THE CYCLIC INDEX OF ADJACENCY TENSOR OF GENERALIZED POWER HYPERGRAPHS

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Abstract. Let G be a t-uniform hypergraph, and let $c(G)$ denote the cyclic index of the adjacency tensor of G. Let m,s,t be positive integers such that $t \geq 2$, $s \geq 2$ and $m = st$. The generalized power $G^{m,s}$ of G is obtained from G by blowing up each vertex into an s-set and preserving the adjacency relation. It was conjectured that $c(G^{m,s}) = s \cdot c(G)$. In this paper we show that the conjecture is false by giving a counterexample, and give some sufficient conditions for the conjecture holding. Finally we give an equivalent characterization of the equality in the conjecture by using a matrix equation over $\mathbb{Z}_m$.

1. Introduction

A hypergraph $G = (V(G), E(G))$ consists of a set of vertices, say $V(G) = \{v_1, v_2, \cdots v_n\}$, and a set of edges, say $E(G) = \{e_1, e_2, \cdots e_k\}$, where $e_j \subseteq V(G)$ for $j \in [k] := \{1, 2, \ldots, k\}$. If $|e_j| = m$ for each $j \in [k]$, then G is called an m-uniform hypergraph. A *walk* W in G is a sequence of alternating vertices and edges: $v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_l}, v_{i_l}$, where $\{v_{i_j}, v_{i_{j+1}}\} \subseteq e_{i_{j+1}}$ for $j = 0, 1, \ldots, l-1$. The hypergraph G is *connected* if every two vertices of G are connected by a walk. The *adjacency tensor* $\mathcal{A}(G)$ of the hypergraph G is defined as $\mathcal{A}(G) = (a_{i_1 i_2 \ldots i_k})$ [4], an m-th order n-dimensional tensor, where

$$a_{i_1 i_2 \ldots i_m} = \begin{cases} \frac{1}{(m-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

In general, A *tensor* (also called *hypermatrix*) $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ of order m and dimension n over a field $\mathbb{F}$ refers to a multiarray of entries $a_{i_1 i_2 \ldots i_m} \in \mathbb{F}$ for all $i_j \in [n]$ and $j \in [m]$, which can be viewed to be the coordinates of the classical tensor (as a multilinear function) under an orthonormal basis. If $m = 2$, then $\mathcal{A}$ is a square matrix of dimension n.

In 2005, independently, Lim [13] and Qi [17] introduced eigenvalues for tensors $\mathcal{A}$. Denote by $\rho(\mathcal{A})$ the spectral radius of $\mathcal{A}$, and by $\mathrm{Spec}(\mathcal{A})$ the spectrum of $\mathcal{A}$. If $\mathcal{A}$ is further nonnegative, then by Perron-Frobenius theorem of nonnegative tensors, $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is weakly irreducible and has k eigenvalues of $\mathcal{A}$ with modulus $\rho(\mathcal{A})$, then those k eigenvalues are equally distributed on the spectral circle. As for nonnegative matrices, the number k is called the cyclic index of $\mathcal{A}$ [2]. The cyclic index reflects the spectral symmetry of

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nonnegative weakly irreducible tensors, which was generalized and investigated in the paper [5].

Definition 1.1 ([5])**.** Let $\mathcal{A}$ be an m-th order n-dimensional tensor, and let ℓ be a positive integer. The tensor $\mathcal{A}$ is called *spectral ℓ-symmetric* if

$$\mathrm{Spec}(\mathcal{A}) = e^{\mathbf{i}\frac{2\pi}{\ell}}\mathrm{Spec}(\mathcal{A}). \tag{1.1}$$

The maximum number ℓ such that (1.1) holds is called the *cyclic index* of $\mathcal{A}$ and denoted by $c(\mathcal{A})$, and $\mathcal{A}$ is called *spectral $c(\mathcal{A})$-cyclic*.

When we say a hypergraph is *spectral ℓ-symmetric* or *spectral ℓ-cyclic*, this is always referring to its adjacency tensor. In particular, for a uniform hypergraph G, denote $c(G) := c(\mathcal{A}(G))$, called the cyclic index of G.

For a general tensor $\mathcal{A}$, if it is spectral ℓ-symmetric, then $\ell | c(\mathcal{A})$ by [5, Lemma 2.7]. It was also proved that if a connected m-uniform hypergraph is spectral ℓ-symmetric, then $\ell | m$, and hence $c(G)|m$; see [5, Lemma 3.2, Corollary 4.3], [6, Lemma 2.5] or [21, Theorem 2.15]. In the paper [5] the authors use the construction of generalized power hypergraphs to show that for every positive integer m and any positive integer ℓ such that $\ell | m$, there always exists an m-uniform hypergraph G such that G is spectral ℓ-symmetric. They posed the following conjecture.

Conjecture 1.2 ([5])**.** let G be a t-uniform hypergraph, and let $G^{m,s}$ be the generalized power of G, where $m = st$. Then

$$c(G^{m,s}) = s \cdot c(G). \tag{1.2}$$

The generalized power of a hypergraph is defined as follows.

Definition 1.3 ([10])**.** Let $G = (V, E)$ be a t-uniform hypergraph. For any integers m, s such that $m > t$ and $1 \leq s \leq \frac{m}{t}$, the generalized power of G, denoted by $G^{m,s}$, is defined as the m-uniform hypergraph with the vertex set $(\cup_{v \in V}\mathbf{v}) \cup (\cup_{e \in E}\mathbf{e})$, and the edge set $\{\mathbf{u}_1 \cup \cdots \cup \mathbf{u}_t \cup \mathbf{e} : e = \{u_1, \ldots, u_t\} \in E(G)\}$, where $\mathbf{v}$ denotes an s-set corresponding to v and $\mathbf{e}$ denotes an $(m - ts)$-set corresponding to e, and all those sets are pairwise disjoint.

In this paper, we only consider the power hypergraphs $G^{m,s}$ with $m = st$, i.e. $G^{m,s}$ is obtained from G by blowing up each vertex into an s-set and preserving the adjacency relation. The generalized power hypergraphs include some special cases, such as the powers of simple graphs introduced by Hu, Qi and Shao [9], the generalized powers of simple graphs introduced by Khan and Fan [11]. Peng [16] introduced s-paths and s-cycles with uniformity m on discussing the Ramsey number, which are exactly the generalized pows of paths and cycles (as simple graphs) respectively if $1 \leq s \leq \frac{m}{2}$. The spectral results on generalized power hypergraphs can be found in [9, 24, 11, 23, 12, 10].

For the conjecture 1.2, it was shown that it is true if $c(G) = 1$ [5]. In this paper we show that the conjecture is false by giving a counterexample, and give some sufficient conditions for the conjecture holding. We finally give an equivalent characterization of Eq. (1.2) by using a matrix equation over $\mathbb{Z}_m$.

2. Preliminaries

2.1. **Notions.** Let $\mathcal{A}$ be a real tensor of order m and dimension n. The tensor $\mathcal{A}$ is called *symmetric* if its entries are invariant under any permutation of their indices. So, the adjacency tensor of a uniform hypergraph is symmetric.

Given a vector $x \in \mathbb{C}^n$, $\mathcal{A}x^m \in \mathbb{C}$ and $\mathcal{A}x^{m-1} \in \mathbb{C}^n$, which are defined as follows:

$$\mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in [n]} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$
$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in [n]} a_{i i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, i \in [n].$$

Let $\mathcal{I} = (i_{i_1 i_2 \ldots i_m})$ be the *identity tensor* of order m and dimension n, that is, $i_{i_1 i_2 \ldots i_m} = 1$ if and only if $i_1 = i_2 = \cdots = i_m \in [n]$ and $i_{i_1 i_2 \ldots i_m} = 0$ otherwise.

Definition 2.1 ([13, 17])**.** Let $\mathcal{A}$ be an m-th order n-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{A})x^{m-1} = 0$, or equivalently $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$, has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then λ is called an eigenvalue of $\mathcal{A}$ and x is an eigenvector of $\mathcal{A}$ associated with λ, where $x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})$.

The *determinant* of $\mathcal{A}$, denoted by $\det \mathcal{A}$, is defined as the resultant of the polynomials $\mathcal{A}\mathbf{x}^{m-1}$ [8], and the *characteristic polynomial* $\varphi_{\mathcal{A}}(\lambda)$ of $\mathcal{A}$ is defined as $\det(\lambda\mathcal{I} - \mathcal{A})$ [17, 3]. It is known that λ is an eigenvalue of $\mathcal{A}$ if and only if it is a root of $\varphi_{\mathcal{A}}(\lambda)$. The *spectrum* of $\mathcal{A}$ is the multi-set of the roots of $\varphi_{\mathcal{A}}(\lambda)$.

The spectral symmetry of a connected hypergraph is closed related to a certain coloring of the hypergraph.

Definition 2.2 ([5])**.** Let $m \geq 2$ and $\ell \geq 2$ be integers such that $\ell \mid m$. An m-uniform hypergraph G on n vertices is called (m, ℓ)-colorable if there exists a map $\phi : [n] \to [m]$ such that if $\{i_1, \ldots, i_m\} \in E(G)$, then

$$\phi(i_1) + \cdots + \phi(i_m) \equiv \frac{m}{\ell} \mod m. \tag{2.1}$$

Such ϕ is called an (m, ℓ)-coloring of G.

If m is even, an m-uniform hypergraph with an $(m, 2)$-coloring was called *odd-colorable* by Nikiforov [14].

Theorem 2.3. [5] *Let G be a connected m-uniform hypergraph. Then G is spectral ℓ-symmetric if and only if G is (m, ℓ)-colorable.*

The *edge-vertex incidence matrix* $B_G = (b_{ev})$ of an m-uniform hypergraph G is defined by

$$b_{ev} = \begin{cases} 1, & \text{if } \; v \in e \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

We may view B_G as one over $\mathbb{Z}_m$, where $\mathbb{Z}_m$ is the ring of integers modulo m. Now Eq. (2.1) is equivalent to

$$B_G \phi = \frac{m}{\ell}\mathbf{1} \text{ over } \mathbb{Z}_m, \tag{2.2}$$

where $\phi = (\phi(1), \ldots, \phi(n))$ is considered as a column vector, and $\mathbf{1}$ is an all-ones vector of dimension n. So, Theorem 2.3 can be rewritten as follows.

Corollary 2.4. Let G be a connected m-uniform hypergraph. Then G is spectral ℓ-symmetric if and only if the equation

$$B_G x = \frac{m}{\ell}\mathbf{1} \text{ over } \mathbb{Z}_m \tag{2.3}$$

has a solution.

In Corollary 2.4 and other places of the paper, the number of coordinates of $\mathbf{1}$ is implicated from context, which is equal to the number of vertices of the hypergraph under discussion.

3. Cyclic index of generalized power hypergraphs

Let G be a t-uniform hypergraph, and let $G^{m,s}$ be a generalized power hypergraph of G, where $1 \le s \le \frac{m}{t}$. If $m > st$, then each edge of G contains a vertex of degree 1, and hence G is a 1-hm bipartite hypergraph [19]. By [19, Theorem 3.2] or [5, Theorem 4.5], $c(G^{m,s}) = m$.

So, in the following, we always assume that G is a connected t-uniform hypergraph, $m = st$, namely, $G^{m,s}$ is considered to be obtained from G by blowing each vertex v into an s-set $\mathbf{v}$ and preserving the adjacency relation. We also assume that the vertex v is contained in $\mathbf{v}$ for each $v \in V(G)$.

Lemma 3.1. *If G is spectral ℓ-symmetric, then $G^{m,s}$ is also spectral ℓ-symmetric. In particular, $G^{m,s}$ is spectral $c(G)$-symmetric and hence $c(G)|c(G^{m,s})$.*

Proof. Suppose that G is spectral ℓ-symmetric. By Corollary 2.4, the equation $B_G x = \frac{t}{\ell}\mathbf{1}$ has a solution ϕ over $\mathbb{Z}_t$. Now define a map Φ on $G^{m,s}$ such that $\Phi|_{\mathbf{v}} = \phi(v)$ for each vertex $v \in V(G)$. Then

$$B_{G^{m,s}}\Phi = s \cdot B_G\phi = \frac{st}{\ell}\mathbf{1} = \frac{m}{\ell}\mathbf{1} \text{ over } \mathbb{Z}_m,$$

which implies that $G^{m,s}$ is spectral ℓ-symmetric also by Corollary 2.4. $\square$

Lemma 3.2. *$G^{m,s}$ is spectral s-symmetric.*

Proof. For each vertex $v \in V(G)$, v is blowing into an s-set $\mathbf{v}$ of vertices of $G^{m,s}$, and is assumed to be contained in $\mathbf{v}$. Define a map Φ on $G^{m,s}$ such that $\Phi(v) = 1$ and $\Phi|_{\mathbf{v}\setminus\{v\}} = 0$ for each vertex $v \in V(G)$. Then

$$B_{G^{m,s}}\Phi = t\mathbf{1} = \frac{m}{s}\mathbf{1} \text{ over } \mathbb{Z}_m,$$

which implies that $G^{m,s}$ is spectral s-symmetric by Corollary 2.4. $\square$

Lemma 3.3. *If $G^{m,s}$ is spectral $s \cdot \ell'$-symmetric, then G is spectral ℓ'-symmetric.*

Proof. By Corollary 2.4, there exists a map Φ defined on $G^{m,s}$ such that

$$B_{G^{m,s}}\Phi = \frac{m}{s \cdot \ell'}\mathbf{1} = \frac{t}{\ell'}\mathbf{1} \text{ over } \mathbb{Z}_m.$$

Now define a map ϕ on G such that $\phi(v) = \sum_{u \in \mathbf{v}} \Phi(u)$ for each $v \in V(G)$. So we have

$$B_G\phi = B_{G^{m,s}}\Phi = \frac{t}{\ell'}\mathbf{1} \text{ over } \mathbb{Z}_m.$$

As m is a multiple of t,

$$B_G\phi = \frac{t}{\ell'}\mathbf{1} \text{ over } \mathbb{Z}_t,$$

which implies that G is spectral ℓ'-symmetric by Corollary 2.4. $\square$

By Lemma 3.2, we may assume $c(G^{m,s}) = s \cdot \ell'$, where ℓ' is a positive integer. By Lemma 3.3, we know that G is spectral ℓ'-symmetric and hence $\ell'|c(G)$ by [5, Lemma 2.7]. So we get the following result immediately.

Corollary 3.4. $c(G^{m,s})|s \cdot c(G)$.

Corollary 3.5. $G^{m,s}$ is spectral $\frac{s\cdot c(G)}{(s,c(G))}$-symmetric. In particular, if $(s,c(G))=1$ or $(s,t)=1$, then $c(G^{m,s})=s\cdot c(G)$.

Proof. By Lemma 3.1 and Lemma 3.2, we know that $c(G)|c(G^{m,s})$ and $s|c(G^{m,s})$, implying that $\frac{s\cdot c(G)}{(s,c(G))}|c(G^{m,s})$. So, $G^{m,s}$ is spectral $\frac{s\cdot c(G)}{(s,c(G))}$-symmetric. As $c(G)|t$, if $(s,t)=1$, then $(s,c(G))=1$. If $(s,c(G))=1$, then $s\cdot c(G)|c(G^{m,s})$. Then result follows by Corollary 3.4. □

By Corollary 3.5, Conjecture 1.2 holds in some special cases, including the case of $c(G)=1$. However, Conjecture 1.2 does not hold in general. Now we give a counterexample to show the negative answer to the conjecture.

Definition 3.6 ([14])**.** Let $n\geq 16k$ and let partition $[n]$ into three sets A,B,C such that $|A|\geq 6k$, $|B|\geq 6k$ and $|C|\geq 4k$. Define the four families of $4k$-subsets of $[n]$.

$$E_1:=\{e:e\subset[n],|e\cap A|=2k,|e\cap C|=2k\}.$$
$$E_2:=\{e:e\subset[n],|e\cap B|=2k,|e\cap C|=2k\}.$$
$$E_3:=\{e:e\subset[n],|e\cap A|=k,|e\cap B|=3k\}.$$
$$E_4:=\{e:e\subset[n],|e\cap A|=3k,|e\cap B|=k\}.$$

Now define a $4k$-uniform hypergraph G by setting $V(G)=[n]$ and $E(G)=E_1\cup E_2\cup E_3\cup E_4$. We call G a *Nikiforov's hypergraph* as it is introduced by Nikiforov.

Nikiforov [14] showed that Nikiforov's hypergraphs G are odd-colorable, or $(4k,2)$-colorable in terms our definition, by defining a function ϕ on G such that $\phi|_A=1$, $\phi|_B=4k-1$ and $\phi|_C=0$. By Theorem 2.3, G is spectral 2-symmetric.

By the following result, if G is a Nikiforov's hypergraph and s is even, then

$$c(G^{m,s})\neq s\cdot c(G).$$

So we give a negative answer to Conjecture 1.2.

Theorem 3.7. *Let G be a $4k$-uniform Nikiforov's hypergraph. Then the following results hold.*

(1) $c(G)=2$.
(2) *If s is even, then* $c(G^{m,s})=s$.

Proof. (1) We first show that $c(G)=2$. Suppose that G is spectral ℓ-symmetric. Then there exists a $\phi:[n]\to[4k]$ such that $B_G\phi=\frac{4k}{\ell}$ over $\mathbb{Z}_{4k}$. It is easily seen that ϕ is constant on each of A,B,C by the equation. So, let $\phi|_A:=a$, $\phi|_B:=b$ and $\phi|_C:=c$. Then, by considering the edges in E_1, we have

$$2ka+2kc=\frac{4k}{\ell} \mod 4k,$$

which implies that ℓ equals 1 or 2, and hence $c(G)=2$ as G is spectral 2-symmetric.

(2) By Corollary 3.4, $c(G^{m,s})|2s$, where $m=4ks$. By Lemma 3.2, $G^{m,s}$ is spectral s-symmetric, and hence $s|c(G^{m,s})$. We will show that if s is even, then $G^{m,s}$ is not spectral $2s$-symmetric so that $c(G^{m,s})=s$.

Assume to the contrary that $G^{m,s}$ is spectral $2s$-symmetric. Then there exists a $\Phi:V(G^{m,s})\to[4ks]$ such that

$$B_{G^{m,s}}\Phi=\frac{4ks}{2s}=2k \text{ over } \mathbb{Z}_{4ks}.$$

For each $v \in V(G)$, define $\phi(v) := \sum_{u \in \mathbf{v}} \Phi(u)$. So we have

$$B_{G^{m,s}}\Phi = B_G\phi = 2k \text{ over } \mathbb{Z}_{4ks}.$$

It is also easily seen that $\phi|_A := \alpha$, $\phi|_B := \beta$ and $\phi|_C := \iota$. By considering the edges in E_3 and E_4 respectively, we have

$$\alpha + 3\beta = 2 \mod 4s, \; 3\alpha + \beta = 2 \mod 4s.$$

So

$$\alpha - \beta = 0 \mod 2s, \; \alpha + \beta = 1 \mod s,$$

which yields a contradiction as s is an even number. $\square$

Finally we give an equivalent characterization of Eq. (1.2) in Conjecture 1.2.

Theorem 3.8. *$c(G^{m,s}) = s \cdot c(G)$ if and only if the equation*

$$B_G x = \frac{t}{c(G)}\mathbf{1} \text{ over } \mathbb{Z}_m \tag{3.1}$$

has a solution.

Proof. Suppose that $c(G^{m,s}) = s \cdot c(G)$. Then $G^{m,s}$ is spectral $s \cdot c(G)$-symmetric, and by Corollary 2.4, there exists a map $\Phi : V(G^{m,s}) \to [m]$ such that

$$B_{G^{m,s}}\Phi = \frac{m}{s \cdot c(G)}\mathbf{1} = \frac{t}{c(G)}\mathbf{1} \text{ over } \mathbb{Z}_m.$$

For each $v \in V(G)$, define $\phi(v) := \sum_{u \in \mathbf{v}} \Phi(u)$. So we have $B_{G^{m,s}}\Phi = B_G\phi$, and get the necessity.

On the other hand, if $B_G x = \frac{t}{c(G)}\mathbf{1}$ has a solution ϕ over $\mathbb{Z}_m$. Define a map $\Psi : V(G^{m,s}) \to [m]$ such that

$$\sum_{u \in \mathbf{v}} \Phi(u) = \phi(v), \text{ for each } v \in V(G).$$

There are $|V(G)|$ independent linear equations; such Φ is easily got (e.g. for each $v \in V(G)$, take $\Phi(v) = \phi(v)$ and $\Phi(u) = 0$ for each $u \in \mathbf{v}\backslash\{v\}$). So we have

$$B_{G^{m,s}}\Phi = B_G\phi = \frac{t}{c(G)}\mathbf{1} = \frac{m}{s \cdot c(G)}\mathbf{1} \text{ over } \mathbb{Z}_m.$$

So $G^{m,s}$ is spectral $s \cdot c(G)$-symmetric. The sufficiency follows by Corollary 3.4. $\square$

As G is spectral $c(G)$-symmetric, by Corollary 2.4 the equation

$$B_G x = \frac{t}{c(G)}\mathbf{1} \text{ over } \mathbb{Z}_t \tag{3.2}$$

has a solution. Obviously, if the equation (3.1) has a solution, then the equation (3.2) has a solution as m is a multiple of t. However, the converse does not hold in general; see the previous counterexample.

4. Remark

For a nonnegative weakly irreducible tensor $\mathcal{A}$, its cyclic index $c(\mathcal{A})$ is exactly the number of eigenvalues with modulus $\rho(\mathcal{A})$. The is implied by Perron-Frobenius theorem for nonnegative tensors, where an eigenvalue of $\mathcal{A}$ is called H^+-*eigenvalue* (respectively H^{++}-*eigenvalue*) if it is associated with a nonnegative (respectively positive) eigenvector. For the notion of irreducible or weakly irreducible tensors, one can refer to [1] and [7]. It is known that the adjacency tensor of a uniform hypergraph G is weakly irreducible if and only if G is connected [15, 22].

Theorem 4.1 (The Perron-Frobenius Theorem for nonnegative tensors)**.**

(1) (Yang and Yang [22]) *If $\mathcal{A}$ is a nonnegative tensor, then $\rho(\mathcal{A})$ is an H^+-eigenvalue of $\mathcal{A}$.*
(2) (Friedland, Gaubert and Han [7]) *If furthermore $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique H^{++}-eigenvalue of $\mathcal{A}$, with a unique positive eigenvector, up to a positive scalar.*
(3) (Chang, Pearson and Zhang [1]) *If moreover $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A})$ is the unique H^+-eigenvalue of $\mathcal{A}$, with a unique nonnegative eigenvector, up to a positive scalar.*

According to the definition of tensor product in [18], for a tensor $\mathcal{A}$ of order m and dimension n, and two diagonal matrices P, Q both of dimension n, the product $P\mathcal{A}Q$ has the same order and dimension as $\mathcal{A}$, whose entries are defined by

$$(P\mathcal{A}Q)_{i_1 i_2 \ldots i_m} = p_{i_1 i_1} a_{i_1 i_2 \ldots i_m} q_{i_2 i_2} \cdots q_{i_m i_m}.$$

If $P = Q^{-1}$, then $\mathcal{A}$ and $P^{m-1}\mathcal{A}Q$ are called *diagonal similar*. It is proved that two diagonal similar tensors have the same spectrum [18].

Theorem 4.2 ([22])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two m-th order n-dimensional real tensors with $|\mathcal{B}| \le \mathcal{A}$, namely, $|b_{i_1 i_2 \ldots i_m}| \le a_{i_1 i_2 \ldots i_m}$ for each $i_j \in [n]$ and $j \in [m]$. Then*

(1) $\rho(\mathcal{B}) \le \rho(\mathcal{A})$.
(2) *Furthermore, if $\mathcal{A}$ is weakly irreducible and $\rho(\mathcal{B}) = \rho(\mathcal{A})$, where $\lambda = \rho(\mathcal{A})e^{\mathbf{i}\theta}$ is an eigenvalue of $\mathcal{B}$ corresponding to an eigenvector y, then y contains no zero entries, and $\mathcal{B} = e^{-\mathbf{i}\theta} D^{-(m-1)} \mathcal{A} D$, where $D = diag\{\frac{y_1}{|y_1|}, \ldots, \frac{y_n}{|y_n|}\}$.*

Theorem 4.3 ([22])**.** *Let $\mathcal{A}$ be an m-th order n-dimensional weakly irreducible nonnegative tensor. Suppose $\mathcal{A}$ has k distinct eigenvalues with modulus $\rho(\mathcal{A})$ in total. Then these eigenvalues are $\rho(\mathcal{A})e^{\mathbf{i}\frac{2\pi j}{k}}$, $j = 0, 1, \ldots, k-1$. Furthermore,*

$$\mathcal{A} = e^{-\mathbf{i}\frac{2\pi}{k}} D^{-(m-1)} \mathcal{A} D, \tag{4.1}$$

and the spectrum of $\mathcal{A}$ remains invariant under a rotation of angle $\frac{2\pi}{k}$ (but not a smaller positive angle) of the complex plane.

Suppose $\mathcal{A}$ be as in Theorem 4.3. If $\mathrm{Spec}(\mathcal{A})$ is invariant under a rotation of angle θ of the complex plane, i.e. $\mathrm{Spec}(\mathcal{A}) = e^{\mathbf{i}\theta}\mathrm{Spec}(\mathcal{A})$, then $\rho(\mathcal{A})e^{\mathbf{i}\theta}$ is an eigenvalue of $\mathcal{A}$ by Theorem 4.1. By Theorem 4.3, $\theta = \frac{2\pi j}{k}$ for some $j \in [k]$, and hence by Theorem 4.2 (and taking $\mathcal{B} = \mathcal{A}$), $\mathrm{Spec}(\mathcal{A}) = e^{\mathbf{i}\frac{2\pi j}{k}}\mathrm{Spec}(\mathcal{A})$. So, for some positive integer ℓ, $\ell | k$,

$$\mathrm{Spec}(\mathcal{A}) = e^{\mathbf{i}\frac{2\pi}{\ell}}\mathrm{Spec}(\mathcal{A}). \tag{4.2}$$

The number k in Theorem 4.3 is exactly the cyclic index of $\mathcal{A}$. In addition, if $\mathcal{A}$ is spectral ℓ-symmetric, Then $\ell \mid c(\mathcal{A})$ by Theorem 4.3.

Now return to a connected t-uniform hypergraph G and its power $G^{m,s}$, where $m = st$. By Lemma 3.1, $G^{m,s}$ is spectral $c(G)$-symmetric; and by Lemma 3.2, $G^{m,s}$ is also spectral s-symmetric. So $G^{m,s}$ has eigenvalues

$$\rho(G^{m,s})e^{\mathbf{i}\frac{2\pi i}{c(G)}}e^{\mathbf{i}\frac{2\pi j}{s}},\ i \in [c(G)], j \in [s].$$

In particular, $\rho(G^{m,s})e^{\mathbf{i}\frac{2\pi}{d}}$ is an eigenvalue of $G^{m,s}$, where $d = \frac{s\cdot c(G)}{(s,c(G))}$. So by Theorem 4.2, $G^{m,s}$ is spectral d-symmetric, which is consistent with Corollary 3.5.

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