A note on a Caro-Wei bound for the bipartite independence number in graphs

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Abstract

A bi-hole of size t in a bipartite graph G is a copy of $K_{t,t}$ in the bipartite complement of G. Given an $n \times n$ bipartite graph G, let $\beta(G)$ be the largest k for which G has a bi-hole of size k. We prove that

$$\beta(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \frac{1}{d(v)+1} \right\rfloor .$$

Furthermore, we prove the following generalization of the result above. Given an $n \times n$ bipartite graph G, let $\beta_d(G)$ be the largest k for which G has a $k \times k$ induced d-degenerate subgraph. We prove that

$$\beta_d(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \min\left(1, \frac{d+1}{d(v)+1}\right) \right\rfloor .$$

Notice that $\beta_0(G) = \beta(G)$.

1 Introduction

A bi-hole of size t in a bipartite graph G is a copy of $K_{t,t}$ in the bipartite complement of G. Given an $n \times n$ bipartite graph G, let $\beta(G)$ be the largest k for which G has a bi-hole of size k. Denote by $d(v)$ the degree of vertex v in graph G. The following theorem is proven in [EMR20].

Theorem 1.1. *Given an $n \times n$ bipartite graph G with average degree d we have*

$$\beta(G) \geq \frac{n}{d+1} - 2.$$

We generalize and improve the theorem above by proving the following.

Theorem 1.2. *Given an $n \times n$ bipartite graph G we have*

$$\beta(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \frac{1}{d(v)+1} \right\rfloor.$$

Notice that Theorem 1.1 follows from Theorem 1.2 by Jensen's inequality. The result above is somewhat similar to the well known Caro-Wei bound [Car79, Wei81], stated below.

Theorem 1.3. *Let $\alpha(G)$ denote the maximum number of vertices of an independent set of graph G. Then given a graph G, we have the following inequality.*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}.$$

We note that better results than the ones in Theorem 1.2 exist when the average degree is large. For example, in [FK10] the following theorem was proven.

Theorem 1.4. *For any $0 < \epsilon < 1$ there is a constant d_0 such that the following holds. Given an $n \times n$ bipartite graph G with average degree $d \geq d_0$ and $n \geq (1+\epsilon)d$ we have*

$$\beta(G) \geq \frac{\epsilon}{2} \cdot \frac{n \ln d}{d}.$$

Theorems similar to Theorem 1.4 were proven in [ASSW20] and [EMR20].
Now we shall discuss a generalization of Theorem 1.2. A graph H is d-degenerate if every non-empty subgraph of it contains a vertex of degree at most d. Thus 0-degenerate graphs are independent sets and 1-degenerate graphs are forests. Let $\alpha_d(G)$ denote the maximum number of vertices of an induced d-degenerate subgraph of G. The following theorem was proven in [AKS87].

Theorem 1.5. *Let G be a graph. Then*

$$\alpha_d(G) \geq \sum_{v \in V(G)} \min\left(1, \frac{d+1}{d(v)+1}\right).$$

The special case $d = 0$ of Theorem 1.5 is the well known Caro-Wei bound [Car79, Wei81]. We prove an analogous theorem to Theorem 1.5 in bipartite graphs for biholes. Given an $n \times n$ bipartite graph G, let $\beta_d(G)$ be the largest k for which G has an induced $k \times k$ d-degenerate subgraph. We prove the following.

Theorem 1.6. *Given an $n \times n$ bipartite graph G and an integer $d \geq 0$ we have*

$$\beta_d(G) \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in V(G)} \min\left(1, \frac{d+1}{d(v)+1}\right) \right\rfloor .$$

Notice that Theorem 1.2 is the special case of $d = 0$ in Theorem 1.6. We prove Theorem 1.2 and Theorem 1.6 in the next two sections.

2 Proof of Theorem 1.2

We shall prove the following slightly stronger result. Given an $n \times n$ bipartite graph $G = (A, B, E)$ where Δ_A is the maximum degree in G of the vertices of A and Δ_B is the maximum degree in G of the vertices of B, we have

$$\beta(G) \geq \frac{1}{2}\left(\frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} + \sum_{v \in A \cup B} \frac{1}{d(v)+1}\right) - 1.$$

Notice that Theorem 1.2 will follow from the fact that

$$\left\lceil \frac{1}{2}\left(\frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} + \sum_{v \in A \cup B} \frac{1}{d(v)+1}\right) - 1 \right\rceil \geq \left\lfloor \frac{1}{2} \cdot \sum_{v \in A \cup B} \frac{1}{d(v)+1} \right\rfloor$$

As $\lceil x - \xi \rceil \geq \lfloor x \rfloor$ for any $\xi \in [0, 1)$, and in our case $0 \leq \xi = 1 - \frac{1}{2}\left(\frac{1}{\Delta_A+1} + \frac{1}{\Delta_B+1}\right) < 1$ and $x = \frac{1}{2} \cdot \sum_{v \in A \cup B} \frac{1}{d(v)+1}$.
Define the potential function $f(d) = \frac{1}{d+1}$ and let the degree sequence of graph G be $d_1, d_2, \ldots, d_{2n}$, hence we need to prove that $\beta(G) \geq S$ where

$$S = \frac{1}{2}\left(f(\Delta_A) + f(\Delta_B) + \sum_{i=1}^{2n} f(d_i)\right) - 1.$$

We prove this claim by induction on n. The base case $n = 1$ is trivially correct. Furthermore we may assume that $\Delta_A \geq 1$ and $\Delta_B \geq 1$ for otherwise the graph is an independent set and the claim follows once again trivially.

Now we shall consider two cases.

Case 1: There is a vertex $a \in A$ such that $d(a) = \Delta_A$ and a vertex $b \in B$ such that $d(b) = \Delta_B$ and there is no edge between vertices a and b.

Let graph $H(A', B', E')$ be the $(n-1) \times (n-1)$ bipartite graph formed from G by removing vertices a and b, and let $d'_1, d'_2, \ldots, d'_{2n-2}$ be the degree sequence of graph H. Let

$$Q = \frac{1}{2}\left(f(\Delta_{A'}) + f(\Delta_{B'}) + \sum_{i=1}^{2n-2} f(d'_i)\right) - 1,$$

where $\Delta_{A'}$ is the maximum degree in H of the vertices of A' and $\Delta_{B'}$ is the maximum degree in H of the vertices of B'. Now notice that

$$\begin{aligned}
Q &\geq S - \frac{1}{2}\left(f(\Delta_A) + f(\Delta_B) - \Delta_A(f(\Delta_B - 1) - f(\Delta_B)) - \Delta_B(f(\Delta_A - 1) - f(\Delta_A))\right) \\
&= S - \frac{1}{2}\left(\frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} - \frac{\Delta_A}{\Delta_B(\Delta_B + 1)} - \frac{\Delta_B}{\Delta_A(\Delta_A + 1)}\right) \\
&= S + \frac{1}{2}\frac{(\Delta_A - \Delta_B)^2(\Delta_A + \Delta_B + 1)}{\Delta_A(\Delta_A + 1)\Delta_B(\Delta_B + 1)} \\
&\geq S
\end{aligned}$$

And thus we are done by applying the induction hypothesis to graph H.

Case 2: Each vertex $a \in A$ such that $d(a) = \Delta_A$ and each vertex $b \in B$ such that $d(b) = \Delta_B$ are joined by an edge.

Pick an arbitrary vertex $a \in A$ such that $d(a) = \Delta_A$ and an arbitrary vertex $b \in B$ such that $d(b) = \Delta_B$. Note that there is an edge between a and b. Let graph $H(A', B', E')$ be the $(n-1) \times (n-1)$ bipartite graph formed from G by removing vertices a and b, and let $d'_1, d'_2, \ldots, d'_{2n-2}$ be the degree sequence of graph H. Let

$$Q = \frac{1}{2}\left(f(\Delta_{A'}) + f(\Delta_{B'}) + \sum_{i=1}^{2n-2} f(d'_i)\right) - 1,$$

where $\Delta_{A'}$ is the maximum degree in H of the vertices of A' and $\Delta_{B'}$ is the maximum degree in H of the vertices of B'. Notice that by the definition of case 2 we have $\Delta_{A'} \leq \Delta_A - 1$ and

$\Delta_{B'} \leq \Delta_B - 1$. Hence we have

$$\begin{aligned}
Q &\geq S - \frac{1}{2}\left(f(\Delta_A) + f(\Delta_B) - (\Delta_A - 1)(f(\Delta_B - 1) - f(\Delta_B)) - (\Delta_B - 1)(f(\Delta_A - 1) - f(\Delta_A))\right) \\
&\quad + \frac{1}{2}\left((f(\Delta_{A'}) - f(\Delta_A)) + (f(\Delta_{B'}) - f(\Delta_B))\right) \\
&\geq S - \frac{1}{2}\left(\frac{1}{\Delta_A + 1} + \frac{1}{\Delta_B + 1} - \frac{\Delta_A}{\Delta_B(\Delta_B + 1)} - \frac{\Delta_B}{\Delta_A(\Delta_A + 1)}\right) \\
&= S + \frac{1}{2}\frac{(\Delta_A - \Delta_B)^2(\Delta_A + \Delta_B + 1)}{\Delta_A(\Delta_A + 1)\Delta_B(\Delta_B + 1)} \\
&\geq S
\end{aligned}$$

And thus we are done by applying the induction hypothesis to graph H.

3 Proof of Theorem 1.6

We shall prove the following slightly stronger result. Set a fixed integer $d \geq 1$ (the $d = 0$ case is Theorem 1.2). Define the potential function

$$f(x) = \min\left(1, \frac{d+1}{x+1}\right).$$

Let the degree sequence of the $n \times n$ bipartite graph $G = (A, B, E)$ be $d_1, d_2, \ldots, d_{2n}$. Furthermore let Δ_A be the maximum degree in G of the vertices of A and Δ_B be the maximum degree in G of the vertices of B. We claim that $\beta_d(G) \geq S$ where

$$S = \frac{1}{2}\left(f(\Delta_A) + f(\Delta_B) + \sum_{i=1}^{2n} f(d_i)\right) - 1.$$

We prove this claim by induction on n. The base case is $n = 1$ is trivially correct. Now we assume that the claim holds for all bipartite graphs on $(n-1) \times (n-1)$ vertices and prove it for bipartite graphs on $n \times n$ vertices. We do this by induction on the number of edges in the $n \times n$ bipartite graph G. The base case when G has no edges follows trivially from the fact that such a graph is an independent set. Now if G contains a vertex v such that $1 \leq d(v) \leq d$ then we delete the edges incident to v and apply the edge induction hypothesis on the resulting graph G'. Hence G' contains a $k \times k$ d-degenerate subgraph H such that $k \geq S$. If H does not contain vertex v then H is also a $k \times k$ d-degenerate subgraph of G. If H contains vertex v then we add back the edges that are incident to v in G and are inside H. The resulting graph H' is

a d-degenerate subgraph of G since we took a vertex v of degree 0 in a d-degenerate subgraph and added to it at most d edges.
Hence we can assume that the minimum degree of each vertex which is not of degree 0 in G is at least $d+1$ and furthermore $\Delta_A \geq d+1$ and $\Delta_B \geq d+1$. The rest of the proof is identical to the proof of Theorem 1.2 and thus omitted (in particular we do the same case analysis of the two cases in the proof of Theorem 1.2 but with potential function $f(x) = \min\left(1, \frac{d+1}{x+1}\right)$).

4 Concluding remarks

It would be interesting to improve Theorem 1.2 for bipartite graphs without cycles of length 4. Furthermore, it would be interesting to generalize the results for k-partite graphs where $k \geq 3$.

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