# On the spectral radius of graphs without a star forest* 

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#### Abstract

In this paper, we determine the maximum spectral radius and all extremal graphs for (bipartite) graphs of order $n$ without a star forest, extending Theorem 1.4 (iii) and Theorem 1.5 for large $n$. As a corollary, we determine the minimum least eigenvalue of $A(G)$ and all extremal graphs for graphs of order $n$ without a star forest, extending Corollary 1.6 for large $n$.


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## 1 Introduction

Let $G$ be an undirected simple graph with vertex set $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ and edge set $E(G)$, where $n$ is called the order of $G$. The adjacency matrix $A(G)$ of $G$ is the $n \times n$ matrix $\left(a_{i j}\right)$, where $a_{i j}=1$ if $v_{i}$ is adjacent to $v_{j}$, and 0 otherwise. The spectral radius of $G$ is the largest eigenvalue of $A(G)$, denoted by $\rho(G)$. The least eigenvalue of $A(G)$ is denoted by $\rho_{n}(G)$. For $v \in V(G)$, the neighborhood $N_{G}(v)$ of $v$ is $\{u: u v \in E(G)\}$ and the degree $d_{G}(v)$ of $v$ is $\left|N_{G}(v)\right|$. We write $N(v)$ and $d(v)$ for $N_{G}(v)$ and $d_{G}(v)$ respectively if there is no ambiguity. Denote by $\Delta(G)$ the maximum degree of $G$. Let $S_{n-1}$ be a star of order $n$. The center of a star is the vertex of maximum degree in the star. The centers of a star forest are the centers of the stars in the star forest. A graph $G$ is $H$-free if it does not contain $H$ as a subgraph. For two vertex disjoint graphs $G$ and $H$, we denote by $G \cup H$ and $G \nabla H$ the union of $G$ and $H$, and the join of $G$ and $H$ which is obtained by joining every vertex of $G$ to every vertex of $H$, respectively. Denote by $k G$ the the union of $k$ disjoint copies of $G$. For graph notation and terminology undefined here, readers are referred to [2].

[^0]Recall that the problem of maximizing the number of edges over all graphs without fixed subgraphs is one of the cornerstones of graph theory.

Problem 1.1 Given a graph $H$, what is the maximum number of edges of a graph $G$ of order $n$ without $H$ ?

Many instances of Problems 1.1 have been solved. For example, Lidický, Liu, and Palmer [9] determined the maximum number of edges of graphs without a forest forest if the order of a graph is sufficiently large.

Theorem 1.2 [9] Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 2$. If $G$ is an $F$-free graph of sufficiently large order $n$, then

$$
e(G) \leq \max _{1 \leq i \leq k}\left\{(i-1)(n-i+1)+\binom{i-1}{2}+\left\lfloor\frac{\left(d_{i}-1\right)(n-i+1)}{2}\right\rfloor\right\}
$$

In spectral extremal graph theory, a similar central problem is of the following type:
Problem 1.3 Given a graph $H$, what is the maximum $\rho(G)$ of a graph $G$ of order $n$ without $H$ ?
Many instances of Problem 1.3 have been solved, for example, see [4, 6, 8, 12, 13, 14, 16]. In addition, if $H$ is a linear forest, Problem 1.3 was solved in [3]. For $H=k P_{3}$, the bipartite version of Problem 1.3 was also proved in [3]. In order to state these results, we need some symbols for given graphs.

Let $S_{n, h}=K_{h} \nabla \bar{K}_{n-h}$. Furthermore, $S_{n, h}^{+}=K_{h} \nabla\left(K_{2} \cup \bar{K}_{n-h-2}\right)$. Let $F_{n, k}=K_{k-1} \nabla\left(\left(p K_{2}\right) \cup\right.$ $K_{s}$ ), where $n-(k-1)=2 p+s$ and $0 \leq s<2$. In addition, for $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$, define

$$
f\left(k, d_{1}, \ldots, d_{k}\right)=\frac{k^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{4 k-2}+2(k-2)\left(\sum_{i=1}^{k} d_{i}\right)}{k-2}
$$

Theorem 1.4 [3] Let $F=\cup_{i=1}^{k} P_{a_{i}}$ be a linear forest with $k \geq 2$ and $a_{1} \geq \cdots \geq a_{k} \geq 2$. Denote $h=\sum_{i=1}^{k}\left\lfloor\frac{a_{i}}{2}\right\rfloor-1$ and suppose that $G$ is an $F$-free graph of sufficiently large order $n$.
(i) If there exists an even $a_{i}$, then $\rho(G) \leq \rho\left(S_{n, h}\right)$ with equality if and only if $G=S_{n, h}$;
(ii) If all $a_{i}$ are odd and there exists at least one $a_{i}>3$, then $\rho(G) \leq \rho\left(S_{n, h}^{+}\right)$with equality if and only if $G=S_{n, h}^{+}$.
(iii) If all $a_{i}$ are 3, i.e., $F=k P_{3}$, then $\rho(G) \leq \rho\left(F_{n, k}\right)$ with equality if and only if $G=F_{n, k}$.

Theorem 1.5 [3] Let $G$ be a $k P_{3}$-free bipartite graph of order $n \geq 11 k-4$ with $k \geq 2$. Then

$$
\rho(G) \leq \sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.
Corollary 1.6 [3] Let $G$ be a $k P_{3}$-free graph of order $n \geq 11 k-4$ with $k \geq 2$. Then

$$
\rho_{n}(G) \geq-\sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.

In Theorem 1.4, the extremal graph for $k P_{3}$ varies form other linear forests. Note that $k P_{3}$ is also a star forest $k S_{2}$. Motivated by Problem [1.3, Theorems 1.2, 1.4 and 1.5, we determine the maximum spectral radius and all extremal graphs for all (bipartite) graphs of order $n$ without a star forest. As a corollary, we determine the minimum least eigenvalue of $A(G)$ and all extremal graphs for graphs of order $n$ without a star forest, extending Corollary 1.6 for large $n$. The main results of this paper are stated as follows.

Theorem 1.7 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ be an $F$-free graph of order $n \geq \frac{\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right)^{4}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{4}}{k-2}$, then

$$
\rho(G) \leq \frac{k+d_{k}-3+\sqrt{\left(k-d_{k}-1\right)^{2}+4(k-1)(n-k+1)}}{2}
$$

with equality if and only if $G=K_{k-1} \nabla H$, where $H$ is a $\left(d_{k}-1\right)$-regular graph of order $n-k+1$. In particular, if $d_{k}=2$, then

$$
\rho(G) \leq \rho\left(F_{n, k}\right)
$$

with equality if and only if $G=F_{n, k}$.

Remark 1. The extremal graph in Theorem 1.7 only depends on the number of the components of $F$ and the minimum order of the stars in $F$.

Theorem 1.8 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ is an $F$-free bipartite graph of order $n \geq \frac{f^{2}\left(k, d_{1}, \ldots, d_{k}\right)}{4 k-8}$, then

$$
\rho(G) \leq \sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.
Corollary 1.9 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ is an $F$-free graph of order $n \geq \frac{f^{2}\left(k, d_{1}, \ldots, d_{k}\right)}{4 k-8}$, then

$$
\rho_{n}(G) \geq-\sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.

Remark 2. For sufficiently large $n$, the extremal graphs in Theorem 1.8 and Corollary 1.9 only depend on the number of the components of $F$.

## 2 Preliminary

We first give a very rough estimation on the number of edges for a graph of order $n \geq \sum_{i=1}^{k} d_{i}+k$ without a star forest.

Lemma 2.1 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ is an $F$-free graph of order $n \geq \sum_{i=1}^{k} d_{i}+k$, then

$$
e(G) \leq\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n-(k-1)\left(\sum_{i=1}^{k} d_{i}+k-1\right)
$$

Proof. Let $C=\left\{v \in V(G): d(v) \geq \sum_{i=1}^{k} d_{i}+k-1\right\}$. Since $G$ is $F$-free, $|C| \leq k-1$, otherwise we can embed an $F$ in $G$ by the definition of $C$. Hence

$$
\begin{aligned}
e(G) & =\sum_{v \in C} d(v)+\sum_{v \in V(G) \backslash C} d(v) \\
& \leq(n-1)|C|+(n-|C|)\left(\sum_{i=1}^{k} d_{i}+k-2\right) \\
& =\left(n-\sum_{i=1}^{k} d_{i}-k+1\right)|C|+\left(\sum_{i=1}^{k} d_{i}+k-2\right) n \\
& \leq(k-1)\left(n-\sum_{i=1}^{k} d_{i}-k+1\right)+\left(\sum_{i=1}^{k} d_{i}+k-2\right) n \\
& =\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n-(k-1)\left(\sum_{i=1}^{k} d_{i}+k-1\right)
\end{aligned}
$$

Lemma 2.2 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. Let $G$ be an $F$-free connected bipartite graph of order $n \geq \frac{d_{1}^{2}}{k-1}+k-1$ with the maximum spectral radius $\rho(G)$ and $\mathbf{x}=\left(x_{u}\right)_{u \in V(G)}$ be a positive eigenvector of $\rho(G)$ such that $\max \left\{x_{u}: u \in V(G)\right\}=1$. Then $x_{u} \geq \frac{1}{\rho(G)}$ for all $u \in V(G)$.

Proof. Set for short $\rho=\rho(G)$. Choose a vertex $w \in V(G)$ such that $x_{w}=1$. Since $K_{k-1, n-k+1}$ is $F$-free, we have

$$
\rho \geq \rho\left(K_{k-1, n-k+1}\right)=\sqrt{(k-1)(n-k+1)}
$$

If $u=w$, then $x_{u}=1 \geq \frac{1}{\rho}$. So we next suppose that $u \neq w$. We consider the following two cases.
Case 1. $u$ is adjacent to $w$. By eigenequation of $A(G)$ on $u$,

$$
\rho x_{u}=\sum_{u v \in E(G)} x_{v} \geq x_{w}=1
$$

which implies that

$$
x_{u} \geq \frac{1}{\rho} .
$$

Case 2. $u$ is not adjacent to $w$. Let $G_{1}$ be a graph obtained from $G$ by deleting all edges incident with $u$ and adding an edge $u w$. Note that $u w$ is a pendant edge in $G_{1}$.

Claim. $G_{1}$ is also $F$-free.
Suppose that $G_{1}$ contains an $F$ as a subgraph. Since $G$ is $F$-free and $G_{1}$ contains an $F$ as a subgraph, we have $u w \in E(F)$. Since $u w$ is a pendant edge in $G_{1}, w$ is a center of $F$ with $d_{F}(w)=d_{j}$, where $1 \leq j \leq k$. Let $G_{2}$ be the subgraph of $G_{1}$ by deleting $w$ and all its neighbors in $F$. Note that $G_{2}$ is also a subgraph of $G$. Since $G_{1}$ contains an $F$ as a subgraph, $G_{2}$ contains $\cup_{i \neq j} S_{d_{i}}$ as a subgraph. By eigenequation of $G$ on $w$,

$$
d(w) \geq \sum_{v w \in E(G)} x_{v}=\rho x_{w}=\rho \geq \sqrt{(k-1)(n-k+1)} \geq d_{1} \geq d_{j}
$$

This implies that $G$ contains an $F$ as a subgraph, a contradiction.
By Claim, $G_{1}$ is $F$-free. Then

$$
\begin{aligned}
0 & \geq \rho\left(G_{1}\right)-\rho \geq \frac{\mathbf{x}^{\mathrm{T}} A\left(G_{1}\right) \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}-\frac{\mathbf{x}^{\mathbf{T}} A(G) \mathbf{x}}{\mathbf{x}^{\mathbf{T}} \mathbf{x}} \\
& =\frac{2}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}\left(x_{u} x_{w}-x_{u} \sum_{u v \in E(G)} x_{v}\right) \\
& =\frac{2 x_{u}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}\left(1-\rho x_{u}\right)
\end{aligned}
$$

which implies that

$$
x_{u} \geq \frac{1}{\rho}
$$

This completes the proof.
Lemma 2.3 Let $d \geq 1, k \geq 1, n \geq \frac{(d-1)^{2}+(k-1)^{2}}{k-1}$, and $H$ be a graph of order $n-k+1$. If $G=K_{k-1} \nabla H$ and $\Delta(H) \leq d-1$, then

$$
\rho(G) \leq \frac{k+d-3+\sqrt{(k-d-1)^{2}+4(k-1)(n-k+1)}}{2}
$$

with equality if and only if $H$ is a $(d-1)$-regular graph.
Proof. If $d=1$, then $G=K_{k-1} \nabla \bar{K}_{n-k+1}$. it is easy to calculate that

$$
\rho\left(K_{k-1} \nabla \bar{K}_{n-k+1}\right)=\frac{k-2+\sqrt{(k-2)^{2}+4(k-1)(n-k+1)}}{2}
$$

Next suppose that $d \geq 2$. Let $u_{1}, u_{2}, \cdots, u_{k-1}$ be the vertex of $G$ corresponding to $K_{k-1}$ in the representation $G:=K_{k-1} \nabla H$. Set for short $\rho=\rho(G)$ and let $\mathbf{x}=\left(x_{v}\right)_{v \in E(G)}$ be a positive eigenvector of $\rho$. By symmetry, $x_{u_{1}}=\cdots=x_{u_{k-1}}$. Choose a vertex $v \in V(H)$ such that

$$
x_{v}=\max _{w \in V(H)} x_{w}
$$

By eigenequation of $A(G)$ on $u_{1}$ and $v$, we have

$$
\begin{gather*}
\rho x_{u_{1}}=(k-2) x_{u_{1}}+\sum_{u u_{1} \in V(H)} x_{u} \leq(k-2) x_{u_{1}}+(n-k+1) x_{v}  \tag{1}\\
\rho x_{v} \leq(k-1) x_{u_{1}}+\sum_{u v \in E(H)} x_{u} \leq(k-1) x_{u_{1}}+(d-1) x_{v} \tag{2}
\end{gather*}
$$

which implies that

$$
\begin{aligned}
(\rho-k+2) x_{u_{1}} & \leq(n-k+1) x_{v} \\
(\rho-d+1) x_{v} & \leq(k-1) x_{u_{1}}
\end{aligned}
$$

Since

$$
\rho>\rho\left(K_{k-1}\right)=k-2
$$

and

$$
\rho>\rho\left(K_{k-1, n-k+1}\right)=\sqrt{(k-1)(n-k+1)} \geq d-1
$$

we have

$$
\rho^{2}-(k+d-3) \rho+(k-2)(d-1)-(k-1)(n-k+1) \leq 0 .
$$

Hence

$$
\rho \leq \frac{k+d-3+\sqrt{(k-d-1)^{2}+4(k-1)(n-k+1)}}{2} .
$$

If equality holds, then all equalities in (1) and (2) hold. So $d(v)=k+d-2$ and $x_{u}=x_{v}$ for any vertex $u \in V(H)$. Since for any $u \in V(H)$,

$$
\rho x_{u}=(k-1) x_{u_{1}}+\sum_{u z \in E(H)} x_{z} \leq(k-1) x_{u_{1}}+(d-1) x_{v}=\rho x_{v}
$$

we have $d(u)=d(v)=d+k-2$. So $H$ is $(d-1)$-regular.

## 3 Proof of Theorem 1.7

Before proving Theorem 1.7, we first prove the following important result for connected graphs without a star forest.

Theorem 3.1 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ be an $F$-free connected graph of order $n \geq\left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}$, then

$$
\rho(G) \leq \frac{k+d_{k}-3+\sqrt{\left(k-d_{k}-1\right)^{2}+4(k-1)(n-k+1)}}{2}
$$

with equality if and only if $G=K_{k-1} \nabla H$, where $H$ is a $\left(d_{k}-1\right)$-regular graph of order $n-k+1$. In particular, if $d_{k}=2$, then

$$
\rho(G) \leq \rho\left(F_{n, k}\right)
$$

with equality if and only if $G=F_{n, k}$.
Proof. Let $G$ be an $F$-free connected graph of order $n$ with the maximum spectral radius. Set for short $V=V(G), E=E(G), A=A(G)$, and $\rho=\rho(G)$. Let $\mathbf{x}=\left(x_{v}\right)_{v \in V(G)}$ be a positive eigenvector of $\rho$ such that

$$
x_{w}=\max \left\{x_{u}: u \in V(G)\right\}=1 .
$$

Since $K_{k-1, n-k+1}$ is $F$-free, we have

$$
\rho \geq \rho\left(K_{k-1, n-k+1}\right)=\sqrt{(k-1)(n-k+1)} .
$$

Let $L=\left\{v \in V: x_{v}>\epsilon\right\}$ and $S=\left\{v \in V: x_{v} \leq \epsilon\right\}$, where $\epsilon=\frac{1}{\sum_{i=1}^{k} 2 d_{i}+5 k-7}$.
Claim. $|L|=k-1$.
If $|L| \neq k-1$, then $|L| \geq k$ or $|L| \leq k-2$.
First suppose that $|L| \geq k$. By eigenequation of $A$ on any vertex $u \in L$, we have

$$
\sum_{i=1}^{k} d_{i}+k-2 \leq \frac{\sqrt{(k-1)(n-k+1)}}{\sum_{i=1}^{k} 2 d_{i}+5 k-7}=\sqrt{(k-1)(n-k+1)} \epsilon<\rho x_{u}=\sum_{u v \in E} x_{v} \leq d(u)
$$

where the first inequality holds because $n \geq\left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}$. Hence

$$
d(u) \geq \sum_{i=1}^{k} d_{i}+k-1
$$

Then we can embed an $F$ with all centers in $L$ in $G$, a contradiction.
Next suppose that $|L| \leq k-2$. Then

$$
e(L) \leq\binom{|L|}{2} \leq \frac{1}{2}(k-2)(k-3)
$$

and

$$
e(L, S) \leq(k-2)(n-k+2)
$$

In addition, by Lemma 2.1

$$
e(S) \leq e(G) \leq\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n
$$

By eigenequation of $A^{2}$ on $w$, we have

$$
\begin{aligned}
(k-1)(n-k+1) & \leq \rho^{2}=\rho^{2} x_{w}=\sum_{v w \in E} \sum_{u v \in E} x_{u} \leq \sum_{u v \in E}\left(x_{u}+x_{v}\right) \\
& =\sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right)+\sum_{u v \in E(S)}\left(x_{u}+x_{v}\right)+\sum_{u v \in E(L)}\left(x_{u}+x_{v}\right) \\
& \leq \sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right)+2 \epsilon e(S)+2 e(L) \\
& \leq \sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right)+2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n+(k-2)(k-3)
\end{aligned}
$$

Hence

$$
\sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right) \geq(k-1)(n-k+1)-2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n-(k-2)(k-3)
$$

On the other hand, by the definition of $L$ and $S$, we have

$$
\sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right) \leq(1+\epsilon) e(L, S) \leq(1+\epsilon)(k-2)(n-k+2)
$$

Thus

$$
(1+\epsilon)(k-2)(n-k+2) \geq(k-1)(n-k+1)-2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n-(k-2)(k-3)
$$

which implies that

$$
\left(\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right) \epsilon-1\right) n \geq \epsilon(k-2)^{2}-\left(k^{2}-3 k+3\right)
$$

Since $\epsilon=\frac{1}{\sum_{i=1}^{k} 2 d_{i}+5 k-7}$, we have

$$
\begin{aligned}
n \leq & \left(k^{2}-3 k+3\right)\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right)\left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)- \\
& (k-2)^{2}\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right) \\
\leq & \left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}
\end{aligned}
$$

a contradiction. This proves the Claim.
By Claim, $|L|=k-1$ and thus $|S|=n-k+1$. Then the subgraph $H$ induced by $S$ in $G$ is $S_{d_{k}}$-free. Otherwise, we can embed $F$ in $G$ with $k-1$ centers in $L$ and a center in $S$ as $d(u) \geq \sum_{i=1}^{k} d_{i}+k-1$ for any $u \in L$, a contradiction. Now $\Delta(H) \leq d_{k}-1$. Note that the resulting graph obtained from $G$ by adding all edges in $L$ and all edges with one end in $L$ and the other in $S$ are also $F$-free and its spectral radius increases strictly. By the extremality of $G$, we have $G=K_{k-1} \nabla H$. By Lemma 2.3 and the extremality of $G$, it follows that $H$ is a $\left(d_{k}-1\right)$-regular graph and

$$
\rho=\frac{k+d_{k}-3+\sqrt{\left(k-d_{k}-1\right)^{2}+4(k-1)(n-k+1)}}{2} .
$$

In particular, if $d_{k}=2$ then $\Delta(H) \leq 1$, i.e., $H=p K_{2} \cup q K_{1}$, where $2 p+q=n-k+1$. By the extremality of $G, G=F_{n, k}$. This completes the proof.

Proof of Theorems 1.7. Let $G$ be an $F$-free graph of order $n$ with the maximum spectral radius.
If $G$ is connected, then the result follows directly from Theorem 3.1. Next we suppose that $G$ is not connected. Since $K_{k-1, n-k+1}$ is $F$-free, we have

$$
\rho(G) \geq \sqrt{(k-1)(n-k+1)}
$$

Let $G_{1}$ be a component of $G$ such that $\rho\left(G_{1}\right)=\rho(G)$ and $n_{1}=\left|V\left(G_{1}\right)\right|$. Then

$$
\begin{aligned}
n_{1}-1 & \geq \rho\left(G_{1}\right)=\rho(G) \geq \sqrt{(k-1)(n-k+1)} \geq \sqrt{(k-2) n} \\
& \geq\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right)^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}
\end{aligned}
$$

which implies that

$$
n_{1} \geq\left(\sum_{i=1}^{k} 2 d_{i}+5 k-8\right)^{2}\left(\sum_{i=1}^{k} d_{i}+k-2\right)^{2}+1
$$

By Theorem 3.1 again,

$$
\begin{aligned}
\rho(G)=\rho\left(G_{1}\right) & \leq \frac{k+d_{k}-3+\sqrt{\left(k-d_{k}-1\right)^{2}+4(k-1)\left(n_{1}-k+1\right)}}{2} \\
& <\frac{k+d_{k}-3+\sqrt{\left(k-d_{k}-1\right)^{2}+4(k-1)(n-k+1)}}{2}
\end{aligned}
$$

In particular, if $d_{k}=2$ then it follows from By Theorem 3.1 again,

$$
\rho(G)=\rho\left(G_{1}\right)=\rho\left(F_{n_{1}, k}\right)<\rho\left(F_{n, k}\right)
$$

Hence the result follows.
Note that extremal graph in Theorem 1.4 (iii) also holds for signless Laplacian special radius $q(G)$ [5]. We conjecture the extremal graph in Theorem 1.7 also holds for signless Laplacian spectral radius $q(G)$.

Conjecture 3.2 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ be an $F$-free graph of large order $n$, then

$$
q(G) \leq \frac{n+2 k+2 d_{k}-6+\sqrt{\left(n+2 k-2 d_{k}-2\right)^{2}-8(k-1)\left(k-d_{k}-1\right)}}{2}
$$

with equality if and only if $G=K_{k-1} \nabla H$, where $H$ is a $\left(d_{k}-1\right)$-regular graph of order $n-k+1$. In particular, if $d_{k}=2$, then

$$
q(G) \leq q\left(F_{n, k}\right)
$$

with equality if and only if $G=F_{n, k}$.

## 4 Proofs of Theorem 1.8 and Corollary 1.9

Before proving Theorem 1.8 and Corollary 1.9 , we first prove the following important result for bipartite connected graphs without a star forest.

Theorem 4.1 Let $F=\cup_{i=1}^{k} S_{d_{i}}$ be a star forest with $k \geq 2$ and $d_{1} \geq \cdots \geq d_{k} \geq 1$. If $G$ is an $F$-free connected bipartite graph of order $n \geq f\left(k, d_{1}, \ldots, d_{k}\right)$, then

$$
\rho(G) \leq \sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.
Proof. Let $G$ be an $F$-free connected bipartite graph of order $n$ with the maximum spectral radius. Set for short $V=V(G), E=E(G), A=A(G)$, and $\rho=\rho(G)$. Let $\mathbf{x}=\left(x_{v}\right)_{v \in V(G)}$ be a positive eigenvector of $\rho$ such that

$$
x_{w}=\max \left\{x_{u}: u \in V(G)\right\}=1
$$

Since $K_{k-1, n-k+1}$ is $F$-free, we have

$$
\begin{equation*}
\rho \geq \rho\left(K_{k-1, n-k+1}\right)=\sqrt{(k-1)(n-k+1)} \tag{3}
\end{equation*}
$$

Let $L=\left\{v \in V: x_{v}>\epsilon\right\}$ and $S=\left\{v \in V: x_{v} \leq \epsilon\right\}$, where

$$
\frac{\sum_{i=1}^{k} d_{i}+k-2}{\sqrt{(k-1)(n-k+1)}} \leq \epsilon \leq \frac{1}{k \sum_{i=1}^{k}\left(2 d_{i}+5 k-4\right)^{2 k-1}}\left(1-\frac{\sum_{i=1}^{k} d_{i}}{n}\right)
$$

Claim 1. $|L| \leq k-1$.

Suppose that $|L| \geq k$. By eigenequation of $A$ on any vertex $u \in L$, we have

$$
\sum_{i=1}^{k} d_{i}+k-2 \leq \sqrt{(k-1)(n-k+1)} \epsilon<\rho x_{u}=\sum_{u v \in E} x_{v} \leq d(u)
$$

Hence

$$
d(u) \geq \sum_{i=1}^{k} d_{i}+k-1
$$

Then we can embed an $F$ in $G$ with all centers in $L$, a contradiction. This proves Claim 1.
Since $|L| \leq k-1$, we have

$$
e(L) \leq\binom{|L|}{2} \leq \frac{1}{2}(k-1)(k-2)
$$

and

$$
e(L, S) \leq(k-1)(n-k+1)
$$

In addition, by Lemma 2.1

$$
e(S) \leq e(G) \leq\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n
$$

We next show that for any vertex in $L$ has large degree.
Claim 2. Let $u \in L$ and $x_{u}=1-\delta$. Then

$$
d(u) \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)(\delta+\epsilon)\right) n
$$

Let $B_{u}=\{v \in V: u v \notin E\}$. We first sum of eigenvector over all vertices of $G$.

$$
\begin{aligned}
\rho \sum_{v \in V} x_{v} & =\sum_{v \in V} \rho x_{v}=\sum_{v \in V} \sum_{v z \in E} x_{z}=\sum_{v \in V} d(v) x_{v} \leq \sum_{v \in L} d(v) x_{v}+\sum_{v \in S} d(v) x_{v} \\
& \leq \sum_{v \in L} d(v)+\epsilon \sum_{v \in S} d(v)=2 e(L)+e(L, S)+\epsilon(2 e(S)+e(L, S)) \\
& =2 e(L)+2 \epsilon e(S)+(1+\epsilon) e(L, S)
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\sum_{v \in V} x_{v} \leq \frac{2 e(L)+2 \epsilon e(S)+(1+\epsilon) e(L, S)}{\rho} \tag{4}
\end{equation*}
$$

Next we sum of eigenvector over all vertices in $B_{u}$ by E.q. (4) and Lemma 2.2. Since

$$
\begin{aligned}
\frac{1}{\rho}\left|B_{u}\right| & \leq \sum_{v \in B_{u}} x_{v} \leq \sum_{v \in V(G)} x_{v}-\sum_{u v \in E(G)} x_{v}=\sum_{v \in V(G)} x_{v}-\rho x_{u} \\
& \leq \frac{2 e(L)+2 \epsilon e(S)+(1+\epsilon) e(L, S)}{\rho}-\rho x_{u}
\end{aligned}
$$

we have

$$
\left|B_{u}\right| \leq 2 e(L)+2 \epsilon e(S)+(1+\epsilon) e(L, S)-\rho^{2} x_{u}
$$

$$
\begin{aligned}
\leq & 2 e(L)+2 \epsilon e(S)+(1+\epsilon) e(L, S)-(k-1)(n-k+1)(1-\delta) \\
\leq & (k-1)(k-2)+2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n+(1+\epsilon)(k-1)(n-k+1)- \\
& (k-1)(n-k+1)(1-\delta) \\
= & \left(2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right)+(\delta+\epsilon)(k-1)\right) n+(k-1)(k-2)-(\delta+\epsilon)(k-1)^{2} \\
\leq & \left(\sum_{i=1}^{k} 2 d_{i}+4 k-6+(k-1)+1\right)(\delta+\epsilon) n \\
= & \left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)(\delta+\epsilon) n,
\end{aligned}
$$

where the last second inequality holds since $(k-1)(k-2) \leq \epsilon n<(\delta+\epsilon) n$ by the definition of $\epsilon$ and n. Hence

$$
d(u) \geq n-1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)(\delta+\epsilon) n \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)(\delta+\epsilon)\right) n .
$$

This completes Claim 2.
Claim 3. Let $1 \leq s<k-1$. Suppose that there is a set $X$ of $s$ vertices such that $X=\{v \in V$ : $x_{v} \geq 1-\eta$ and $\left.d(v) \geq(1-\eta) n\right\}$. Then there exists a vertex $u \in L \backslash X$ such that

$$
x_{u} \geq 1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)^{2}(\eta+\epsilon)
$$

and

$$
d(u) \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)^{2}(\eta+\epsilon)\right) n
$$

By eigenequation of $A^{2}$ on $w$, we have

$$
\begin{aligned}
\rho^{2} & =\rho^{2} x_{w}=\sum_{v w \in E} \sum_{u v \in E} x_{u} \leq \sum_{u v \in E}\left(x_{u}+x_{v}\right) \\
& =\sum_{u v \in E(S)}\left(x_{u}+x_{v}\right)+\sum_{u v \in E(L)}\left(x_{u}+x_{v}\right) \sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right) \\
& \leq 2 \epsilon e(S)+2 e(L)+\sum_{u v \in E(L, S)}\left(x_{u}+x_{v}\right) \\
& \leq 2 \epsilon e(S)+2 e(L)+\epsilon e(L, S)+\sum_{\substack{u v \in E(L \backslash X, S) \\
u \in L \backslash X}} x_{u}+\sum_{\substack{u v \in E(L \cap X, S) \\
u \in L \cap X}} x_{u},
\end{aligned}
$$

which implies that

$$
\sum_{\substack{u v \in E(L \backslash X, S) \\ u \in L \backslash X}} x_{u}
$$

$$
\begin{aligned}
\geq & \rho^{2}-2 \epsilon e(S)-2 e(L)-\epsilon e(L, S)-\sum_{\substack{u v \in E(L \cap X, S) \\
u \in L \cap X}} x_{u} \\
\geq & (k-1)(n-k+1)-2 \epsilon\left(\sum_{i=1}^{k} d_{i}+2 k-3\right) n-(k-1)(k-2)- \\
& \epsilon(k-1)(n-k+1)-s n \\
= & \left(k-1-s-\epsilon\left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)\right) n-(k-1)(2 k-3)+\epsilon(k-1)^{2} \\
\geq & \left(k-1-s-\epsilon\left(\sum_{i=1}^{k} 2 d_{i}+5 k-7\right)\right) n-\epsilon n \\
= & \left(k-1-s-\epsilon\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)\right) n,
\end{aligned}
$$

where the last third inequality holds since $(k-1)(2 k-3) \leq \epsilon n$ by the definition of $\epsilon$ and $n$. In addition,

$$
\begin{aligned}
e(L \backslash X, S) & =e(L, S)-e(L \cap X, S) \\
& \leq(k-1)(n-k+1)-s(1-\eta) n+\binom{s}{2} \\
& \leq(k-1-s(1-\eta)) n-\left((k-1)^{2}-\binom{k-2}{2}\right) \\
& \leq(k-1-s(1-\eta)) n .
\end{aligned}
$$

Let

$$
g(s)=\frac{k-1-s-\epsilon\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)}{k-1-s(1-\eta)} .
$$

It is easy to see that $g(s)$ is decreasing with respect to $1 \leq s \leq k-2$. Then

$$
\begin{aligned}
\frac{\sum_{u v \in E(L \backslash X, S)} x_{u}}{e(L \backslash X, S)} & \geq g(s) \geq g(k-2)=\frac{1-\epsilon\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)}{1+(k-2) \eta} \\
& \geq 1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)(\eta+\epsilon) .
\end{aligned}
$$

Hence there exists a vertex $u \in L \backslash X$ such that

$$
x_{u} \geq 1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)(\eta+\epsilon) .
$$

By Claim 2,

$$
\begin{aligned}
d(u) & \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)\left(\left(\sum_{i=1}^{k} 2 d_{i}+5 k-6\right)(\eta+\epsilon)+\epsilon\right)\right) n \\
& \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)^{2}(\eta+\epsilon)\right) n
\end{aligned}
$$

This completes Claim 3.

Claim 4. $|L|=k-1$. Furthermore, for all $u \in L$,

$$
x_{u} \geq 1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon
$$

and

$$
d(u) \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon\right) n
$$

Note that $w \in L$ and $x_{w}=1$. By Claim 2,

$$
d(w) \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right) \epsilon\right) n
$$

Applying Claim 5 iteratively for $k-2$ times, we can find a set $X \subseteq L \backslash\{w\}$ of $k-2$ vertices such that for any $u \in X$,

$$
\begin{aligned}
x_{u} & \geq 1-\left(\sum_{j=1}^{k-2}\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)^{2 j}+\left(\sum_{i=1}^{k} 2 d_{i}+5 k-5\right)^{2 k-2}\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)\right) \epsilon \\
& \geq 1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon
\end{aligned}
$$

and

$$
d(u) \geq\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon\right) n
$$

Noting $|L| \leq k-1$, we have $L=X \cup\{w\}$. Hence $|L|=k-1$. This proves Claim 4.
Let $T$ be the common neighborhood of $L$ and $R=S \backslash T$. By Claim 4,

$$
|L|=k-1
$$

and

$$
|T| \geq\left(1-k\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon\right) n \geq \sum_{i=1}^{k} d_{i}
$$

Since $G$ is bipartite, $L$ and $T$ are both independent sets of $G$.
Claim 5. $R$ is empty.
Suppose that $R$ is not empty, i.e., there is a vertex $v \in R$. Then $v$ has at most $d_{k}-1$ neighbors in $S$, otherwise we can embed an $F$ in $G$. Let $H$ be a graph obtained from $G$ by removing all edges incident with $v$ and then connecting $v$ to each vertex in $L$. Clearly, $H$ is still $F$-free. By the definition of $R, v$ can be adjacent to at most $k-2$ vertices in $L$. Let $u \in L$ be the vertex not adjacent to $v$. Then By Claims 4 and 5, we have

$$
\rho(H)-\rho \geq \frac{\mathbf{x}^{T} A(H) \mathbf{x}}{\mathbf{x}^{T} \mathbf{x}}-\frac{\mathbf{x}^{T} A \mathbf{x}}{\mathbf{x}^{T} \mathbf{x}}
$$

$$
\begin{aligned}
& \geq \frac{2 x_{v}}{\mathbf{x}^{T} \mathbf{x}}\left(x_{u}-\sum_{\substack{u z \in E \\
z \in S}} x_{z}\right) \\
& \geq \frac{2 x_{v}}{\mathbf{x}^{T} \mathbf{x}}\left(1-\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1} \epsilon-\left(d_{k}-1\right) \epsilon\right) \\
& =\frac{2 x_{v}}{\mathbf{x}^{T} \mathbf{x}}\left(1-\left(\left(\sum_{i=1}^{k} 2 d_{i}+5 k-4\right)^{2 k-1}+d_{k}-1\right) \epsilon\right) \\
& >0
\end{aligned}
$$

Hence $\rho(H)>\rho$, a contradiction. This proves Claim 5.
By Claim $5, S=T$. By the definition of $T$, we have $G=K_{k-1, n-k+1}$. This completes the proof.

Proof of Theorem 1.8, Let $G$ be an $F$-free bipartite graph of order $n$ with the maximum spectral radius.

If $G$ is connected, then the result follows directly from Theorem 4.1. Next we suppose that $G$ is not connected. Since $K_{k-1, n-k+1}$ is $F$-free,

$$
\rho(G) \geq \sqrt{(k-1)(n-k+1)}
$$

Let $G_{1}$ be a component of $G$ such that $\rho\left(G_{1}\right)=\rho(G)$ and $n_{1}=\left|V\left(G_{1}\right)\right|$. Note that $G$ is triangle-free. By Wilf theorem [15, Theorem 2], we have

$$
\frac{n_{1}^{2}}{4} \geq \rho^{2}\left(G_{1}\right)=\rho(G)^{2} \geq(k-1)(n-k+1) \geq(k-2) n \geq \frac{f^{2}\left(k, d_{1}, \ldots, d_{k}\right)}{4}
$$

which implies that

$$
n_{1} \geq f\left(k, d_{1}, \ldots, d_{k}\right)
$$

By Theorem 4.1 again,

$$
\rho\left(G_{1}\right) \leq \sqrt{(k-1)\left(n_{1}-k+1\right)}<\sqrt{(k-1)(n-k+1)}
$$

a contradiction. This completes the proof.
Proof of Corollary 1.9. By a result of Favaron et al. [7, $\rho_{n}(G) \geq \rho_{n}(H)$ for some spanning bipartite subgraph $H$. Moreover, the equality holds if and only if $G=H$, which can be deduced by its original proof. By Theorem 1.8,

$$
\rho(H) \leq \sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $H=K_{k-1, n-k+1}$. Since the spectrum of a bipartite graph is symmetric [10],

$$
\rho_{n}(H) \geq-\sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $H=K_{k-1, n-k+1}$. Thus we have

$$
\rho_{n}(G) \geq-\sqrt{(k-1)(n-k+1)}
$$

with equality if and only if $G=K_{k-1, n-k+1}$.

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