1], for the Grassmann graphs in [4, 5, 18], for the Hamming graphs in [15, 20, 25, 27], for the Johnson graphs in [28], for the Paley graphs in [11] and for the Star graph in [14].

The Hamming graph H(n,q) is a graph whose vertices are the words of length n over the alphabet $\{0, 1, \ldots, q-1\}$; and two vertices are adjacent if they differ in exactly one coordinate. The adjacency matrix of H(n,q) has n+1 eigenvalues $\lambda_i(n,q) = n(q-1) - q \cdot i$, where $0 \leq i \leq n$. Let $U_{[i,j]}(n,q)$, where $0 \leq i \leq j \leq n$, denote a direct sum of eigenspaces of H(n,q) corresponding to consecutive eigenvalues from $n(q-1) - q \cdot i$ to $n(q-1) - q \cdot j$. In this work we consider the following generalization of Problem 1 for the Hamming graph.

Problem 2. Let $n \ge 1$, $q \ge 2$ and $0 \le i \le j \le n$. Find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n,q)$.

In [26] Valyuzhenich and Vorob'ev solved Problem 2 for arbitrary $q \ge 3$ except the case when q = 3 and i + j > n. Moreover, in [26] a characterization of functions from the space $U_{[i,j]}(n,q)$ with the minimum cardinality of the support was obtained for $q \ge 3$, $i + j \le n$ and $q \ge 5$, i = j, $i > \frac{n}{2}$. In this work we solve Problem 2 for q = 2 and q = 3, i + j > n. In particular, we find the minimum cardinality of the support of a $\lambda_i(n,3)$ -eigenfunction of H(n,3) for $i > \frac{n}{2}$. Thus, Problem 2 is now completely solved. As we see below, in the case q = 3 the eigenfunctions attaining the minimum cardinality of the support have more complicated structure than for q > 3, so the case remaining after the preceding work and solved in the current paper is really exceptional.

Bitrades are used for constructing and studying combinatorial designs and codes (see [2, 3, 12]). One of important problems in the theory of bitrades is the problem of finding the minimum sizes of bitrades. This problem was investigated for null designs [4, 5, 9], for combinatorial bitrades [13], for Latin bitrades [21] and for q-ary Steiner bitrades [17, 18]. In this work we study 1-perfect bitrades in the Hamming graph. The problems of the existence and classification of 1-perfect bitrades and extended 1-perfect bitrades in the Hamming graphs were studied in [19, 27] and in [16] respectively. In this work we consider the following problem for 1-perfect bitrades in the Hamming graph.

Problem 3. Let $n \ge 3$ and $q \ge 2$. Find the minimum size of a 1-perfect bitrade in H(n,q).

For q = 2 Problem 3 was essentially solved by Etzion and Vardy [8] and Solov'eva [22] (the results were formulated for more special cases of 1-perfect bitrades embedded into perfect binary codes, but both proofs work in the general case). In [19] Mogilnykh and Solov'eva for arbitrary $q \ge 2$ showed the existence of 1-perfect bitrades in H(n,q) of size $2 \cdot (q!)^{\frac{n-1}{q}}$. This fact implies that a lower bound $2^{n-\frac{n-1}{q}} \cdot (q-1)^{\frac{n-1}{q}}$ for the size of 1-perfect bitrades in H(n,q) proved in [26] is sharp for q = 4, i.e. Problem 3 is solved for q = 4. In [19] Mogilnykh and Solov'eva found the minimum size of a 1-perfect bitrade in H(q+1,q)for arbitrary $q \ge 2$. In this work, using the correspondence between 1-perfect bitrades and (-1)-eigenfunctions, we solve Problem 3 for q = 3. The paper is organized as follows. In Section 2, we introduce basic definitions and notations. In Section 3, we give some preliminary results. In Section 4, we define four families of functions that have the minimum cardinality of the support in the space $U_{[i,j]}(n,q)$ for q = 2 and for q = 3 and i + j > n respectively. In Section 5, we find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n,2)$. In Section 6, we find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n,3)$ for $\frac{i}{2} + j \leq n$ and i + j > n. In Section 7, we find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n,3)$ for $\frac{i}{2} + j \leq n$ and i + j > n. In Section 7, we find the minimum cardinality of the support of functions from the space $U_{[i,j]}(n,3)$ for $\frac{i}{2} + j > n$. In Section 8, we prove that the minimum size of a 1-perfect bitrade in H(3m + 1, 3), where $m \geq 1$, is $2^{m+1} \cdot 3^m$.

2. Basic definitions

Let G = (V, E) be a graph with the adjacency matrix A(G). The set of neighbors of a vertex x is denoted by N(x). Let λ be an eigenvalue of the matrix A(G). A function $f: V \longrightarrow \mathbb{R}$ is called a λ -eigenfunction of G if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y) \tag{1}$$

holds for any vertex $x \in V$. The set of functions $f: V \longrightarrow \mathbb{R}$ satisfying the equality (1) for any vertex $x \in V$ is called a λ -eigenspace of G. The support of a function $f: V \longrightarrow \mathbb{R}$ is the set $Supp(f) = \{x \in V \mid f(x) \neq 0\}$. Denote |f| = |Supp(f)|.

Let $\Sigma_q = \{0, 1, \ldots, q-1\}$. The vertex set of the Hamming graph H(n, q) is Σ_q^n and two vertices are adjacent if they differ in exactly one coordinate. It is well known that the set of eigenvalues of the adjacency matrix of H(n,q) is $\{\lambda_i(n,q) = n(q-1) - q \cdot i \mid i = 0, 1, \ldots, n\}$. Denote by $U_i(n,q)$ the $\lambda_i(n,q)$ -eigenspace of H(n,q). The direct sum of subspaces

$$U_i(n,q) \oplus U_{i+1}(n,q) \oplus \ldots \oplus U_j(n,q)$$

for $0 \le i \le j \le n$ is denoted by $U_{[i,j]}(n,q)$.

The Cartesian product $G \Box H$ of graphs G and H is a graph with the vertex set $V(G) \times V(H)$; and any two vertices (u, u') and (v, v') are adjacent if and only if either u = v and u' is adjacent to v' in H, or u' = v' and u is adjacent to v in G.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, let $f_1 : V_1 \longrightarrow \mathbb{R}$ and $f_2 : V_2 \longrightarrow \mathbb{R}$. Let $G = G_1 \square G_2$. Define the *tensor product* $f_1 \cdot f_2 : V_1 \times V_2 \longrightarrow \mathbb{R}$ by the following rule: $(f_1 \cdot f_2)(x, y) = f_1(x)f_2(y)$ for $(x, y) \in V(G) = V_1 \times V_2$.

Let $y = (y_1, \ldots, y_{n-1})$ be a vertex of H(n-1,q), $k \in \Sigma_q$ and $r \in \{1, 2, \ldots, n\}$. We consider the vector $x = (y_1, \ldots, y_{r-1}, k, y_r, \ldots, y_{n-1})$ of length n. Given a function $f : \Sigma_q^n \longrightarrow \mathbb{R}$, we define the function $f_k^r : \Sigma_q^{n-1} \longrightarrow \mathbb{R}$ by the rule $f_k^r(y) = f(x)$. A function $f : \Sigma_q^n \longrightarrow \mathbb{R}$ is called *uniform* if for any $r \in \{1, 2, \ldots, n\}$ there exists $l(r) \in \Sigma_q$ such that $f_k^r = f_m^r$ for all $k, m \in \Sigma_q \setminus \{l(r)\}$.

Let Sym(X) denote the symmetric group on a finite set X and let Sym_n denote the symmetric group on the set $\{1, \ldots, n\}$.

Let $f(x_1, x_2, \ldots, x_n)$ be a function, let $\pi \in \text{Sym}_n$ and let $\sigma_1, \ldots, \sigma_n \in \text{Sym}(\Sigma_q)$. We define the function $f_{\pi,\sigma_1,\ldots,\sigma_n}$ by the following rule:

$$f_{\pi,\sigma_1,\ldots,\sigma_n}(x_1,\ldots,x_n) = f(\sigma_1(x_{\pi(1)}),\ldots,\sigma_n(x_{\pi(n)})).$$

Let G = (V, E) be a graph. For a vertex $x \in V$ denote $B(x) = N(x) \cup \{x\}$. Let T_0 and T_1 be two disjoint nonempty subsets of V. The ordered pair (T_0, T_1) is called a 1-perfect bitrade in G if for any vertex $x \in V$ the set B(x) either contains one element from T_0 and one element from T_1 or does not contain elements from $T_0 \cup T_1$.

Remark 1. If (T_0, T_1) is a 1-perfect bitrade in a graph G, then the following properties hold:

- T_0 and T_1 are independent sets in G and $|T_0| = |T_1|$.
- the subgraph of G induced by $T_0 \cup T_1$ is a perfect matching.

The size of a 1-perfect bitrade (T_0, T_1) is $|T_0| + |T_1|$.

Example 1. Let $T_0 = \{000, 111\}$ and $T_1 = \{001, 110\}$. Then (T_0, T_1) is a 1-perfect bitrade of size 4 in H(3, 2) (see Figure 1).

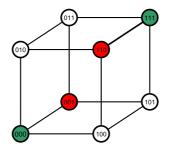


Figure 1: 1-perfect bitrade in H(3, 2).

Example 2. Let G = (V, E) be a graph. Recall that a set $C \subseteq V$ is called a 1-perfect code in G if for any vertex $x \in V$ the set B(x) contains one vertex from C. Let C_1 and C_2 be two 1-perfect codes in G ($C_1 \neq C_2$). Then ($C_1 \setminus C_2, C_2 \setminus C_1$) is a 1-perfect bitrade in G.

Let (T_0, T_1) be a 1-perfect bitrade in a graph G = (V, E). We define the function $f_{(T_0,T_1)}: V \longrightarrow \{-1,0,1\}$ by the following rule:

$$f_{(T_0,T_1)}(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

3. Preliminaries

In this section we give useful preliminary results. The following result is a corollary of well known result for so-called NEPS of graphs (see [6], Theorem 2.3.4).

Lemma 1 ([26], Corollary 1). Let $f_1 \in U_i(m,q)$ and $f_2 \in U_j(n,q)$. Then $f_1 \cdot f_2 \in U_{i+j}(m+n,q)$.

We will use Lemma 1 in Section 4. The following two results were proved in [26].

Lemma 2 ([26], Lemma 4). Let $f \in U_{[i,j]}(n,q)$ and $r \in \{1, 2, ..., n\}$. Then the following statements are true:

$$\begin{split} &1. \ f_k^r - f_m^r \in U_{[i-1,j-1]}(n-1,q) \ for \ k, m \in \Sigma_q. \\ &2. \ \sum_{k=0}^{q-1} f_k^r \in U_{[i,j]}(n-1,q). \\ &3. \ f_k^r \in U_{[i-1,j]}(n-1,q) \ for \ k \in \Sigma_q. \end{split}$$

Lemma 3 ([26], Lemma 5). Let $f \in U_{[i,j]}(n,q)$, let $r \in \{1, 2, ..., n\}$, and let $m \in \Sigma_q$. If $f_k^r \equiv 0$ for any $k \in \Sigma_q \setminus \{m\}$, then $f_m^r \in U_{[i,j-1]}(n-1,q)$.

In Sections 5, 6 and 7 we will use Lemmas 2 and 3 for inductive arguments. The following two results were proved in [26].

Theorem 1 ([26], Theorem 2). Let f be a uniform function from $U_{[i,j]}(n,q)$, where $i+j \ge n$, $q \ge 3$ and $f \ne 0$. Then $|f| \ge 2^{n-j}(q-1)^{n-j}q^{i+j-n}$.

Theorem 2 ([26], Theorem 1). Let $f \in U_{[i,j]}(n,q)$, $i + j \leq n$, $q \geq 3$ and $f \not\equiv 0$. Then $|f| \geq 2^i (q-1)^i q^{n-i-j}$.

We will use Theorems 1 and 2 in the proof of Theorem 5. The following result was obtained in [27].

Lemma 4 ([27], Proposition 2). Let n = qm + 1 and $q = p^k$, where p is a prime, $m \ge 1$ and $k \ge 1$. Then there exist a 1-perfect bitrade in H(n,q) of size $2^{m+1} \cdot q^{m(q-2)}$.

We will use Lemma 4 in the proof of Theorem 7.

4. Constructions of functions with the minimum cardinality of the support

In this section we give constructions of functions that have the minimum cardinality of the support in the space $U_{[i,j]}(n,q)$ for q=2 and q=3, i+j>n.

We define the function $a_{q,k,m}: \Sigma_q^2 \longrightarrow \mathbb{R}$ for $k, m \in \Sigma_q$ by the following rule:

$$a_{q,k,m}(x,y) = \begin{cases} 1, & \text{if } x = k \text{ and } y \neq m; \\ -1, & \text{if } y = m \text{ and } x \neq k; \\ 0, & \text{otherwise.} \end{cases}$$

We note that $|a_{q,k,m}| = 2(q-1)$ and $a_{q,k,m} \in U_1(2,q)$ for any $k,m \in \Sigma_q$. Denote $A_q = \{a_{q,k,m} \mid k,m \in \Sigma_q\}$.

We define the function $\varphi_1: \Sigma_3^2 \longrightarrow \mathbb{R}$ by the following rule:

$$\varphi_1(x,y) = \begin{cases} 1, & \text{if } x = y = 0; \\ -1, & \text{if } x = 1 \text{ and } y = 2; \\ 0, & \text{otherwise.} \end{cases}$$

For $a, b \in \Sigma_3$ denote by $a \oplus b$ the sum of a and b modulo 3. We define the function $\varphi : \Sigma_3^3 \longrightarrow \mathbb{R}$ by the following rule:

$$\varphi(x, y, z) = \begin{cases} \varphi_1(x, y), & \text{if } z = 0; \\ \varphi_1(x \oplus 1, y \oplus 1), & \text{if } z = 1; \\ \varphi_1(x \oplus 2, y \oplus 2), & \text{if } z = 2. \end{cases}$$

We note that $|\varphi| = 6$. By the definition of an eigenfunction we see that $\varphi \in U_2(3,3)$. Denote

$$B = \{\varphi_{\pi,\sigma_1,\sigma_2,\sigma_3} \mid \pi \in \operatorname{Sym}_3, \sigma_1, \sigma_2, \sigma_3 \in \operatorname{Sym}(\Sigma_3)\}.$$

We define the function $c_{q,k,m}: \Sigma_q \longrightarrow \mathbb{R}$ for $k, m \in \Sigma_q$ and $k \neq m$ by the following rule:

$$c_{q,k,m}(x) = \begin{cases} 1, & \text{if } x = k; \\ -1, & \text{if } x = m; \\ 0, & \text{otherwise} \end{cases}$$

We note that $|c_{q,k,m}| = 2$ and $c_{q,k,m} \in U_1(1,q)$ for any $k,m \in \Sigma_q$ and $k \neq m$. Denote $C_q = \{c_{q,k,m} \mid k, m \in \Sigma_q, k \neq m\}.$

We define the function $d_{q,k}: \Sigma_q \longrightarrow \mathbb{R}$ for $k \in \Sigma_q$ by the following rule:

$$d_{q,k}(x) = \begin{cases} 1, & \text{if } x = k; \\ 0, & \text{otherwise} \end{cases}$$

We note that $|d_{q,k}| = 1$ and $d_{q,k} \in U_{[0,1]}(1,q)$ for any $k \in \Sigma_q$. Denote $D_q = \{d_{q,k} \mid k \in \Sigma_q\}$. Let $e_q : \Sigma_q \longrightarrow \mathbb{R}$ and $e_q \equiv 1$. We note that $|e_q| = q$ and $e_q \in U_0(1,q)$. Denote $E_q = \{e_q\}$.

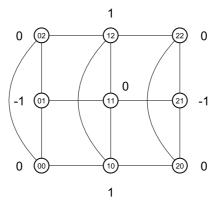


Figure 2: Function $a_{3,1,1}$ in H(2,3).

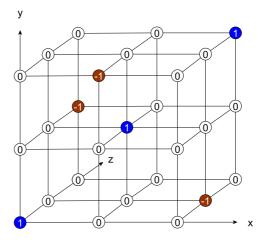


Figure 3: Function $\varphi(x, y, z)$ in H(3, 3).

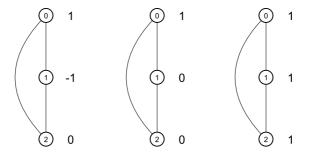


Figure 4: Functions $c_{3,0,1}$, $d_{3,0}$ and e_3 in H(1,3).

Let $i + j \leq n$. We say that a function $f: \Sigma_2^n \longrightarrow \mathbb{R}$ belongs to the class $F_1(n, i, j)$ if

$$f = c \cdot \prod_{k=1}^{i} g_k \cdot \prod_{k=1}^{n-i-j} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in A_2$ for $k \in [1, i]$, $h_k \in E_2$ for $k \in [1, n - i - j]$ and $v_k \in D_2$ for $k \in [1, j - i]$.

Let i + j > n. We say that a function $f: \Sigma_2^n \longrightarrow \mathbb{R}$ belongs to the class $F_2(n, i, j)$ if

$$f = c \cdot \prod_{k=1}^{n-j} g_k \cdot \prod_{k=1}^{i+j-n} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in A_2$ for $k \in [1, n-j]$, $h_k \in C_2$ for $k \in [1, i+j-n]$ and $v_k \in D_2$ for $k \in [1, j-i]$.

Let $\frac{i}{2} + j \leq n$ and i + j > n. We say that a function $f : \Sigma_3^n \longrightarrow \mathbb{R}$ belongs to the class $F_3(n, i, j)$ if

$$f = c \cdot \prod_{k=1}^{2n-i-2j} g_k \cdot \prod_{k=1}^{i+j-n} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in A_3$ for $k \in [1, 2n - i - 2j]$, $h_k \in B$ for $k \in [1, i + j - n]$ and $v_k \in D_3$ for $k \in [1, j - i]$.

Let $\frac{i}{2} + j > n$. We say that a function $f: \Sigma_3^n \longrightarrow \mathbb{R}$ belongs to the class $F_4(n, i, j)$ if

$$f = c \cdot \prod_{k=1}^{n-j} g_k \cdot \prod_{k=1}^{i+2j-2n} h_k \cdot \prod_{k=1}^{j-i} v_k,$$

where c is a constant, $g_k \in B$ for $k \in [1, n - j]$, $h_k \in C_3$ for $k \in [1, i + 2j - 2n]$ and $v_k \in D_3$ for $k \in [1, j - i]$.

Lemma 5. The following statements are true:

- 1. Let $i + j \leq n$ and $f \in F_1(n, i, j)$. Then $f \in U_{[i,j]}(n, 2)$ and $|f| = 2^{n-j}$.
- 2. Let i + j > n and $f \in F_2(n, i, j)$. Then $f \in U_{[i,j]}(n, 2)$ and $|f| = 2^i$.
- 3. Let $\frac{i}{2} + j \leq n$, i + j > n and $f \in F_3(n, i, j)$. Then $f \in U_{[i,j]}(n, 3)$ and $|f| = 2^{3(n-j)-i} \cdot 3^{i+j-n}$.
- 4. Let $\frac{i}{2} + j > n$ and $f \in F_4(n, i, j)$. Then $f \in U_{[i,j]}(n, 3)$ and $|f| = 2^{i+j-n} \cdot 3^{n-j}$.

Proof. As we noted above $A_q \subset U_1(2,q)$, $B \subset U_2(3,3)$, $C_q \subset U_1(1,q)$, $D_q \subset U_{[0,1]}(1,q)$ and $E_q \subset U_0(1,q)$. Hence using Lemma 1 and the fact that $|f_1 \cdot f_2| = |f_1| \cdot |f_2|$, we obtain the statement of this lemma.

In Section 5 we prove that functions from $F_1(n, i, j)$ and $F_2(n, i, j)$ have the minimum cardinality of the support in the space $U_{[i,j]}(n, 2)$ for $i + j \leq n$ and i + j > n respectively. In Sections 6 and 7 we prove that functions from $F_3(n, i, j)$ and $F_4(n, i, j)$ have the minimum cardinality of the support in the space $U_{[i,j]}(n, 3)$ for $\frac{i}{2} + j \leq n$, i + j > n and $\frac{i}{2} + j > n$ respectively.

5. Problem 2 for q = 2

In this section we consider Problem 2 for q = 2. The first main result of this section is the following.

Theorem 3. Let
$$f \in U_{[i,j]}(n,2)$$
, where $i+j \leq n$ and $f \not\equiv 0$. Then
 $|f| \geq 2^{n-j}$ (2)

and this bound is sharp.

Proof. Let us prove the bound (2) by induction on n, i and j. If j = 0, then i = 0 and $f \in U_0(n, 2)$. So, in this case f is a constant. Hence $|f| = 2^n$ and the claim of the theorem holds. So, we can assume that $j \ge 1$. If n = 1 and $j \ge 1$, then i = 0 and j = 1. In this case $f \in U_{[0,1]}(1, 2)$ and the claim of the theorem holds.

Let us prove the induction step for $n \ge 2$ and $j \ge 1$. Since $j \ge 1$, f is not constant. Then there exists $r \in \{1, 2, ..., n\}$ such that $f_0^r \ne f_1^r$. Without loss of generality, we assume that r = n. Denote $f_k = f_k^n$ for $k \in \Sigma_2$. Lemma 2 implies that $f_0 - f_1 \in U_{[i-1,j-1]}(n-1,2)$. By the induction assumption we obtain that

$$|f_0 - f_1| \ge 2^{n-j}$$

Then we have

$$|f| = |f_0| + |f_1| \ge |f_0 - f_1| \ge 2^{n-j}.$$

Lemma 5 implies that if $f \in F_1(n, i, j)$, then $f \in U_{[i,j]}(n, 2)$ and $|f| = 2^{n-j}$. Thus, the bound (2) is sharp.

The second main result of this section is the following.

Theorem 4. Let $f \in U_{[i,j]}(n,2)$, where i + j > n and $f \not\equiv 0$. Then

$$|f| \ge 2^i \tag{3}$$

and this bound is sharp.

Proof. Since i + j > n, we have $i \ge 1$. Let us prove the bound (3) by induction on n, i and j. If n = 1 and i + j > n, then i = j = 1. In this case $f \in U_1(1, 2)$ and the bound $|f| \ge 2$ holds. Let us prove the induction step for $n \ge 2$. Denote $f_k = f_k^n$ for $k \in \Sigma_2$. Let us consider two cases.

Suppose that $f_0 \neq 0$ and $f_1 \neq 0$. Lemma 2 implies that $f_k \in U_{[i-1,j]}(n-1,2)$ for any $k \in \Sigma_2$. By the induction assumption we obtain that $|f_k| \geq 2^{i-1}$ for any $k \in \Sigma_2$. Then we have

$$|f| = |f_0| + |f_1| \ge 2^i$$
.

Suppose that $f_k \equiv 0$ for some $k \in \Sigma_2$. Without loss of generality, we assume that $f_0 \equiv 0$ and $f_1 \not\equiv 0$. Lemma 3 implies that $f_1 \in U_{[i,j-1]}(n-1,2)$. By the induction assumption we obtain that $|f_1| \ge 2^i$. Then we have

$$|f| = |f_0| + |f_1| = |f_1| \ge 2^i.$$

Lemma 5 implies that if $f \in F_2(n, i, j)$, then $f \in U_{[i,j]}(n, 2)$ and $|f| = 2^i$. Thus, the bound (3) is sharp.

6. Problem 2 for q = 3, i + j > n and $\frac{i}{2} + j \le n$

In this section we consider Problem 2 for q = 3, i + j > n and $\frac{i}{2} + j \le n$. The main result of this section is the following.

Theorem 5. Let $f \in U_{[i,j]}(n,3)$, where $\frac{i}{2} + j \leq n$, i + j > n and $f \not\equiv 0$. Then

$$|f| \ge 2^{3(n-j)-i} \cdot 3^{i+j-n} \tag{4}$$

and this bound is sharp.

Proof. Let us prove the bound (4) by induction on n, i and j. If $n \leq 3$, i + j > n and $\frac{i}{2} + j \leq n$, then n = 3 and i = j = 2. Then $f \in U_2(3,3)$. In this case the proof of the theorem can be carried out in the same way as for the induction step.

Let us prove the induction step for $n \ge 4$. If f is uniform, then applying Theorem 1 for q = 3 we obtain that

$$|f| \ge 2^{2(n-j)} \cdot 3^{i+j-n} > 2^{3(n-j)-i} \cdot 3^{i+j-n}.$$

So, we can assume that f is non-uniform. Then there exists a number $r \in \{1, \ldots, n\}$ such that $f_k^r \not\equiv f_m^r$ for any $k, m \in \Sigma_3$ and $k \neq m$. Without loss of generality, we assume that r = n. Denote $f_k = f_k^n$ for $k \in \Sigma_3$.

Lemma 2 implies that $f_k - f_m \in U_{[i-1,j-1]}(n-1,3)$ for any $k, m \in \Sigma_3$ and $k \neq m$. Since $\frac{i}{2} + j \leq n$ and i + j > n, we see that $i \geq 2$. Moreover, we have $\frac{i-1}{2} + j - 1 \leq n - 1$. Then applying the induction assumption for i + j > n + 1 and Theorem 2 for i + j = n + 1, we obtain that

$$|f_k - f_m| \ge 2^{3(n-j)-i+1} \cdot 3^{i+j-n-1}$$

for any $k, m \in \Sigma_3$ and $k \neq m$. Hence

$$|f_k| + |f_m| \ge 2^{3(n-j)-i+1} \cdot 3^{i+j-n-1}$$

for any $k, m \in \Sigma_3$ and $k \neq m$. Then we have

$$|f| = |f_0| + |f_1| + |f_2| = \frac{1}{2}((|f_0| + |f_1|) + (|f_0| + |f_2|) + (|f_1| + |f_2|)) \ge 2^{3(n-j)-i} \cdot 3^{i+j-n}.$$

Lemma 5 implies that if $f \in F_3(n, i, j)$, then $f \in U_{[i,j]}(n, 3)$ and $|f| = 2^{3(n-j)-i} \cdot 3^{i+j-n}$. Thus, the bound (4) is sharp.

Using Theorem 5 for i = j, we immediately obtain the following result.

Corollary 1. Let $f \in U_i(n,3)$, where $\frac{n}{2} < i \leq \frac{2n}{3}$ and $f \not\equiv 0$. Then

$$|f| \ge 2^{3n-4i} \cdot 3^{2i-i}$$

and this bound is sharp.

7. Problem 2 for q = 3 and $\frac{i}{2} + j > n$

In this section we consider Problem 2 for q = 3 and $\frac{i}{2} + j > n$. The main result of this section is the following.

Theorem 6. Let $f \in U_{[i,j]}(n,3)$, where $\frac{i}{2} + j > n$ and $f \not\equiv 0$. Then

$$|f| \ge 2^{i+j-n} \cdot 3^{n-j} \tag{5}$$

and this bound is sharp.

Proof. Let us prove the bound (5) by induction on n, i and j. If n = 1 and $\frac{i}{2} + j > n$, then i = j = 1. In this case $f \in U_1(1,3)$ and the inequality $|f| \ge 2$ holds.

Let us prove the induction step for $n \ge 2$. Let us consider the functions f_0^n , f_1^n and f_2^n . Denote $f_k = f_k^n$ for $k \in \Sigma_3$. Let $S = \{s \in \Sigma_3 \mid f_s \neq 0\}$. Let us consider three cases depending on |S|.

In the first case we suppose |S| = 1. Without loss of generality, we assume that $S = \{0\}$. Thus $f_1 \equiv 0$ and $f_2 \equiv 0$. Then Lemma 3 implies that $f_0 \in U_{[i,j-1]}(n-1,3)$. We note that $\frac{i}{2} + j - 1 > n - 1$. Applying the induction assumption for f_0 , we obtain that

$$|f_0| \ge 2^{i+j-n} \cdot 3^{n-j}.$$

Then we have

$$|f| = |f_0| \ge 2^{i+j-n} \cdot 3^{n-j}.$$

In the second case we suppose |S| = 2. Without loss of generality, we assume that $S = \{0, 1\}$. So, $f_2 \equiv 0$ and $f_k \not\equiv 0$ for any $k \in \{0, 1\}$. Lemma 2 implies that $f_k - f_2 \in U_{[i-1,j-1]}(n-1,3)$ for any $k \in \{0, 1\}$. Consequently, $f_k \in U_{[i-1,j-1]}(n-1,3)$ for any $k \in \{0, 1\}$. Applying the induction assumption for i + 2j > 2n + 1 and Theorem 5 for i + 2j = 2n + 1, we obtain that

$$|f_k| > 2^{i+j-n-1} \cdot 3^{n-j}$$

for any $k \in \{0, 1\}$. Then we have

$$|f| = |f_0| + |f_1| \ge 2^{i+j-n} \cdot 3^{n-j}.$$

In the third case we suppose |S| = 3. So $f_k \neq 0$ for any $k \in \Sigma_3$. Lemma 2 implies that $f_k \in U_{[i-1,j]}(n-1,3)$ for j < n and any $k \in \Sigma_3$ and $f_k \in U_{[i-1,n-1]}(n-1,3)$ for j = n and any $k \in \Sigma_3$. We note that $\frac{i-1}{2} + j > n-1$ for j < n. Applying the induction assumption, we obtain that

$$|f_k| \ge 2^{i+j-n} \cdot 3^{n-j-1}$$

for j < n and any $k \in \Sigma_3$ and $|f_k| \ge 2^{i-1}$ for j = n and any $k \in \Sigma_3$. Then we have

$$|f| = |f_0| + |f_1| + |f_2| \ge 2^{i+j-n} \cdot 3^{n-j}$$

for j < n and

$$|f| = |f_0| + |f_1| + |f_2| \ge 3 \cdot 2^{i-1} > 2^i$$

for j = n.

Lemma 5 implies that if $f \in F_4(n, i, j)$, then $f \in U_{[i,j]}(n, 3)$ and $|f| = 2^{i+j-n} \cdot 3^{n-j}$. Thus, the bound (5) is sharp.

Using Theorem 6 for i = j, we immediately obtain the following result.

Corollary 2. Let $f \in U_i(n,3)$, where $i > \frac{2}{3}n$ and $f \neq 0$. Then

$$|f| \ge 2^{2i-n} \cdot 3^{n-i}$$

and this bound is sharp.

8. 1-perfect bitrades in the Hamming graph H(n,3)

In this section we study 1-perfect bitrades in H(n,3). Firstly, we prove the following result.

Lemma 6. Let (T_0, T_1) be a 1-perfect bitrade in a graph G. Then $f_{(T_0,T_1)}$ is a (-1)-eigenfunction of G.

Proof. Let x be a vertex of G. By the definition of a 1-perfect bitrade we obtain that

$$\sum_{y \in B(x)} f_{(T_0, T_1)}(y) = 0$$

Therefore

$$f_{(T_0,T_1)}(x) = -\sum_{y \in N(x)} f_{(T_0,T_1)}(y),$$

i.e. $f_{(T_0,T_1)}$ is a (-1)-eigenfunction of G.

Corollary 3. Let $q = p^k$, where p is a prime and $k \ge 1$. Then H(n,q) has a 1-perfect bitrade if and only if n = qm + 1 for some $m \ge 1$.

Proof. Firstly, we note that there are several constructions of 1-perfect bitrades in H(qm + 1, q) (for example, see Lemma 4).

Suppose that (T_0, T_1) is a 1-perfect bitrade in H(n, q). By Lemma 6 we obtain that $f_{(T_0,T_1)}$ is a (-1)-eigenfunction of H(n,q). Then $-1 = \lambda_i(n,q)$ for some *i*. Hence n = qm+1 for some $m \ge 1$.

Thus, we can consider Problem 3 only for n = qm + 1, where $m \ge 1$. Now we prove the main result of this section.

Theorem 7. The minimum size of a 1-perfect bitrade in H(3m + 1, 3), where $m \ge 1$, is $2^{m+1} \cdot 3^m$.

Proof. Let (T_0, T_1) be a 1-perfect bitrade in H(3m + 1, 3). Firstly, let us prove the bound

$$|T_0| + |T_1| \ge 2^{m+1} \cdot 3^m \tag{6}$$

Lemma 6 implies that $f_{(T_0,T_1)}$ is a (-1)-eigenfunction of H(3m+1,3). We note that $-1 = \lambda_{2m+1}(3m+1,3)$. Applying Corollary 2 for n = 3m+1 and i = 2m+1 $(i > \frac{2}{3}n)$, we obtain that

$$|f_{(T_0,T_1)}| \ge 2^{m+1} \cdot 3^m$$

Then we have

$$|T_0| + |T_1| = |f_{(T_0,T_1)}| \ge 2^{m+1} \cdot 3^m.$$

So, it remains to prove that the bound (6) is sharp. Using Lemma 4 for q = 3, we see that the bound (6) is sharp.

Remark 2. We note that for m = 1 the claim of Theorem 7 was recently proved by Mogilnykh and Solov'eva in [19].

Remark 3. Let $f \in F_4(3m + 1, 2m + 1, 2m + 1)$. Denote $T_0 = \{x \in \Sigma_3^{3m+1} \mid f(x) = c\}$ and $T_1 = \{x \in \Sigma_3^{3m+1} \mid f(x) = -c\}$. It is easy to verify that (T_0, T_1) is a 1-perfect bitrade in H(3m + 1, 3). Moreover, this 1-perfect bitrade coincides with the 1-perfect bitrades (for q = 3) constructed by Krotov and Vorob'ev in [27] and Mogilnykh and Solov'eva in [19].

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