# On the spectral radius of graphs without a star forest<sup>\*</sup>

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#### Abstract

In this paper, we determine the maximum spectral radius and all extremal graphs for (bipartite) graphs of order n without a star forest, extending Theorem 1.4 (iii) and Theorem 1.5 for large n. As a corollary, we determine the minimum least eigenvalue of A(G) and all extremal graphs for graphs of order n without a star forest, extending Corollary 1.6 for large n.

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## 1 Introduction

Let G be an undirected simple graph with vertex set  $V(G) = \{v_1, \ldots, v_n\}$  and edge set E(G), where n is called the order of G. The adjacency matrix A(G) of G is the  $n \times n$  matrix  $(a_{ij})$ , where  $a_{ij} = 1$  if  $v_i$  is adjacent to  $v_j$ , and 0 otherwise. The spectral radius of G is the largest eigenvalue of A(G), denoted by  $\rho(G)$ . The least eigenvalue of A(G) is denoted by  $\rho_n(G)$ . For  $v \in V(G)$ , the neighborhood  $N_G(v)$  of v is  $\{u : uv \in E(G)\}$  and the degree  $d_G(v)$  of v is  $|N_G(v)|$ . We write N(v)and d(v) for  $N_G(v)$  and  $d_G(v)$  respectively if there is no ambiguity. Denote by  $\Delta(G)$  the maximum degree of G. Let  $S_{n-1}$  be a star of order n. The center of a star is the vertex of maximum degree in the star. The centers of a star forest are the centers of the stars in the star forest. A graph G is H-free if it does not contain H as a subgraph. For two vertex disjoint graphs G and H, we denote by  $G \cup H$  and  $G \nabla H$  the union of G and H, and the join of G and H which is obtained by joining every vertex of G to every vertex of H, respectively. Denote by kG the the union of k disjoint copies of G. For graph notation and terminology undefined here, readers are referred to [2].

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Recall that the problem of maximizing the number of edges over all graphs without fixed subgraphs is one of the cornerstones of graph theory.

**Problem 1.1** Given a graph H, what is the maximum number of edges of a graph G of order n without H?

Many instances of Problems 1.1 have been solved. For example, Lidický, Liu, and Palmer [9] determined the maximum number of edges of graphs without a forest forest if the order of a graph is sufficiently large.

**Theorem 1.2** [9] Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 2$ . If G is an F-free graph of sufficiently large order n, then

$$e(G) \le \max_{1 \le i \le k} \left\{ (i-1)(n-i+1) + \binom{i-1}{2} + \left\lfloor \frac{(d_i-1)(n-i+1)}{2} \right\rfloor \right\}.$$

In spectral extremal graph theory, a similar central problem is of the following type:

**Problem 1.3** Given a graph H, what is the maximum  $\rho(G)$  of a graph G of order n without H?

Many instances of Problem 1.3 have been solved, for example, see [4, 6, 8, 12, 13, 14, 16]. In addition, if H is a linear forest, Problem 1.3 was solved in [3]. For  $H = kP_3$ , the bipartite version of Problem 1.3 was also proved in [3]. In order to state these results, we need some symbols for given graphs.

Let  $S_{n,h} = K_h \nabla \overline{K}_{n-h}$ . Furthermore,  $S_{n,h}^+ = K_h \nabla (K_2 \cup \overline{K}_{n-h-2})$ . Let  $F_{n,k} = K_{k-1} \nabla ((pK_2) \cup K_s)$ , where n - (k-1) = 2p + s and  $0 \le s < 2$ . In addition, for  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ , define

$$f(k, d_1, \dots, d_k) = \frac{k^2 (\sum_{i=1}^k d_i + k - 2)^2 (\sum_{i=1}^k 2d_i + 5k - 4)^{4k-2} + 2(k-2) (\sum_{i=1}^k d_i)}{k-2}.$$

**Theorem 1.4** [3] Let  $F = \bigcup_{i=1}^{k} P_{a_i}$  be a linear forest with  $k \ge 2$  and  $a_1 \ge \cdots \ge a_k \ge 2$ . Denote  $h = \sum_{i=1}^{k} \lfloor \frac{a_i}{2} \rfloor - 1$  and suppose that G is an F-free graph of sufficiently large order n. (i) If there exists an even  $a_i$ , then  $\rho(G) \le \rho(S_{n,h})$  with equality if and only if  $G = S_{n,h}$ ; (ii) If all  $a_i$  are odd and there exists at least one  $a_i > 3$ , then  $\rho(G) \le \rho(S_{n,h})$  with equality if and only if  $G = S_{n,h}$ ; (iii) If all  $a_i$  are 3, i.e.,  $F = kP_3$ , then  $\rho(G) \le \rho(F_{n,k})$  with equality if and only if  $G = F_{n,k}$ .

**Theorem 1.5** [3] Let G be a  $kP_3$ -free bipartite graph of order  $n \ge 11k - 4$  with  $k \ge 2$ . Then

$$\rho(G) \le \sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

**Corollary 1.6** [3] Let G be a  $kP_3$ -free graph of order  $n \ge 11k - 4$  with  $k \ge 2$ . Then

$$\rho_n(G) \ge -\sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

In Theorem 1.4, the extremal graph for  $kP_3$  varies form other linear forests. Note that  $kP_3$  is also a star forest  $kS_2$ . Motivated by Problem 1.3, Theorems 1.2, 1.4 and 1.5, we determine the maximum spectral radius and all extremal graphs for all (bipartite) graphs of order n without a star forest. As a corollary, we determine the minimum least eigenvalue of A(G) and all extremal graphs for graphs of order n without a star forest, extending Corollary 1.6 for large n. The main results of this paper are stated as follows.

**Theorem 1.7** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G be an *F*-free graph of order  $n \ge \frac{(\sum_{i=1}^{k} 2d_i + 5k - 8)^4 (\sum_{i=1}^{k} d_i + k - 2)^4}{k-2}$ , then

$$\rho(G) \le \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}$$

with equality if and only if  $G = K_{k-1}\nabla H$ , where H is a  $(d_k - 1)$ -regular graph of order n - k + 1. In particular, if  $d_k = 2$ , then

$$\rho(G) \le \rho(F_{n,k})$$

with equality if and only if  $G = F_{n,k}$ .

**Remark 1.** The extremal graph in Theorem 1.7 only depends on the number of the components of F and the minimum order of the stars in F.

**Theorem 1.8** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G is an F-free bipartite graph of order  $n \ge \frac{f^2(k, d_1, \dots, d_k)}{4k-8}$ , then

$$\rho(G) \le \sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

**Corollary 1.9** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G is an *F*-free graph of order  $n \ge \frac{f^2(k, d_1, \dots, d_k)}{4k-8}$ , then

$$\rho_n(G) \ge -\sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

**Remark 2.** For sufficiently large n, the extremal graphs in Theorem 1.8 and Corollary 1.9 only depend on the number of the components of F.

### 2 Preliminary

We first give a very rough estimation on the number of edges for a graph of order  $n \ge \sum_{i=1}^{k} d_i + k$  without a star forest.

**Lemma 2.1** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G is an F-free graph of order  $n \ge \sum_{i=1}^{k} d_i + k$ , then

$$e(G) \le \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n - (k-1)\left(\sum_{i=1}^{k} d_i + k - 1\right).$$

**Proof.** Let  $C = \{v \in V(G) : d(v) \ge \sum_{i=1}^{k} d_i + k - 1\}$ . Since G is F-free,  $|C| \le k - 1$ , otherwise we can embed an F in G by the definition of C. Hence

$$\begin{split} e(G) &= \sum_{v \in C} d(v) + \sum_{v \in V(G) \setminus C} d(v) \\ &\leq (n-1)|C| + (n-|C|) \left(\sum_{i=1}^{k} d_i + k - 2\right) \\ &= \left(n - \sum_{i=1}^{k} d_i - k + 1\right)|C| + \left(\sum_{i=1}^{k} d_i + k - 2\right)n \\ &\leq (k-1) \left(n - \sum_{i=1}^{k} d_i - k + 1\right) + \left(\sum_{i=1}^{k} d_i + k - 2\right)n \\ &= \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n - (k-1) \left(\sum_{i=1}^{k} d_i + k - 1\right) \end{split}$$

**Lemma 2.2** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . Let G be an *F*-free connected bipartite graph of order  $n \ge \frac{d_1^2}{k-1} + k - 1$  with the maximum spectral radius  $\rho(G)$ and  $\mathbf{x} = (x_u)_{u \in V(G)}$  be a positive eigenvector of  $\rho(G)$  such that  $\max\{x_u : u \in V(G)\} = 1$ . Then  $x_u \ge \frac{1}{\rho(G)}$  for all  $u \in V(G)$ .

**Proof.** Set for short  $\rho = \rho(G)$ . Choose a vertex  $w \in V(G)$  such that  $x_w = 1$ . Since  $K_{k-1,n-k+1}$  is *F*-free, we have

$$\rho \ge \rho(K_{k-1,n-k+1}) = \sqrt{(k-1)(n-k+1)}.$$

If u = w, then  $x_u = 1 \ge \frac{1}{\rho}$ . So we next suppose that  $u \ne w$ . We consider the following two cases.

**Case 1.** u is adjacent to w. By eigenequation of A(G) on u,

$$\rho x_u = \sum_{uv \in E(G)} x_v \ge x_w = 1,$$

which implies that

$$x_u \ge \frac{1}{\rho}.$$

**Case 2.** u is not adjacent to w. Let  $G_1$  be a graph obtained from G by deleting all edges incident with u and adding an edge uw. Note that uw is a pendant edge in  $G_1$ .

Claim.  $G_1$  is also F-free.

Suppose that  $G_1$  contains an F as a subgraph. Since G is F-free and  $G_1$  contains an F as a subgraph, we have  $uw \in E(F)$ . Since uw is a pendant edge in  $G_1$ , w is a center of F with  $d_F(w) = d_j$ , where  $1 \leq j \leq k$ . Let  $G_2$  be the subgraph of  $G_1$  by deleting w and all its neighbors in F. Note that  $G_2$  is also a subgraph of G. Since  $G_1$  contains an F as a subgraph,  $G_2$  contains  $\bigcup_{i \neq j} S_{d_i}$  as a subgraph. By eigenequation of G on w,

$$d(w) \ge \sum_{vw \in E(G)} x_v = \rho x_w = \rho \ge \sqrt{(k-1)(n-k+1)} \ge d_1 \ge d_j.$$

This implies that G contains an F as a subgraph, a contradiction.

By Claim,  $G_1$  is *F*-free. Then

$$0 \geq \rho(G_1) - \rho \geq \frac{\mathbf{x}^T A(G_1) \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^T A(G) \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$
$$= \frac{2}{\mathbf{x}^T \mathbf{x}} \Big( x_u x_w - x_u \sum_{uv \in E(G)} x_v \Big)$$
$$= \frac{2x_u}{\mathbf{x}^T \mathbf{x}} \Big( 1 - \rho x_u \Big),$$

which implies that

This completes the proof.

**Lemma 2.3** Let  $d \ge 1$ ,  $k \ge 1$ ,  $n \ge \frac{(d-1)^2 + (k-1)^2}{k-1}$ , and H be a graph of order n - k + 1. If  $G = K_{k-1} \nabla H$  and  $\Delta(H) \le d-1$ , then

 $x_u \ge \frac{1}{\rho}.$ 

$$\rho(G) \le \frac{k+d-3+\sqrt{(k-d-1)^2+4(k-1)(n-k+1)}}{2}$$

with equality if and only if H is a (d-1)-regular graph.

**Proof.** If d = 1, then  $G = K_{k-1} \nabla \overline{K}_{n-k+1}$ . it is easy to calculate that

$$\rho(K_{k-1}\nabla \overline{K}_{n-k+1}) = \frac{k-2+\sqrt{(k-2)^2+4(k-1)(n-k+1)}}{2}.$$

Next suppose that  $d \ge 2$ . Let  $u_1, u_2, \dots, u_{k-1}$  be the vertex of G corresponding to  $K_{k-1}$  in the representation  $G := K_{k-1} \nabla H$ . Set for short  $\rho = \rho(G)$  and let  $\mathbf{x} = (x_v)_{v \in E(G)}$  be a positive eigenvector of  $\rho$ . By symmetry,  $x_{u_1} = \dots = x_{u_{k-1}}$ . Choose a vertex  $v \in V(H)$  such that

$$x_v = \max_{w \in V(H)} x_w.$$

By eigenequation of A(G) on  $u_1$  and v, we have

$$\rho x_{u_1} = (k-2)x_{u_1} + \sum_{uu_1 \in V(H)} x_u \le (k-2)x_{u_1} + (n-k+1)x_v \tag{1}$$

$$\rho x_v \le (k-1)x_{u_1} + \sum_{uv \in E(H)} x_u \le (k-1)x_{u_1} + (d-1)x_v, \tag{2}$$

which implies that

$$(\rho - k + 2)x_{u_1} \leq (n - k + 1)x_v$$
  
 $(\rho - d + 1)x_v \leq (k - 1)x_{u_1}.$ 

Since

$$\rho > \rho(K_{k-1}) = k - 2,$$

and

$$\rho > \rho(K_{k-1,n-k+1}) = \sqrt{(k-1)(n-k+1)} \ge d-1,$$

we have

$$\rho^2 - (k+d-3)\rho + (k-2)(d-1) - (k-1)(n-k+1) \le 0$$

Hence

$$\rho \le \frac{k+d-3+\sqrt{(k-d-1)^2+4(k-1)(n-k+1)}}{2}.$$

If equality holds, then all equalities in (1) and (2) hold. So d(v) = k + d - 2 and  $x_u = x_v$  for any vertex  $u \in V(H)$ . Since for any  $u \in V(H)$ ,

$$\rho x_u = (k-1)x_{u_1} + \sum_{uz \in E(H)} x_z \le (k-1)x_{u_1} + (d-1)x_v = \rho x_v$$

we have d(u) = d(v) = d + k - 2. So H is (d - 1)-regular.

# 3 Proof of Theorem 1.7

Before proving Theorem 1.7, we first prove the following important result for connected graphs without a star forest.

**Theorem 3.1** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G be an *F*-free connected graph of order  $n \ge (\sum_{i=1}^{k} 2d_i + 5k - 7)^2 (\sum_{i=1}^{k} d_i + k - 2)^2$ , then

$$\rho(G) \le \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}$$

with equality if and only if  $G = K_{k-1}\nabla H$ , where H is a  $(d_k - 1)$ -regular graph of order n - k + 1. In particular, if  $d_k = 2$ , then

$$\rho(G) \le \rho(F_{n,k})$$

with equality if and only if  $G = F_{n,k}$ .

**Proof.** Let G be an F-free connected graph of order n with the maximum spectral radius. Set for short V = V(G), E = E(G), A = A(G), and  $\rho = \rho(G)$ . Let  $\mathbf{x} = (x_v)_{v \in V(G)}$  be a positive eigenvector of  $\rho$  such that

$$x_w = \max\{x_u : u \in V(G)\} = 1.$$

Since  $K_{k-1,n-k+1}$  is F-free, we have

$$\rho \ge \rho(K_{k-1,n-k+1}) = \sqrt{(k-1)(n-k+1)}.$$

Let  $L = \{v \in V : x_v > \epsilon\}$  and  $S = \{v \in V : x_v \le \epsilon\}$ , where  $\epsilon = \frac{1}{\sum_{i=1}^k 2d_i + 5k - 7}$ .

**Claim.** |L| = k - 1.

If  $|L| \neq k-1$ , then  $|L| \geq k$  or  $|L| \leq k-2$ .

First suppose that  $|L| \ge k$ . By eigenequation of A on any vertex  $u \in L$ , we have

$$\sum_{i=1}^{k} d_i + k - 2 \le \frac{\sqrt{(k-1)(n-k+1)}}{\sum_{i=1}^{k} 2d_i + 5k - 7} = \sqrt{(k-1)(n-k+1)}\epsilon < \rho x_u = \sum_{uv \in E} x_v \le d(u),$$

where the first inequality holds because  $n \ge (\sum_{i=1}^{k} 2d_i + 5k - 7)^2 (\sum_{i=1}^{k} d_i + k - 2)^2$ . Hence

$$d(u) \ge \sum_{i=1}^{k} d_i + k - 1.$$

Then we can embed an F with all centers in L in G, a contradiction.

Next suppose that  $|L| \le k - 2$ . Then

$$e(L) \le \binom{|L|}{2} \le \frac{1}{2}(k-2)(k-3)$$

and

$$e(L,S) \le (k-2)(n-k+2).$$

In addition, by Lemma 2.1,

$$e(S) \le e(G) \le \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n.$$

By eigenequation of  $A^2$  on w, we have

$$\begin{aligned} (k-1)(n-k+1) &\leq \rho^2 = \rho^2 x_w = \sum_{vw \in E} \sum_{uv \in E} x_u \leq \sum_{uv \in E} (x_u + x_v) \\ &= \sum_{uv \in E(L,S)} (x_u + x_v) + \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v) \\ &\leq \sum_{uv \in E(L,S)} (x_u + x_v) + 2\epsilon e(S) + 2e(L) \\ &\leq \sum_{uv \in E(L,S)} (x_u + x_v) + 2\epsilon \left(\sum_{i=1}^k d_i + 2k - 3\right) n + (k-2)(k-3) \end{aligned}$$

Hence

$$\sum_{uv \in E(L,S)} (x_u + x_v) \ge (k-1)(n-k+1) - 2\epsilon \left(\sum_{i=1}^k d_i + 2k - 3\right)n - (k-2)(k-3).$$

On the other hand, by the definition of L and S, we have

$$\sum_{uv \in E(L,S)} (x_u + x_v) \le (1+\epsilon)e(L,S) \le (1+\epsilon)(k-2)(n-k+2).$$

Thus

$$(1+\epsilon)(k-2)(n-k+2) \ge (k-1)(n-k+1) - 2\epsilon \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n - (k-2)(k-3),$$

which implies that

$$\left(\left(\sum_{i=1}^{k} 2d_i + 5k - 8\right)\epsilon - 1\right)n \ge \epsilon(k-2)^2 - (k^2 - 3k + 3).$$

Since  $\epsilon = \frac{1}{\sum_{i=1}^{k} 2d_i + 5k - 7}$ , we have

$$n \leq (k^{2} - 3k + 3) \left( \sum_{i=1}^{k} 2d_{i} + 5k - 8 \right) \left( \sum_{i=1}^{k} 2d_{i} + 5k - 7 \right) - (k - 2)^{2} \left( \sum_{i=1}^{k} 2d_{i} + 5k - 8 \right)$$
$$\leq \left( \sum_{i=1}^{k} 2d_{i} + 5k - 7 \right)^{2} \left( \sum_{i=1}^{k} d_{i} + k - 2 \right)^{2},$$

a contradiction. This proves the Claim.

By Claim, |L| = k-1 and thus |S| = n-k+1. Then the subgraph H induced by S in G is  $S_{d_k}$ -free. Otherwise, we can embed F in G with k-1 centers in L and a center in S as  $d(u) \ge \sum_{i=1}^k d_i + k - 1$  for any  $u \in L$ , a contradiction. Now  $\Delta(H) \le d_k - 1$ . Note that the resulting graph obtained from G by adding all edges in L and all edges with one end in L and the other in S are also F-free and its spectral radius increases strictly. By the extremality of G, we have  $G = K_{k-1}\nabla H$ . By Lemma 2.3 and the extremality of G, it follows that H is a  $(d_k - 1)$ -regular graph and

$$\rho = \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}$$

In particular, if  $d_k = 2$  then  $\Delta(H) \leq 1$ , i.e.,  $H = pK_2 \cup qK_1$ , where 2p + q = n - k + 1. By the extremality of  $G, G = F_{n,k}$ . This completes the proof.

**Proof of Theorems 1.7.** Let G be an F-free graph of order n with the maximum spectral radius.

If G is connected, then the result follows directly from Theorem 3.1. Next we suppose that G is not connected. Since  $K_{k-1,n-k+1}$  is F-free, we have

$$\rho(G) \ge \sqrt{(k-1)(n-k+1)}.$$

Let  $G_1$  be a component of G such that  $\rho(G_1) = \rho(G)$  and  $n_1 = |V(G_1)|$ . Then

$$n_1 - 1 \ge \rho(G_1) = \rho(G) \ge \sqrt{(k-1)(n-k+1)} \ge \sqrt{(k-2)n}$$
$$\ge \left(\sum_{i=1}^k 2d_i + 5k - 8\right)^2 \left(\sum_{i=1}^k d_i + k - 2\right)^2,$$

which implies that

$$n_1 \geq \left(\sum_{i=1}^k 2d_i + 5k - 8\right)^2 \left(\sum_{i=1}^k d_i + k - 2\right)^2 + 1.$$

By Theorem 3.1 again,

$$\rho(G) = \rho(G_1) \leq \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n_1 - k + 1)}}{2} \\ < \frac{k + d_k - 3 + \sqrt{(k - d_k - 1)^2 + 4(k - 1)(n - k + 1)}}{2}.$$

In particular, if  $d_k = 2$  then it follows from By Theorem 3.1 again,

$$\rho(G) = \rho(G_1) = \rho(F_{n_1,k}) < \rho(F_{n,k}).$$

Hence the result follows.

Note that extremal graph in Theorem 1.4 (iii) also holds for signless Laplacian special radius q(G) [5]. We conjecture the extremal graph in Theorem 1.7 also holds for signless Laplacian spectral radius q(G).

**Conjecture 3.2** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G be an F-free graph of large order n, then

$$q(G) \le \frac{n+2k+2d_k-6+\sqrt{(n+2k-2d_k-2)^2-8(k-1)(k-d_k-1)}}{2}$$

with equality if and only if  $G = K_{k-1}\nabla H$ , where H is a  $(d_k - 1)$ -regular graph of order n - k + 1. In particular, if  $d_k = 2$ , then

$$q(G) \le q(F_{n,k})$$

with equality if and only if  $G = F_{n,k}$ .

# 4 Proofs of Theorem 1.8 and Corollary 1.9

Before proving Theorem 1.8 and Corollary 1.9, we first prove the following important result for bipartite connected graphs without a star forest.

**Theorem 4.1** Let  $F = \bigcup_{i=1}^{k} S_{d_i}$  be a star forest with  $k \ge 2$  and  $d_1 \ge \cdots \ge d_k \ge 1$ . If G is an F-free connected bipartite graph of order  $n \ge f(k, d_1, \ldots, d_k)$ , then

$$\rho(G) \le \sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

**Proof.** Let G be an F-free connected bipartite graph of order n with the maximum spectral radius. Set for short V = V(G), E = E(G), A = A(G), and  $\rho = \rho(G)$ . Let  $\mathbf{x} = (x_v)_{v \in V(G)}$  be a positive eigenvector of  $\rho$  such that

$$x_w = \max\{x_u : u \in V(G)\} = 1.$$

Since  $K_{k-1,n-k+1}$  is *F*-free, we have

$$\rho \ge \rho(K_{k-1,n-k+1}) = \sqrt{(k-1)(n-k+1)}.$$
(3)

Let  $L = \{v \in V : x_v > \epsilon\}$  and  $S = \{v \in V : x_v \le \epsilon\}$ , where

$$\frac{\sum_{i=1}^{k} d_i + k - 2}{\sqrt{(k-1)(n-k+1)}} \le \epsilon \le \frac{1}{k \sum_{i=1}^{k} (2d_i + 5k - 4)^{2k-1}} \left(1 - \frac{\sum_{i=1}^{k} d_i}{n}\right)$$

**Claim 1.**  $|L| \le k - 1$ .

Suppose that  $|L| \ge k$ . By eigenequation of A on any vertex  $u \in L$ , we have

$$\sum_{i=1}^{k} d_i + k - 2 \le \sqrt{(k-1)(n-k+1)}\epsilon < \rho x_u = \sum_{uv \in E} x_v \le d(u).$$

Hence

$$d(u) \ge \sum_{i=1}^{k} d_i + k - 1.$$

Then we can embed an F in G with all centers in L, a contradiction. This proves Claim 1. Since  $|L| \le k - 1$ , we have

$$e(L) \le \binom{|L|}{2} \le \frac{1}{2}(k-1)(k-2)$$

and

$$e(L, S) \le (k-1)(n-k+1).$$

In addition, by Lemma 2.1,

$$e(S) \le e(G) \le \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n.$$

We next show that for any vertex in L has large degree. Claim 2. Let  $u \in L$  and  $x_u = 1 - \delta$ . Then

$$d(u) \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)(\delta + \epsilon)\right)n.$$

Let  $B_u = \{v \in V : uv \notin E\}$ . We first sum of eigenvector over all vertices of G.

$$\begin{split} \rho \sum_{v \in V} x_v &= \sum_{v \in V} \rho x_v = \sum_{v \in V} \sum_{vz \in E} x_z = \sum_{v \in V} d(v) x_v \le \sum_{v \in L} d(v) x_v + \sum_{v \in S} d(v) x_v \\ &\le \sum_{v \in L} d(v) + \epsilon \sum_{v \in S} d(v) = 2e(L) + e(L,S) + \epsilon(2e(S) + e(L,S)) \\ &= 2e(L) + 2\epsilon e(S) + (1 + \epsilon)e(L,S), \end{split}$$

which implies that

$$\sum_{v \in V} x_v \le \frac{2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S)}{\rho}.$$
(4)

Next we sum of eigenvector over all vertices in  $B_u$  by E.q. (4) and Lemma 2.2. Since

$$\begin{aligned} \frac{1}{\rho}|B_u| &\leq \sum_{v \in B_u} x_v \leq \sum_{v \in V(G)} x_v - \sum_{uv \in E(G)} x_v = \sum_{v \in V(G)} x_v - \rho x_u \\ &\leq \frac{2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S)}{\rho} - \rho x_u, \end{aligned}$$

we have

$$|B_u| \leq 2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S) - \rho^2 x_u$$

$$\leq 2e(L) + 2\epsilon e(S) + (1+\epsilon)e(L,S) - (k-1)(n-k+1)(1-\delta) \leq (k-1)(k-2) + 2\epsilon \left(\sum_{i=1}^{k} d_i + 2k - 3\right)n + (1+\epsilon)(k-1)(n-k+1) - (k-1)(n-k+1)(1-\delta) = \left(2\epsilon \left(\sum_{i=1}^{k} d_i + 2k - 3\right) + (\delta+\epsilon)(k-1)\right)n + (k-1)(k-2) - (\delta+\epsilon)(k-1)^2 \leq \left(\sum_{i=1}^{k} 2d_i + 4k - 6 + (k-1) + 1\right)(\delta+\epsilon)n = \left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)(\delta+\epsilon)n,$$

where the last second inequality holds since  $(k-1)(k-2) \le \epsilon n < (\delta + \epsilon)n$  by the definition of  $\epsilon$  and n. Hence

$$d(u) \ge n - 1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)(\delta + \epsilon)n \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)(\delta + \epsilon)\right)n$$

This completes Claim 2.

**Claim 3.** Let  $1 \le s < k - 1$ . Suppose that there is a set X of s vertices such that  $X = \{v \in V : x_v \ge 1 - \eta \text{ and } d(v) \ge (1 - \eta)n\}$ . Then there exists a vertex  $u \in L \setminus X$  such that

$$x_u \ge 1 - \left(\sum_{i=1}^k 2d_i + 5k - 5\right)^2 (\eta + \epsilon)$$

and

$$d(u) \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)^2 (\eta + \epsilon)\right)n.$$

By eigenequation of  $A^2$  on w, we have

$$\begin{split} \rho^2 &= \rho^2 x_w = \sum_{vw \in E} \sum_{uv \in E} x_u \leq \sum_{uv \in E} (x_u + x_v) \\ &= \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v) \sum_{uv \in E(L,S)} (x_u + x_v) \\ &\leq 2\epsilon e(S) + 2e(L) + \sum_{uv \in E(L,S)} (x_u + x_v) \\ &\leq 2\epsilon e(S) + 2e(L) + \epsilon e(L,S) + \sum_{\substack{uv \in E(L \setminus X,S) \\ u \in L \setminus X}} x_u + \sum_{\substack{uv \in E(L \cap X,S) \\ u \in L \cap X}} x_u, \end{split}$$

which implies that

$$\sum_{\substack{uv \in E(L \setminus X,S) \\ u \in L \setminus X}} x_u$$

$$\geq \rho^{2} - 2\epsilon e(S) - 2e(L) - \epsilon e(L, S) - \sum_{\substack{uv \in E(L \cap X, S) \\ u \in L \cap X}} x_{u}$$

$$\geq (k-1)(n-k+1) - 2\epsilon \left(\sum_{i=1}^{k} d_{i} + 2k - 3\right)n - (k-1)(k-2) - \epsilon(k-1)(n-k+1) - sn$$

$$= \left(k-1-s - \epsilon \left(\sum_{i=1}^{k} 2d_{i} + 5k - 7\right)\right)n - (k-1)(2k-3) + \epsilon(k-1)^{2}$$

$$\geq \left(k-1-s - \epsilon \left(\sum_{i=1}^{k} 2d_{i} + 5k - 7\right)\right)n - \epsilon n$$

$$= \left(k-1-s - \epsilon \left(\sum_{i=1}^{k} 2d_{i} + 5k - 6\right)\right)n,$$

where the last third inequality holds since  $(k-1)(2k-3) \leq \epsilon n$  by the definition of  $\epsilon$  and n. In addition,

$$\begin{aligned} e(L \setminus X, S) &= e(L, S) - e(L \cap X, S) \\ &\leq (k - 1)(n - k + 1) - s(1 - \eta)n + \binom{s}{2} \\ &\leq (k - 1 - s(1 - \eta))n - \left((k - 1)^2 - \binom{k - 2}{2}\right) \\ &\leq (k - 1 - s(1 - \eta))n. \end{aligned}$$

Let

$$g(s) = \frac{k - 1 - s - \epsilon \left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)}{k - 1 - s(1 - \eta)}.$$

It is easy to see that g(s) is decreasing with respect to  $1 \le s \le k-2$ . Then

$$\frac{\sum_{\substack{u \in L \setminus X, S \\ u \in L \setminus X}} x_u}{e(L \setminus X, S)} \geq g(s) \geq g(k-2) = \frac{1 - \epsilon \left(\sum_{i=1}^k 2d_i + 5k - 6\right)}{1 + (k-2)\eta}$$
$$\geq 1 - \left(\sum_{i=1}^k 2d_i + 5k - 6\right)(\eta + \epsilon).$$

Hence there exists a vertex  $u \in L \backslash X$  such that

$$x_u \ge 1 - \left(\sum_{i=1}^k 2d_i + 5k - 6\right)(\eta + \epsilon).$$

By Claim 2,

$$d(u) \geq \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right) \left(\left(\sum_{i=1}^{k} 2d_i + 5k - 6\right)(\eta + \epsilon) + \epsilon\right)\right) n$$
  
$$\geq \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)^2(\eta + \epsilon)\right) n$$

This completes Claim 3.

**Claim 4.** |L| = k - 1. Furthermore, for all  $u \in L$ ,

$$x_u \ge 1 - \left(\sum_{i=1}^k 2d_i + 5k - 4\right)^{2k-1} \epsilon$$

and

$$d(u) \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 4\right)^{2k-1} \epsilon\right) n.$$

Note that  $w \in L$  and  $x_w = 1$ . By Claim 2,

$$d(w) \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 5\right)\epsilon\right)n.$$

Applying Claim 5 iteratively for k-2 times, we can find a set  $X \subseteq L \setminus \{w\}$  of k-2 vertices such that for any  $u \in X$ ,

$$x_{u} \geq 1 - \left(\sum_{j=1}^{k-2} \left(\sum_{i=1}^{k} 2d_{i} + 5k - 5\right)^{2j} + \left(\sum_{i=1}^{k} 2d_{i} + 5k - 5\right)^{2k-2} \left(\sum_{i=1}^{k} 2d_{i} + 5k - 4\right)\right) \epsilon$$
$$\geq 1 - \left(\sum_{i=1}^{k} 2d_{i} + 5k - 4\right)^{2k-1} \epsilon$$

and

$$d(u) \ge \left(1 - \left(\sum_{i=1}^{k} 2d_i + 5k - 4\right)^{2k-1} \epsilon\right) n.$$

Noting  $|L| \leq k - 1$ , we have  $L = X \cup \{w\}$ . Hence |L| = k - 1. This proves Claim 4.

Let T be the common neighborhood of L and  $R = S \setminus T$ . By Claim 4,

|L| = k - 1

and

$$|T| \ge \left(1 - k \left(\sum_{i=1}^{k} 2d_i + 5k - 4\right)^{2k-1} \epsilon\right) n \ge \sum_{i=1}^{k} d_i.$$

Since G is bipartite, L and T are both independent sets of G.

Claim 5. R is empty.

Suppose that R is not empty, i.e., there is a vertex  $v \in R$ . Then v has at most  $d_k - 1$  neighbors in S, otherwise we can embed an F in G. Let H be a graph obtained from G by removing all edges incident with v and then connecting v to each vertex in L. Clearly, H is still F-free. By the definition of R, v can be adjacent to at most k - 2 vertices in L. Let  $u \in L$  be the vertex not adjacent to v. Then By Claims 4 and 5, we have

$$\rho(H) - \rho \geq \frac{\mathbf{x}^T A(H) \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

$$\geq \frac{2x_v}{\mathbf{x}^T \mathbf{x}} \left( x_u - \sum_{\substack{uz \in E \\ z \in S}} x_z \right)$$
  
$$\geq \frac{2x_v}{\mathbf{x}^T \mathbf{x}} \left( 1 - \left( \sum_{i=1}^k 2d_i + 5k - 4 \right)^{2k-1} \epsilon - (d_k - 1)\epsilon \right)$$
  
$$= \frac{2x_v}{\mathbf{x}^T \mathbf{x}} \left( 1 - \left( \left( \sum_{i=1}^k 2d_i + 5k - 4 \right)^{2k-1} + d_k - 1 \right) \epsilon \right)$$
  
$$> 0,$$

Hence  $\rho(H) > \rho$ , a contradiction. This proves Claim 5.

By Claim 5, S = T. By the definition of T, we have  $G = K_{k-1,n-k+1}$ . This completes the proof.  $\Box$ 

**Proof of Theorem 1.8.** Let G be an F-free bipartite graph of order n with the maximum spectral radius.

If G is connected, then the result follows directly from Theorem 4.1. Next we suppose that G is not connected. Since  $K_{k-1,n-k+1}$  is F-free,

$$\rho(G) \ge \sqrt{(k-1)(n-k+1)}.$$

Let  $G_1$  be a component of G such that  $\rho(G_1) = \rho(G)$  and  $n_1 = |V(G_1)|$ . Note that G is triangle-free. By Wilf theorem [15, Theorem 2], we have

$$\frac{n_1^2}{4} \ge \rho^2(G_1) = \rho(G)^2 \ge (k-1)(n-k+1) \ge (k-2)n \ge \frac{f^2(k, d_1, \dots, d_k)}{4},$$

which implies that

$$n_1 \ge f(k, d_1, \dots, d_k).$$

By Theorem 4.1 again,

$$\rho(G_1) \leq \sqrt{(k-1)(n_1-k+1)} < \sqrt{(k-1)(n-k+1)},$$

a contradiction. This completes the proof.

**Proof of Corollary 1.9.** By a result of Favaron et al. [7],  $\rho_n(G) \ge \rho_n(H)$  for some spanning bipartite subgraph H. Moreover, the equality holds if and only if G = H, which can be deduced by its original proof. By Theorem 1.8,

$$\rho(H) \le \sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $H = K_{k-1,n-k+1}$ . Since the spectrum of a bipartite graph is symmetric [10],

$$\rho_n(H) \ge -\sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $H = K_{k-1,n-k+1}$ . Thus we have

$$\rho_n(G) \ge -\sqrt{(k-1)(n-k+1)}$$

with equality if and only if  $G = K_{k-1,n-k+1}$ .

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