SAXL CONJECTURE FOR TRIPLE HOOKS

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ABSTRACT. We make some progresses on Saxl Conjecture. Firstly, we show that the probability that a partition is comparable in dominance order to the staircase partition tends to zero as the staircase partition grows. Secondly, for partitions whose Durfee size is k where $k \ge 3$, by semigroup property, we show that there exists a number n_k such that if the tensor squares of the first n_k staircase partitions contain all irreducible representations corresponding to partitions with Durfee size k, then all tensor squares contain partitions with Durfee size k. Specially, we show that $n_3 = 14$ and $n_4 = 28$. Furthermore, with the help of computer we show that the Saxl Conjecture is true for all triple hooks (i.e. partitions with Durfee size 3). Similar results for chopped square and caret shapes are also discussed.

1. INTRODUCTION

In representation theory and related fields, the Kronecker coefficients play a crucial role. For partitions $\lambda, \mu \vdash n$, let $[\lambda]$ and $[\mu]$ be two irreducible representations of S_n . The tensor product $[\lambda] \otimes [\mu]$ is an S_n -representation via the diagonal embedding $\pi \mapsto (\pi, \pi), \pi \in S_n$. This S_n -representation decomposes as follows

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu)[\nu],$$

where $g(\lambda, \mu, \nu)$ are called *Kronecker coefficients*. In spite of their importance, little is known about the Kronecker coefficients, leaving some fundamental questions unanswered. For example, no combinatorial description akin to the Littlewood-Richardson rule is known for the Kronecker coefficients. Another important question is to determine whether they are positive or not, such as the Saxl Conjecture.

In 2012, J. Saxl conjectured that all irreducible representations of the symmetric group occur in the decomposition of the tensor square of the irreducible representation corresponding to the staircase partition [16]. Let ρ_m denote the staircase partition. So the Saxl Conjecture claims that $g(\rho_m, \rho_m, \lambda) > 0$ for each $\lambda \vdash m(m + 1)/2$. Many progresses have been made on this conjecture, see for example [3, 4, 11, 13, 16, 18, 21].

For $\lambda \vdash m(m + 1)/2$, we say that λ satisfies Saxl Conjecture if $g(\rho_m, \rho_m, \lambda) > 0$. In [11, Thm. 2.1], Ikenmeyer showed that if a partition $\nu \vdash m(m + 1)/2$ is comparable in the dominance order to the staircase partition ρ_m , then ν satisfies Saxl Conjecture. From his result, we would like to know the proportion of these partitions in the total partitions of m(m + 1)/2. In Section 3, by the result of [19], we show that the proportion tends to zero as $m \to \infty$ (see Corollary 3.3). Thus the probability that a partition is comparable to the staircase partition tends to zero as $m \to \infty$.

Another criterion to find partitions satisfying Saxl Conjecture is based on nonvanishing irreducible characters, see for example [16, Lemma 1.3] and [3, Cor. 4.4]. Based on the

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character criterion, Bessenrodt showed that all double-hooks (i.e. partitions with Durfee size 2) satisfy Saxl Conjecture [3, Thm. 4.10]. Recently, by the results of 2-modular representation theory, Bessenrodt et al. verified Saxl's conjecture for several large new families of partitions, such as partitions which label height 0 characters and k-Carter-Saxl pairs [4].

In Section 4, we give a generalization of Bessenrodt's result in [3]. For partitions whose Durfee size is k where $k \ge 3$, by semigroup property, we show that there exists a number n_k such that if the tensor squares of the first n_k staircase partitions contain all irreducible representations corresponding to partitions with Durfee size k, then the tensor squares of any staircase partitions contain all partitions (of the same weight) with Durfee size k. For example, we show that $n_3 = 14$ and $n_4 = 28$. With the help of computer, for triple hooks (i.e. partitions with Durfee size 3), we verify that $n_3 = 14$ can be reduced to 9. Combining with the result of [13], we show that all triple hooks satisfy Saxl Conjecture (see Theorem 4.22).

Our technique is elementary and based on the semigroup property of Kronecker coefficients [8, 13]. We also use the technique to discuss the occurrences of hooks and double-hooks for other self-conjugate partitions, such as the chopped square and caret shapes. Our main idea is as follows. Let $\lambda \vdash n$ be a self-conjugate partition, such as ρ_m . For $\mu \vdash n$, we want to determine if $g(\lambda, \lambda, \mu) > 0$. Then we reduce the problem of deciding $g(\lambda, \lambda, \mu) > 0$ to deciding $g(\alpha, \alpha, \nu) > 0$, where $\alpha \subseteq \lambda$ is a smaller self-conjugate partition.

The paper is organized as follows. In Section 2, we summarize basic definitions and results needed in this paper. In Section 3, we show that the probability that a partition is comparable to ρ_m tends to zero as $m \to \infty$. In Section 4, we discuss the occurrences of partitions with fixed Durfee sizes in tensor squares. With the help of computer, we show that triple hooks satisfy Saxl Conjecture. Some remarks, problems and a generalised Saxl Conjecture for each *n* are raised in Section 5.

In this paper, 'with the help of computer' means the results are obtained by using Stembridge's Maple package SF [23].

2. Preliminaries

If *A* is a set, the cardinality of *A* is denoted by |A|. A partition λ of *n*, denoted by $\lambda \vdash n$, is defined to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of non-negative integers such that the sum $\sum_i \lambda_i = n$. We also write $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ if $\lambda_{k+1} = \lambda_{k+2} = \cdots = 0$. Thus for example

$$(3, 3, 2, 1, 0, 0, 0, \ldots) = (3, 3, 2, 1, 0, 0) = (3, 3, 2, 1),$$

as partitions of 9. The *length* of a partition λ is the number of its nonzero entries and is denoted by $\ell(\lambda)$. The set of all partitions of *n* is denoted by P(n). To a partition λ we associate its *Young diagram*, which is a top-aligned and left-aligned array of boxes such that in row *i* we have λ_i boxes. Thus for $\lambda \vdash n$ the corresponding Young diagram has *n* boxes. For example, for $\lambda = (8, 8, 8, 7, 7, 4)$ the corresponding Young diagram is



We do not distinguish between a partition λ and its Young diagram. If we transpose a Young diagram at the main diagonal, then we obtain another Young diagram, which is called the *conjugate partition* of λ and denoted by λ' . The row lengths of λ' are the column lengths of λ . In the example above we have $\lambda' = (6, 6, 6, 6, 5, 5, 5, 3)$. A partition λ is called self-conjugate if $\lambda = \lambda'$. Sometimes we use the notation which indicates the number of times each integer occurs as a part of a partition. For example, we write $\lambda = (3, 3, 3, 2, 2, 1)$ as $(3^3, 2^2, 1)$ which means that 3 parts of λ are equal to 3, and so on. We denote by $d(\lambda)$ the Durfee size of λ , i.e. the number of boxes in the main diagonal of λ . Let $D(n, k) = \{\mu \in P(n) \mid d(\mu) = k\}$ denote partitions in P(n) whose Durfee size is k. Specially, $D\left(\frac{m(m+1)}{2}, k\right)$ will be abbreviated as S(m, k). If the boxes are arranged using matrix coordinates, the *hook* of box (i, j) in a Young diagram is given by the box itself, the boxes to its right and below and is denoted by $h_{i,j}$. The *hook length* is the number of boxes in a hook and denoted by $|h_{i,j}|$. Define the *principal hook partition* by $\widehat{\lambda} = (|h_{1,1}|, ..., |h_{s,s}|)$, where $s = d(\lambda)$. So for $\lambda = (8, 8, 8, 7, 7, 4)$ above, we have $d(\lambda) = 5$ and $\widehat{\lambda} = (13, 11, 9, 6, 3)$. For $m \ge 1$, we call $\rho_m = (m, m - 1, ..., 1)$ the *staircase partition* which is a partition of $\frac{m(m+1)}{2}$.

For $n \in \mathbb{N}$, let S_n denote the symmetric group on *n* symbols. For a partition $\lambda \vdash n$, let $[\lambda]$ denote the irreducible S_n -representation of type λ . The corresponding irreducible character is denoted by χ^{λ} . For $\nu \vdash n$, let $\chi^{\lambda}(\nu)$ denote the value of χ^{λ} on the conjugacy class of cycle type ν of the symmetric group S_n .

A partition λ *dominates* another partition μ , denoted by $\lambda \ge \mu$ if for all k we have $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} \mu_i$. If λ dominates μ or μ dominates λ , we say that λ and μ are *comparable* in the dominance order. For $\lambda \in P(n)$, let $C(\lambda) \subseteq P(n)$ be the set of partitions which are comparable to λ . Let $\Lambda(\lambda) \subseteq C(\lambda)$ (resp. $V(\lambda)$) be the set of partitions which are less than or equal to (resp. greater than or equal to) λ in dominance order. If $\lambda \leq \mu$, then we have $\lambda' \ge \mu'$ [12, Lem. 1.4.11]. For two partitions λ and μ , let $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, ...)$ and $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, ...)$ denote their rowwise sum and difference. We say $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i.

3. DOMINANCE ORDER AND SAXL CONJECTURE

Using the result of [19], in this section we will show that the probability that a partition is comparable to ρ_m tends to zero as $m \to \infty$. It reflects the effectiveness of Ikenmeyer's criterion [11, Thm. 2.1].

Denote by $\Phi(\rho_m)$ the set of $\lambda \vdash \frac{m(m+1)}{2}$ such that $g(\rho_m, \rho_m, \lambda) > 0$. For $\lambda = (\lambda_1, \lambda_2, ...)$, if $a \ge \lambda_1$, then the partition $(a, \lambda_1, \lambda_2, ...)$ is denoted by (a, λ) . Similarly, $(\lambda, 1^a)$ denotes the partition $(\lambda_1, ..., \lambda_{\ell(\lambda)}, 1, ..., 1)$ where there are *a* ones behind $\lambda_{\ell(\lambda)}$. Comparing with Proposition 4.14 of [16], the following gives another lower bound of $|\Phi(\rho_m)|$.

Proposition 3.1. For $m \ge 3$, there exist at least 2^m partitions that are comparable to ρ_m . In particular, we have $|\Phi(\rho_m)| > 2^m$.

Proof. We will show by induction that there are 2^{m-1} partitions less than ρ_m . Then by taking transpose, we obtain another 2^{m-1} partitions that are greater than ρ_m .

There are five partitions less than ρ_3 : (2,2,1,1), (2,2,2), (2,1⁴),(3,1³),(1⁶). Assume that there are at least 2^{m-2} partitions less than ρ_{m-1} . For each $\lambda \in \Lambda(\rho_{m-1})$, define two partitions by (m, λ) and $(\lambda, 1^m)$. Then it is not hard to see that they belong to $\Lambda(\rho_m)$. Moreover, we can see that for $\lambda, \mu \in \Lambda(\rho_{m-1})$ if $\lambda \neq \mu$, then $(m, \lambda), (\lambda, 1^m), (m, \mu)$ and $(\mu, 1^m)$ are pairwise different. Thus, for each partition in $\Lambda(\rho_{m-1})$ we obtain two new partitions in $\Lambda(\rho_m)$ which are pairwise different. So by induction, there are at least 2^{m-1} partitions less than ρ_m .

The lower bound $|\Phi(\rho_m)| > 2^m$ follows from Theorem 2.1 of [11].

For $\lambda \in P(n)$, if $\sum_{j=1}^{i} \lambda'_{j} \ge \sum_{j=1}^{i} \lambda_{j} + i$, then λ is said to be *graphical* [19]. If $\lambda \le \lambda'$, then λ is said to be *conjugate-upward*. Let G(n) and U(n) denote the set of all graphical and conjugate-upward partitions, respectively. The following theorem gives an upper bound

for |G(n)|/|P(n)|, which is also suitable for |U(n)|/|P(n)| (see the discussions in [19, Sect. 1]).

Theorem 3.2. [19, Thm. 3.1] For G(n), U(n) and n large enough, we have

$$\frac{|U(n)|}{|P(n)|}, \ \frac{|G(n)|}{|P(n)|} \le \exp\left(-\frac{0.11\log n}{\log\log n}\right).$$

For $C(\lambda)$ defined in Section 2, by Theorem 3.2 we have the following corollary.

Corollary 3.3. Suppose that $\lambda \in P(n)$ is self-conjugate. Then

$$\lim_{n \to +\infty} \frac{|C(\lambda)|}{|P(n)|} = 0.$$

In particular, for ρ_m we have

$$\lim_{m \to +\infty} \frac{|C(\rho_m)|}{|P(\frac{m(m+1)}{2})|} = 0.$$

That is, the probability that a partition is comparable to ρ_m is zero as $m \to \infty$.

Proof. By definition we have $C(\lambda) = V(\lambda) \cup \Lambda(\lambda)$. For any $\mu \in \Lambda(\lambda)$, we have that $\mu \leq \lambda$ and therefore $\mu' \geq \lambda'$. Since $\lambda' = \lambda$, we have $\mu \leq \mu'$. Thus, $\Lambda(\lambda) \subseteq U(n)$ and there is a bijection between $V(\lambda)$ and $\Lambda(\lambda)$ by taking transpose. So we have that $|C(\lambda)| = 2|\Lambda(\lambda)| - 1$.

By Theorem 3.2 we have $\lim_{n\to+\infty} \frac{|U(n)|}{|P(n)|} = 0$. Since $\Lambda(\lambda) \subseteq U(n)$, we have $\lim_{n\to+\infty} \frac{|\Lambda(\lambda)|}{|P(n)|} = 0$ and

$$\lim_{n \to +\infty} \frac{|C(\lambda)|}{|P(n)|} = \lim_{n \to +\infty} \frac{2|\Lambda(\lambda)| - 1}{|P(n)|} = 0.$$

4. PARTITIONS WITH FIXED DURFEE SIZES IN TENSOR SQUARES

In this section, for partitions whose Durfee size is k where $k \ge 3$, by semigroup property, we show that there exists a number n_k such that if the tensor squares of the first n_k staircase partitions contain all irreducible representations corresponding to partitions with Durfee size k, then the tensor squares of any staircase partitions contain all partitions (of the same weight) with Durfee size k (see Proposition 4.12 below). Specially, we show $n_3 = 14$ and $n_4 = 28$. After that, with the help of computer, we show that all triple hooks satisfy Saxl Conjecture. We also discuss the occurrences of hooks and double-hooks in the the tensor squares and caret shapes.

4.1. The number n_k via semigroup property.

For $\mu \in D(n, k)$, besides the first *k* columns, let \mathcal{A}_i denote the set of columns with length *i*. Besides the first *k* rows, let \mathcal{B}_i denote the set of rows with length *i*. So we have $1 \le i \le k$. With these notations, we give the following definition.

Definition 4.1. Let a_i (resp. b_i) denote the number of columns (resp. rows) in \mathcal{A}_i (resp. \mathcal{B}_i). The *arm weight* of $\mu \in D(n, k)$ is defined as $A := \sum_{i=1}^k ia_i$. The *leg weight* of μ is defined as $B := \sum_{i=1}^k ib_i$. If we set $A_i = ia_i$ and $B_i = ib_i$, then $A = \sum_{i=1}^k A_i$ and $B = \sum_{i=1}^k B_i$. Moreover, we have $n = k^2 + A + B$.

For Kronecker coefficients, we have the following property which is called the *semi-group property*: if $g(\lambda, \mu, \nu) > 0$ and $g(\alpha, \beta, \gamma) > 0$, then $g(\lambda + \alpha, \mu + \beta, \nu + \gamma) > 0$ [8, 13]. Hence, the set of nonzero Kronecker coefficients is a semigroup.

Definition 4.2. Let *K* be the set of all pairs of partitions (α, β) such that $g(\alpha, \alpha, \beta) > 0$.

From Definition 4.2, we have that if $(\alpha, \beta) \in K$ and $(\lambda, \mu) \in K$, then $(\alpha + \lambda, \beta + \mu) \in K$. The following definition will be used in Subsection 4.2.5 and Subsection 4.3.

Definition 4.3. [14, 13] Let λ and μ be two partitions. Let $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ , arranged in descending order, which is called the *vertical sum* of λ and μ .

For example, if $\lambda = (3, 2, 1)$ and $\mu = (2, 2)$, then $\lambda \cup \mu = (3, 2, 2, 2, 1)$. In [13, Def. 9], the symbol of vertical sum is '+_V'. For Young diagrams, we can see that $\lambda \cup \mu$ means they add together vertically.

Lemma 4.4. [13, Cor. 2.5] *If* $g(\alpha, \beta, \gamma) > 0$ and $g(\lambda, \mu, \nu) > 0$, then we have $g(\alpha \cup \lambda, \beta \cup \mu, \gamma + \nu) > 0$. In particular, if $(\alpha, \beta) \in K$ and $(\lambda, \mu) \in K$, then $(\alpha \cup \lambda, \beta + \mu) \in K$.

In the following, we let $\tau_m^i = (m, m - 1, ..., m - i + 1)$ where $m \ge i - 1$. Let $\sigma_m^i = (i^{m-i+1}, i-1, i-2, ..., 2, 1)$ denote the conjugate of τ_m^i .

Definition 4.5. Suppose that $\mu \in S(m, k)$ and σ_m^i , $v \vdash mi - \frac{i(i-1)}{2}$ for $m \ge i-1$. We say that μ is *i-decomposable* for v if $(\sigma_m^i, v) \in K$ and there exists a partition α such that $\mu = v + \alpha$, that is, $\mu - v$ is still a partition.

It is well known that the Kronecker coefficients are invariant when two of its three partitions are transposed (see e. g. Lemma 2.2 and 2.3 in [13]).

Lemma 4.6. For Kronecker coefficient $g(\lambda, \mu, \nu)$, we have $g(\lambda, \mu, \nu) = g(\lambda', \mu, \nu') = g(\lambda, \mu', \nu') = g(\lambda, \mu', \nu') = g(\lambda', \mu', \nu)$. In particular, we have $g(\tau_m^i, \tau_m^i, \nu) = g(\sigma_m^i, \sigma_m^i, \nu)$ where $\nu \vdash im - \frac{i(i-1)}{2}$.

Lemma 4.7. For $\mu \in S(m, k)$, if μ' is i-decomposable for υ and $(\rho_{m-i}, \mu' - \upsilon) \in K$, then we have $(\rho_m, \mu) \in K$. On the other hand, if μ is i-decomposable for υ and $(\rho_{m-i}, \mu - \upsilon) \in K$, then we also have $(\rho_m, \mu) \in K$.

Proof. Suppose that μ' is *i*-decomposable for ν and $(\rho_{m-i}, \mu' - \nu) \in K$. Then we have $(\sigma_m^i, \nu) \in K$. Since $\rho_m = \rho_{m-i} + \sigma_m^i$, by semigroup property we have

$$(\rho_m,\mu') = \left(\rho_{m-i} + \sigma_m^i, \mu' - \upsilon + \upsilon\right) \in K.$$

By Lemma 4.6, we have $g(\rho_m, \rho_m, \mu) = g(\rho_m, \rho'_m, \mu') = g(\rho_m, \rho_m, \mu')$. Thus, we have

$$(\rho_m,\mu) \in K.$$

Similarly, we have $(\rho_m, \mu) \in K$ if μ is *i*-decomposable for v and $(\rho_{m-i}, \mu - v) \in K$. \Box

The following lemma generalizes Theorem 2.1 of [11].

Lemma 4.8. [13, Thm. 9.1] For partitions $\mu, \nu \vdash n$, if μ has distinct row lengths and $\mu \leq \nu$, then $(\mu, \nu) \in K$.

By Lemma 4.6, Corollary 1.9 of [24] can be reformulated as follows.

Lemma 4.9. [24, Cor. 1.9] For each $\mu \vdash 2m - 1$, if $\ell(\mu) \le 4$, then $(\sigma_m^2, \mu) \in K$.

Lemma 4.10. If $\mu = (m - 1, m - 1, m - 1)$, then (τ_m^3, μ) , $(\sigma_m^3, \mu) \in K$.

Proof. Let m - 2 = 3s + t where $t \in \{0, 1, 2\}$. Then

$$\begin{aligned} \tau_m^3 = (m, m-1, m-2) &= (3s+t+2, 3s+t+1, 3s+t) \\ = (3s, 3s, 3s) + (t+2, t+1, t) \end{aligned}$$

and

$$\mu = (m - 1, m - 1, m - 1) = (3s + t + 1, 3s + t + 1, 3s + t + 1)$$

$$= (3s, 3s, 3s) + (t + 1, t + 1, t + 1)$$

By Theorem 4.6 of [16], we have $((3, 3, 3), (3, 3, 3)) \in K$. Thus, by semigroup property we have

$$((3s, 3s, 3s), (3s, 3s, 3s)) = (s(3, 3, 3), s(3, 3, 3)) \in K.$$

For t = 0, 1, 2, by computer we can check that

$$((t+2, t+1, t), (t+1, t+1, t+1)) \in K.$$

Thus by semigroup property we have

$$(\tau_m^3, \mu) = \left((3s, 3s, 3s) + (t+2, t+1, t), (3s, 3s, 3s) + (t+1, t+1, t+1) \right) \in K.$$

By Lemma 4.6, we have $g(\sigma_m^3, \sigma_m^3, \mu) = g(\tau_m^3, \tau_m^3, \mu)$ which completes the proof.

Lemma 4.11. For each *i*, if $\tau_m^i \leq v$, then $g(\tau_m^i, \tau_m^i, v) = g(\sigma_m^i, \sigma_m^i, v)$ and $(\tau_m^i, v), (\sigma_m^i, v) \in$ K. In particular, we have

- (1) if *m* is odd, then $(\sigma_m^3, v) \in K$ where $v = (\frac{3m-3}{2}, \frac{3m-3}{2})$.

- (1) (i) (i) (ii) (iii) and m > 2i.

Proof. It follows by Lemma 4.8 and 4.6.

Recall that $D\left(\frac{m(m+1)}{2},k\right)$ is abbreviated as S(m,k). The upper bound $4k^2 + 4k - 2$ in the following proposition is not best. For k = 3, 4, we will improve it in Proposition 4.19 and Proposition 4.21.

Proposition 4.12. Suppose that $(\rho_m, \mu) \in K$ for all m such that $1 \le m \le 4k^2 + 4k - 2$ and all $\mu \in S(m,k)$. Then for all $m \ge 4k^2 + 4k - 1$ and $\mu \in S(m,k)$ we also have $(\rho_m, \mu) \in K$.

Proof. With notations in Definition 4.1, we have

$$\frac{m(m+1)}{2} = k^2 + \sum_{i=1}^k ia_i + \sum_{i=1}^k ib_i.$$
(4.1)

Suppose that $a_i \ge 2m - 2i + 1$ for some *i*. Let $\tau = (2m - 2i + 1, 2m - 2i + 1, ..., 2m - 2i + 1) \vdash$ $2im - 2i^2 + i$. Then we can see that $\mu - \tau$ is still a partition. By (5) of Lemma 4.11 we have $(\sigma_m^{2i}, \tau) \in K$. Hence, if $(\rho_{m-2i}, \mu - \tau) \in K$, then by semigroup property we have

$$(\rho_m,\mu) = \left(\rho_{m-2i} + \sigma_m^{2i}, \mu - \tau + \tau\right) \in K.$$

Similarly, suppose that $b_i \ge 2m - 2i + 1$ for some *i*. Then $\mu' - \tau$ is still a partition. If $(\rho_{m-2i}, \mu' - \tau) > 0$, then we have

$$(\rho_m,\mu') = \left(\rho_{m-2i} + \sigma_m^{2i},\mu' - \tau + \tau\right) \in K.$$

Since $g(\rho_m, \rho_m, \mu) = g(\rho_m, \rho_m, \mu')$, we have $(\rho_m, \mu) \in K$. Hence, by semigroup property if a_i or $b_i \ge 2m - 2i + 1$ for some *i*, then the positivity of $g(\rho_m, \rho_m, \mu)$ can be reduced to the positivity of $g(\rho_{m-2i}, \rho_{m-2i}, \mu - \tau)$ or $g(\rho_{m-2i}, \rho_{m-2i}, \mu' - \tau)$.

Suppose that $m \ge 4k^2 + 4k - 1$. Then there exists some *i* such that a_i or $b_i \ge 2m - 2i + 1$. Otherwise, for each *i* both a_i and b_i are less than 2m - 2i + 1. By (4.1) we have

$$\frac{m(m+1)}{2} < k^2 + \sum_{i=1}^k i(2m-2i+1) + \sum_{i=1}^k i(2m-2i+1)$$

$$= k^{2} + 2k(k+1)m - \frac{k(k+1)(4k-1)}{3}$$
$$= 2k(k+1)m - \frac{4k^{3} - k}{3},$$

which is equivalent to

$$m^2 + m(1 - 4k - 4k^2) + \frac{8k^3 - 2k}{3} < 0.$$

It contradicts $m \ge 4k^2 + 4k - 1$.

Hence, it follows from the above considerations that the positivity of $g(\rho_m, \rho_m, \mu)$ for all m such that $1 \le m \le 4k^2 + 4k - 2$ and all $\mu \in S(m, k)$ implies the positivity of $g(\rho_m, \rho_m, \mu)$, where $m \ge 4k^2 + 4k - 1$ and $\mu \in S(m, k)$.

In the following, we give a proof of Corollary 6.1 in [11] without using its Theorem 2.1.

Proposition 4.13. For every $v \vdash m(m + 1)/2$, if d(v) = 1 (i.e. v is a hook), then we have $(\rho_m, v) \in K$.

Proof. Suppose that $v \vdash m(m+1)/2$ and d(v) = 1. Then by the notations in Definition 4.1, we have

$$\frac{m(m+1)}{2} = |v| = 1 + A + B = 1 + a_1 + b_1.$$

Suppose that $(\rho_i, \mu) \in K$ for all μ such that $d(\mu) = 1$ and all *i* such that $1 \le i \le m - 1$. We can show that $(\rho_m, \nu) \in K$ by induction.

(1) If $a_1 \ge m$, then $\tau = \nu - (m)$ is a partition of (m - 1)m/2 which is also a hook. Then by induction we have $(\rho_{m-1}, \tau) \in K$. Since $((1^m), (m)) \in K$, by semigroup property we have

$$(\rho_{m-1} + (1^m), \tau + (m)) = (\rho_m, \nu) \in K.$$

(2) If $b_1 \ge m$, then $(\rho_m, \nu) \in K$ is equivalent to $(\rho_m, \nu') \in K$ by Lemma 4.6. By the discussion in (1) above, we also have $(\rho_m, \nu') \in K$.

(3) Suppose that both a_1 and b_1 are less than *m*. Then we have

$$\frac{m(m+1)}{2} = 1 + a_1 + b_1 \le 1 + m - 1 + m - 1$$

$$\le 2m - 1,$$

which implies that $m \leq 2$. It is easily checked that $(\rho_i, \mu) \in K$, where $d(\mu) = 1$ and i = 1, 2.

Suppose that $\mu \in S(m, 3)$. Let A be the arm weight of μ . By Definition 4.1, we have $A = a_1 + 2a_2 + 3a_3$ where $a_1, a_2, a_3 \ge 0$. Thus, besides $a_1 = a_2 = a_3 = 0$, there are seven cases on the first three rows of μ :

Case (1): $a_1, a_2, a_3 > 0$;	Case (2): $a_1 = 0, a_2, a_3 > 0;$	
Case (3): $a_2 = 0$, $a_1, a_3 > 0$;	Case (4): $a_3 = 0$, $a_1, a_2 > 0$;	(4.2)
Case (5): $a_1 = a_2 = 0$, $a_3 > 0$;	Case (6): $a_1 = a_3 = 0$, $a_2 > 0$;	
Case (7): $a_2 = a_3 = 0$, $a_1 > 0$.		

In the following, we will show the decomposability of partitions μ in S(m, 3) and S(m, 4) under seven cases in (4.2). By Lemma 4.9, Lemma 4.10 and Lemma 4.11, the upper bounds in Proposition 4.12 can be reduced to 14 and 28 for partitions in S(m, 3) and S(m, 4), respectively. Firstly, we give a definition.

Suppose that $l \le \frac{m(m+1)}{2}$ and $\mu \in S(m, k)$. With notations in Definition 4.1, if there exists $0 \le x_i \le a_i$ such that $l = x_1 + 2x_2 + \cdots + kx_k$, then a partition $v \vdash l$ can be obtained from the columns of μ such that $\mu = \tau + v$ and $\ell(v) \le k$, where τ is another partition. In fact, we can select x_i columns in \mathcal{A}_i and put them together in their original order. In this way, we obtain the partition $v \vdash l$, which is defined as

$$\upsilon = \left(\sum_{i=1}^k x_i, \sum_{i=2}^k x_i, \dots, x_k\right)$$

Specially, when k = 3 we have $v = (x_1 + x_2 + x_3, x_2 + x_3, x_3)$.

Definition 4.14. With notations above, we call $(x_1, x_2, ..., x_k)$ the *select vector* (or *S*-vector for short) for v.

Example 4.15. Let m = 8 and $\mu = (14, 11, 8, 3) \in S(8, 3)$. We have $a_3 = 5$, $a_2 = 3$ and $a_1 = 3$. Let $x_1 = 2$, $x_2 = 2$ and $x_3 = 3$. We obtain the *S*-vector (2,2,3) and the corresponding partition $\upsilon = (x_1 + x_2 + x_3, x_2 + x_3, x_3) = (7, 5, 3) + 2m - 1 = 15$. Let $\tau = (7, 6, 5, 3)$. Then we have $\mu = \tau + \upsilon = (7, 6, 5, 3) + (7, 5, 3)$, which is described by Young diagrams below.



Lemma 4.16. With notations in Definition 4.1, suppose that the arm weight $A \ge 2m - 1$ for $\mu \in S(m, 3)$. Then μ is 2-decomposable for some v if μ satisfies any of the following conditions:

- (1) $a_2 = a_3 = 0, a_1 > 0;$
- (2) $a_3 = 0, a_1, a_2 > 0;$
- (3) $a_1, a_2, a_3 > 0$;
- (4) $0 \le A_2 + A_3 \le 2m 1;$
- (5) $a_2 = 0, a_1, a_3 > 0$ and 2m 1 = 3s + t, where $t \in \{0, 1\}$ and $s \in \mathbb{N}$.

We can see that in Lemma 4.16 conditions (1), (2), (3) and (5) correspond to Cases (7), (4), (1) and (3) in (4.2), respectively.

Proof. Suppose that $\mu \in S(m, 3)$. If a partition $v \vdash 2m - 1$ can be obtained from μ such that $\ell(v) \leq 3$ and $\mu = v + \tau$, then $(\sigma_m^2, v) \in K$ by Lemma 4.9 and so μ is 2-decomposable for v. For five conditions above, the way to get v is given below.

(1) In this condition, we can choose 2m - 1 columns in \mathcal{A}_1 . Then we get the partition v = (2m - 1) with S-vector (2m - 1, 0, 0).

(2) In this condition, if $2a_2 \ge 2m - 1$, then we can choose m - 1 columns in \mathcal{A}_2 and one column in \mathcal{A}_1 . Then we obtain the partition v = (m, m - 1) with *S*-vector (1, m - 1, 0). If $2a_2 < 2m - 1$, we can choose a_2 columns in \mathcal{A}_2 and $2m - 1 - 2a_2$ column in \mathcal{A}_1 . Then we obtain the partition $v = (2m - 1 - a_2, a_2)$ with *S*-vector $(2m - 1 - 2a_2, a_2, 0)$.

(3) Since $a_1, a_2, a_3 > 0$, we have $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 are not empty.

- Suppose that $3a_3 \ge 2m 1$. Write 2m 1 as 2m 1 = 3s + t where $s \ge 0$ and $t \in \{0, 1, 2\}$.
 - If t = 0, then extract s columns in \mathcal{A}_3 to form $\upsilon = (s, s, s)$ with S-vector is (0, 0, s).
 - If t = 1, then extract s columns in \mathcal{A}_3 and one column in \mathcal{A}_1 to form $\upsilon = (s + 1, s, s)$ with S-vector (1, 0, s).
 - If t = 2, then extract s columns in \mathcal{A}_3 and one column in \mathcal{A}_2 to form $\upsilon = (s + 1, s + 1, s)$ with S-vector (0, 1, s).

- On the other hand, suppose that $3a_3 < 2m 1$.
 - Assume that $2a_2 \ge 2m 1 3a_3$.

- If
$$2m - 1 - 3a_3$$
 is even, take $v = (\frac{2m - 1 - a_3}{2}, \frac{2m - 1 - a_3}{2}, a_3)$ with S-vector $(0, \frac{2m - 1 - 3a_3}{2}, a_3)$.

- If $2m 1 3a_3$ is odd, take $v = (\frac{2m a_3}{2}, \frac{2m a_3 2}{2}, a_3)$ with *S*-vector (1, $\frac{2m 3a_3 2}{2}, a_3)$.
- If $2a_2 < 2m 1 3a_3$, take $v = (u + a_2 + a_3, a_2 + a_3, a_3)$ with S-vector (u, a_2, a_3) , where $u = 2m 1 3a_3 2a_2$.

(4) In this condition, we can let $v = (a_3 + a_2 + w, a_3 + a_2, a_3)$ with *S*-vector (w, a_2, a_3) where $w = 2m - 1 - A_3 - A_2$. In fact, this condition has been shown in (3). Since it will be used often, we put it here separately.

- (5) Assume that t = 1. Then we have 2m 1 = 3s + 1.
 - If $A_3 \ge 2m 1 = 3s + 1$, take v = (s + 1, s, s) with S-vector (1, 0, s).
 - If $A_3 < 2m 1 = 3s + 1$, take $v = (a_3 + w, a_3, a_3)$ with S-vector $(w, 0, a_3)$, where $w = 2m 1 A_3$.

If t = 0, then we have 2m - 1 = 3s. Similarly, if $A_3 \ge 2m - 1 = 3s$, we can let v = (s, s, s) with *S*-vector (0,0,*s*). If $A_3 < 2m - 1 = 3s$, we can let $v = (a_3 + w, a_3, a_3)$ with *S*-vector $(w, 0, a_3)$, where $w = 2m - 1 - A_3$.

Lemma 4.17. Suppose that $\mu \in S(m, 3)$. With notations in Definition 4.1, μ is 3-decomposable for some v if μ satisfies any of the following conditions:

- (1) $A_3 \ge 3m 3$, especially, when $a_1 = a_2 = 0$, $a_3 > 0$ and $A \ge 3m 3$;
- (2) $a_2 = 0$, $a_1, a_3 > 0$ and $A \ge 3m 3$;
- (3) $a_1 = 0, a_2, a_3 > 0$ and $A \ge 3m$.

We can see that in Lemma 4.17 conditions (1), (2) and (3) correspond to Cases (5), (3) and (2) in (4.2), respectively.

Proof. (1) Suppose that $A_3 \ge 3m-3$. Then we can choose m-1 columns in \mathcal{A}_3 and obtain the partition v = (m-1, m-1, m-1) with *S*-vector (0, 0, m-1). By Lemma 4.10, we can see that μ is 3-decomposable for v.

(2) In this condition, if $3a_3 = A_3 \ge 3m-3$, then by (1) we have that μ is 3-decomposable for $\nu = (m-1, m-1, m-1)$. On the other hand, suppose that $3a_3 < 3m-3$. Then we can choose a_3 columns in \mathcal{A}_3 and $3m-3-3a_3$ columns in \mathcal{A}_1 . In this way, we obtain the partition $\nu = (3m-3-2a_3, a_3, a_3)$ with S-vector $(3m-3-3a_3, 0, a_3)$. We can see that $\tau_m^3 = (m, m-1, m-2) \le \nu$. By Lemma 4.11, we have that μ is 3-decomposable for ν .

(3) In this condition, if $3a_3 = A_3 \ge 3m - 3$, then by (1) we have μ is 3-decomposable for $\nu = (m - 1, m - 1, m - 1)$.

On the other hand, suppose that $3a_3 < 3m - 3$. Firstly, assume that $3m - 3 - 3a_3$ is even. Then we can choose a_3 columns in \mathcal{A}_3 and $\frac{3m - 3 - 3a_3}{2}$ columns in \mathcal{A}_2 . So we obtain the partition $\upsilon = (\frac{3m - 3 - a_3}{2}, \frac{3m - 3 - a_3}{2}, a_3)$ with *S*-vector $(0, \frac{3m - 3 - 3a_3}{2}, a_3)$. We can see that $\tau_m^3 \leq \upsilon$. By Lemma 4.11, we have μ is 3-decomposable for υ .

Secondly, assume that $3m - 3 - 3a_3$ is odd. So $3m - 3 - 3(a_3 - 1) = 3m - 3a_3$ is even. Then we can choose $a_3 - 1$ columns in \mathcal{A}_3 and $\frac{3m - 3a_3}{2}$ columns in \mathcal{A}_2 . So we obtain the partition $v = (\frac{3m - a_3}{2} - 1, \frac{3m - a_3}{2} - 1, a_3 - 1)$ with *S*-vector (0, $\frac{3m - 3a_3}{2}, a_3 - 1)$. We can see that $\tau_m^3 \leq v$. By Lemma 4.11, we have μ is 3-decomposable for v.

Lemma 4.18. For $\mu \in S(m, 3)$ and $m \ge 6$, if its arm (resp. leg) weight is no less than 4m - 6, then μ (resp. μ ') is i-decomposable for some v and $i \in \{1, 2, 3, 4\}$.

Proof. By notations in Definition 4.1, the arm weight of μ is $A = \sum_{i=1}^{3} ia_i$. By taking transpose, the leg weight becomes the arm weight, so we only need to show the case when $A \ge 4m - 6$.

Since $4m-6 \ge 2m-1$, 3m-3 and 3m for $m \ge 6$, by Lemma 4.16 and 4.17 we only need to discuss Case (6) in (4.2). Suppose that $a_1 = a_3 = 0$ and $a_2 > 0$. Then $A = A_2 \ge 4m - 6$ and we can choose 2m-3 columns in \mathcal{A}_2 . In this way, we get the partition v = (2m-3, 2m-3) with *S*-vector (0, 2m-3, 0). We can see that $\tau_m^4 = (m, m-1, m-2, m-3) \le (2m-3, 2m-3)$. Thus, by (2) of Lemma 4.11 we have that μ is 4-decomposable for v.

Proposition 4.19. Suppose that $(\rho_m, \mu) \in K$ for all m such that $1 \leq m \leq 14$ and all $\mu \in S(m, 3)$. Then for all $m \geq 15$ and $\mu \in S(m, 3)$, we also have $(\rho_m, \mu) \in K$.

Proof. Let A (resp. B) be the arm (resp. leg) weight of μ . By definition we have

$$\frac{m(m+1)}{2} = 3^2 + A + B$$

If $m \ge 15$, then either A or $B \ge 4m - 6$. Otherwise, if both A and B are less than 4m - 6, then we have

$$\frac{m(m+1)}{2} = 9 + A + B \le 9 + 4m - 7 + 4m - 7,$$

which is equivalent to

 $m^2 - 15m + 10 \le 0.$

It contradicts $m \ge 15$. Hence, by Lemma 4.18 if $m \ge 15$, then either μ or μ' is *i*-decomposable for some v, where $i \in \{1, 2, 3, 4\}$.

Suppose that $(\rho_m, \mu) \in K$ for all $1 \le m \le 14$ and $\mu \in S(m, 3)$. By Lemma 4.7 and induction, we can see that for all $m \ge 15$ and $\mu \in S(m, 3)$ we also have $(\rho_m, \mu) \in K$.

Lemma 4.20. For $\mu \in S(m, 4)$ and $m \ge 11$, if its arm (resp. leg) weight is no less than 8m - 28, then there exist some *i* and v, where $i \in \{1, 2, 3, ..., 8\}$, such that μ (resp. μ') is *i*-decomposable for v.

Proof. By definition, the arm weight of μ is $A = \sum_{i=1}^{4} ia_i$. As in Lemma 4.18, we only need to show the case when $A \ge 8m - 28$.

Firstly, suppose that $A_4 = 4a_4 \ge 2m-1$. There are 8 cases on a_1, a_2, a_3 , which consist of 7 cases in (4.2) and $a_1 = a_2 = a_3 = 0$. We will discuss the decomposability of $\mu \in S(m, 4)$ under them. One the one hand, for a_1, a_2 and a_3 , if at least two of them are nonzero, then we can find a $\nu \vdash 2m - 1$ such that μ is 2-decomposable for ν . The discussion is given below.

- (1) Suppose that $a_1, a_2, a_3 > 0$. For $A_4 = 4a_4 \ge 2m 1$,
 - if 2m 1 = 4s + 1, then extract *s* columns in \mathcal{A}_4 and 1 column in \mathcal{A}_1 to form v = (s + 1, s, s, s) with *S*-vector (1, 0, 0, s);
 - if 2m 1 = 4s + 3, then extract *s* columns in \mathcal{A}_4 and 1 column in \mathcal{A}_3 to form v = (s + 1, s + 1, s + 1, s) with *S*-vector (0, 0, 1, s).
- (2) Suppose that $a_1 = 0$ and $a_2, a_3 > 0$. For $A_4 = 4a_4 \ge 2m 1$,
 - if 2m-1 = 4s+1 = 4(s-1)+5, take v = (s+1, s+1, s, s-1) with S-vector (0, 1, 1, s-1);

• if 2m - 1 = 4s + 3, take v = (s + 1, s + 1, s + 1, s) with S-vector (0, 0, 1, s).

- (3) Suppose that $a_2 = 0$ and $a_1, a_3 > 0$.
 - If 2m 1 = 4s + 1, take v = (s + 1, s, s, s) with S-vector (1, 0, 0, s).
 - If 2m 1 = 4s + 3, take v = (s + 1, s + 1, s + 1, s) with S-vector (0, 0, 1, s).
- (4) Suppose that $a_3 = 0$ and $a_1, a_2 > 0$.

- If 2m 1 = 4s + 1, take v = (s + 1, s, s, s) with S-vector (1, 0, 0, s).
- If 2m 1 = 4s + 3, take v = (s + 2, s + 1, s, s) with S-vector (1, 1, 0, s).

In four cases above, since $\ell(v) = 4$, by Lemma 4.9 we have $(\sigma_m^2, v) \in K$. Thus, we have μ is 2-decomposable for v. One the other hand, we deal with the remaining four cases.

(5) Suppose that $a_1 = a_2 = 0$ and $a_3 > 0$.

• Firstly, let $a_3 \in \{1, 2\}$. If 2m - 1 = 4s + 3, take v = (s + 1, s + 1, s + 1, s) with *S*-vector (0, 0, 1, *s*). By Lemma 4.9 we have that $(\sigma_m^2, v) \in K$. Thus, μ is 2-decomposable for v. If 2m - 1 = 4s + 1, then m = 2s + 1 is odd and 6m - 15 = 4(3s - 3) + 3. Now we show that $a_4 > 3s - 3$. Since $a_1 = a_2 = 0$ and $a_3 \in \{1, 2\}$, we have $A = A_4 + A_3$ and $A_4 \ge A - 6$. Since $A \ge 8m - 28$, we have

$$4a_4 = A_4 \ge A - 6 \ge 8m - 34 = 16s - 26.$$

By the assumption $m \ge 11$ and $s = \frac{m-1}{2}$, we have

$$a_4 \ge \frac{16s - 26}{4} > 3s - 3.$$

Thus, we can take $v = (3s - 2, 3s - 2, 3s - 2, 3s - 3) \vdash 6m - 15$ with *S*-vector (0, 0, 1, 3s - 3). By direct computation, $\tau_m^6 \leq (3s - 2, 3s - 2, 3s - 2, 3s - 3)$. So by (3) of Lemma 4.11 we have $(\sigma_m^6, v) \in K$. Thus, μ is 6-decomposable for v.

- Secondly, suppose that $a_3 \ge 3$. If 2m 1 = 4s + 1, then 2m 1 = 4(s 2) + 9. So we can take $v = (s + 1, s + 1, s + 1, s - 2) \vdash 2m - 1$ with *S*-vector (0, 0, 3, s - 2). By Lemma 4.9 we have $(\sigma_m^2, v) \in K$. Thus, we have that μ is 2-decomposable for v. Similarly, if 2m - 1 = 4s + 3, then μ is also 2-decomposable for v = (s + 1, s + 1, s + 1, s) with *S*-vector (0, 0, 1, *s*).
- (6) Suppose that $a_1 = a_3 = 0$ and $a_2 > 0$. In this case, we have $A = A_2 + A_4$.
 - If $A_4 \ge 4m 28$, take v = (2m 7, 2m 7, 2m 7, 2m 7) with *S*-vector (0, 0, 0, 2m 7). By (4) of Lemma 4.11 we have $(\sigma_m^8, v) \in K$. Thus, μ is 8-decomposable for v.
 - If $A_4 = 4a_4 < 4m 28$, take $v = (2m a_4 14, 2m a_4 14, a_4, a_4)$ with *S*-vector $(0, 2m - 2a_4 - 14, 0, a_4)$. Since $\tau_m^8 \leq v$, by Lemma 4.11 we have $(\sigma_m^8, v) \in K$. Thus, μ is also 8-decomposable for v.
- (7) Suppose that $a_2 = a_3 = 0$ and $a_1 > 0$. In this case, we have $A = A_1 + A_4$.
 - If A₄ ≥ 4m-28, then just as the discussion in (6), we have μ is 8-decomposable for υ = (2m 7, 2m 7, 2m 7, 2m 7).
 - If $A_4 = 4a_4 < 4m 28$, take $\upsilon = (4m 3a_4 28, a_4, a_4, a_4)$ with *S*-vector $(4m 4a_4 28, 0, 0, a_4)$. Since $\tau_m^8 \leq \upsilon$, by Lemma 4.11 we have $(\sigma_m^8, \upsilon) \in K$. Thus, μ is also 8-decomposable for υ .
- (8) Suppose that $a_1 = a_2 = a_3 = 0$. In this case, we have $A = A_4 \ge 8m 28$. So we also have μ is 8-decomposable for $\upsilon = (2m 7, 2m 7, 2m 7, 2m 7)$.

Secondly, suppose that $4a_4 < 2m - 1$. Then for $m \ge 11$ we have

$$3a_3 + 2a_2 + a_1 = A - 4a_4$$

>8m - 28 - (2m - 1)
>4m - 6.

Hence, besides the columns whose lengths are no less than 4, the remaining columns of μ satisfy the conditions in Lemma 4.18. So by the same discussion, we have μ is *i*-decomposable for some $i \in \{1, 2, 3, 4\}$.

Proposition 4.21. Suppose that $(\rho_m, \mu) \in K$ for all m such that $1 \leq m \leq 28$ and all $\mu \in S(m, 4)$. Then for all $m \geq 29$ and $\mu \in S(m, 4)$, we also have $(\rho_m, \mu) \in K$.

Proof. Let A (resp. B) be the arm (resp. leg) weight of μ . By definition we have

$$\frac{m(m+1)}{2} = 4^2 + A + B$$

As the proof in Proposition 4.19, we have that if $m \ge 29$, then either A or $B \ge 8m - 28$. The proof is completed by Lemma 4.20 and similar discussions as in Proposition 4.19.

4.2. Triple hooks in tensor squares.

In this subsection, we will show that all triple hooks satisfy Saxl Conjecture. Firstly, we discuss the decomposability for $\mu \in S(m, 3)$, where $10 \le m \le 14$.

For $\mu \in S(m, 3)$, let A (resp. B) be the arm (resp. leg) weight of μ . By Definition 4.1 we know that

$$A + B = \frac{m(m+1)}{2} - 9.$$

By taking transpose, the leg weight becomes the arm weight. So in the following we assume that $A \ge \frac{1}{2} \left(\frac{m(m+1)}{2} - 9 \right)$. We will verify the decomposability under the cases in (4.2). We can see that most cases in (4.2) are easy to handle, except Case (2) and Case (6). When m = 10 and 12, there are some partitions which are hard to decompose. We will treat them individually with the help of computer.

4.2.1. The decomposability of S(14, 3).

For m = 14, since A + B = m(m+1)/2 - 9 = 96, we assume that $A \ge 48$. Moreover, we have 2m - 1 = 27, 3m - 3 = 39 and 4m - 6 = 50. So by Lemma 4.16 and 4.17, we find that only Case (6) needs to be discussed.

Case (6): Suppose that μ satisfies Case (6). Then we have $A = A_2 \ge 48$. Firstly, if $A_2 > 48$, then $A_2 \ge 50 = 4m - 6$. By (2) of Lemma 4.11 we have that μ is 4-decomposable for $\nu = (25, 25)$.

Secondly, let $A = A_2 = 48$. Then we have B = A = 48. Now we discuss the decomposability of μ' . Just as A = 48, by Lemma 4.16 and 4.17, for μ' we find that only Case (6) needs to be discussed. That is, $b_1 = b_3 = 0$ and $b_2 = 24$. So by $A = A_2 = 48$ we have $a_1 = a_3 = b_1 = b_3 = 0$ and $a_2 = b_2 = 24$. This implies that $\mu = (27, 27, 3, 2^{24})$. By including the third column of the Durfee square into the arm, we can see that $\mu - \nu = (14, 14, 2^{25})$ is a partition, where $\nu = (13, 13, 1) \vdash 2m - 1 = 27$. By Lemma 4.9 we have $(\sigma_{14}^2, \nu) \in K$. Thus, μ is 2-decomposable for ν .

4.2.2. The decomposability of S(13,3).

For m = 13, we have A + B = m(m + 1)/2 - 9 = 82, 2m - 1 = 25, 3m - 3 = 36 and 4m - 6 = 46. Then we assume that $A \ge 41$. So by Lemma 4.16 and 4.17, we find that only Case (6) needs to be discussed.

Case (6): Suppose that μ satisfies Case (6). Then $A = A_2 \ge 41 \ge 36 = 3m - 3$. Since $\tau_{13}^3 \le v$ where $v = (18, 18) \vdash 36$, by (1) of Lemma 4.11 we have μ is 3-decomposable for v.

4.2.3. The decomposability of S(12, 3).

For m = 12, we have A + B = m(m + 1)/2 - 9 = 69, 2m - 1 = 23, 3m - 3 = 33 and 4m - 6 = 42. Then we assume that $A \ge 35$. So by Lemma 4.16 and 4.17, we find that Case (2) and (6) need to be discussed.

Case (2): Under Case (2), we have $A = A_2 + A_3$.

- If $A_2 \ge 2m 1 = 23$, then extract 10 columns in \mathcal{A}_2 and 1 column in \mathcal{A}_3 to form $\upsilon = (11, 11, 1) \vdash 23$ with *S*-vector (0, 10, 1). So by Lemma 4.9, we can see that μ is 2-decomposable.
- If $A_3 \ge 2m 1 = 23$, take $v = (8, 8, 7) \vdash 23$ with *S*-vector (0, 1, 7). By Lemma 4.9 we have that μ is also 2-decomposable.
- If both $A_2 < 2m 1 = 23$ and $A_3 < 2m 1 = 23$, by the assumption $A \ge 35$ we have $A_2, A_3 > 12$. Thus, $A_3 \in \{15, 18, 21\}$ and $A_2 \in \{14, 16, 18, 20, 22\}$. Under this assumption, we can always choose 5 columns in \mathcal{A}_3 and 4 columns in \mathcal{A}_2 to form $\upsilon = (9, 9, 5)$ with *S*-vector (0, 4, 5). By Lemma 4.9 we have that μ is 2-decomposable.

Case (6): Suppose that μ satisfies Case (6). Then we have $A = A_2 \ge 35$. If $A_2 \ge 4m - 6 = 42$, then we can choose 21 columns in \mathcal{R}_2 to form v = (21, 21) + 42 with S-vector (0, 21, 0). Since $\tau_{12}^4 \le v$, by (2) of Lemma 4.11 we have that μ is 4-decomposable.

On the other hand, suppose that $42 > A_2 \ge 35$. Then $A = A_2 \in \{36, 38, 40\}$ and $B \in \{33, 31, 29\}$. Firstly, suppose that $b_3 = 0$. In this case, we can include the third column of μ which consists of 3 boxes into the arm such that μ is 2-decomposable for $\nu = (11, 11, 1)$. More precisely, we can choose the third column and other 10 columns in \mathcal{R}_2 . Then we get the partition $\nu = (11, 11, 1)$.

Secondly, suppose that $b_3 > 0$ and recall that $B \in \{33, 31, 29\}$. Then we discuss the decomposability of μ' under the following four conditions on b_1 , b_2 , b_3 . Note that arm weight of μ' is B.

(C-1) b_1 , $b_2 > 0$, $b_3 > 0$; (C-2) $b_1 = 0$, $b_2 > 0$, $b_3 > 0$; (C-3) $b_1 > 0$, $b_2 = 0$, $b_3 > 0$; (C-4) $b_1 = b_2 = 0$, $b_3 > 0$.

(C-1): Suppose that μ satisfies (C-1). Since $B \in \{33, 31, 29\}$ which is greater than 2m - 1 = 23, by (3) of Lemma 4.16 we have μ' is 2-decomposable.

B_2	B_3	(b_2, b_3)	S-vector	υ
2	27	(1,9)	(0,1,7)	(8,8,7)
8	21	(4,7)	(0,4,5)	(9,9,5)
14	15	(7,5)	(0,4,5)	(9,9,5)
20	9	(10,3)	(0,10,1)	(11,11,1)
26	3	(13,1)	(0,10,1)	(11,11,1)
TABLE 1. The case when $B = B_2 + B_3 = 29$				

(C-2): Suppose that μ satisfies (C-2). For each $B \in \{33, 31, 29\}$, if we write it as $B = 3b_3 + 2b_2$ in all possible ways, we can see that there exist x_3 and x_2 such that $23 = 3x_3 + 2x_2$ and $0 \le x_2 \le b_2$, $0 \le x_3 \le b_3$, that is, we get the *S*-vector $(0, x_2, x_3)$ for $\upsilon = (x_3 + x_2, x_3 + x_2, x_3) + 2m - 1 = 23$. Then by Lemma 4.9 we have that μ' is 2-decomposable for $\upsilon = (x_3 + x_2, x_3 + x_2, x_3 + x_2, x_3)$. The case when $B = B_2 + B_3 = 29$ is given in Table 1. When B = 31 and 33, the discussions are similar.

(C-3): Suppose that μ satisfies (C-3). Firstly, suppose that $b_1 = 1$. Then for $B \in \{33, 31, 29\}$ we should have B = 31 and therefore $b_3 = 10$. Since $A_3 = 0$, we can include the third row of μ which consists of 3 boxes into the leg such that μ' is 3-decomposable for $\nu = (11, 11, 11)$. In fact, we can choose the third row of μ and other 10 rows in \mathcal{B}_3 . After taking transpose, we get the partition $\nu = (11, 11, 11)$ with S-vector (0, 0, 11).

Secondly, suppose that $b_1 \ge 2$. Then for each *B* if $B_3 \ge 23 = 2m - 1$, by choosing 7 rows in \mathcal{B}_3 and 2 rows in \mathcal{B}_1 and taking transpose, we obtain the partition v = (9, 7, 7) + 23 = 2m - 1 with *S*-vector (2,0,7). By Lemma 4.9 we have μ' is 2-decomposable for v = (9,7,7). On the other hand, suppose that $B_3 < 23 = 2m - 1$ for each *B*. Since

 $B_2 = 0$, we have $B_2 + B_3 = B_3 < 23 = 2m - 1$. So by (4) of Lemma 4.16, we have μ' is 2-decomposable.

(C-4): Suppose that μ satisfies (C-4). Then for $B \in \{33, 31, 29\}$ we should have $B = B_3 = 3m - 3 = 33$. By (1) of Lemma 4.17 we have that μ' is 3-decomposable.

4.2.4. The decomposability of S(11, 3).

For m = 11, we have A + B = m(m + 1)/2 - 9 = 57, 2m - 1 = 21, 3m - 3 = 30 and 4m - 6 = 38. Then we assume that $A \ge 29$. So by Lemma 4.16 and 4.17, we find that Case (2), (5) and (6) need to be discussed.

Case (2): Under Case (2), we have $A = A_2 + A_3$.

- Firstly, if $A_2 \ge 2m 1 = 21$, extract 9 columns in \mathcal{A}_2 and 1 column in \mathcal{A}_3 to form $\upsilon = (10, 10, 1) \vdash 21$ with *S*-vector (0, 9, 1). Then by Lemma 4.9, we can see that μ is 2-decomposable.
- Secondly, if $A_3 \ge 2m 1 = 21$, take $v = (7, 7, 7) \vdash 21$ with *S*-vector (0, 0, 7). By Lemma 4.9, we also have that μ is 2-decomposable.
- Thirdly, suppose that both $A_2 < 2m 1 = 21$ and $A_3 < 2m 1 = 21$. More precisely, it means that $A_2 \le 20$ and $A_3 \le 18$. By the assumption $A_2 + A_3 \ge 29$ we have $A_3 \ge 9$ and $A_2 \ge 11$. Thus, $A_3 \in \{9, 12, 15, 18\}$ and $A_2 \in \{12, 14, 16, 18, 20\}$. In this case, we can always extract 3 columns in \mathcal{A}_3 and 6 columns in \mathcal{A}_2 to form v = (9, 9, 3) with *S*-vector (0, 6, 3). By Lemma 4.9 we have that μ is 2-decomposable.

Case (5): Under Case (5), we have $A = A_3$. Since $A = A_3 \ge 29 > 21 = 2m - 1$, we can choose 7 columns in \mathcal{A}_3 to form $v = (7, 7, 7) \vdash 21$ with *S*-vector (0, 0, 7). Then by Lemma 4.9, μ is 2-decomposable.

Case (6): Under Case (6), we have $A = A_2 \ge 29$. Since A_2 is even, we have $A = A_2 \ge 30$. So we can choose 15 columns in \mathcal{A}_2 to form $v = (15, 15) \vdash 30$ with *S*-vector (0, 15, 0). Since $\tau_{11}^3 \le (15, 15)$, by (1) of Lemma 4.11 we have $(\sigma_{11}^3, v) \in K$. So μ is 3-decomposable.

4.2.5. The decomposability of S(10, 3).

For m = 10, we have A + B = m(m+1)/2 - 9 = 46, $2m - 1 = 19 = 3 \times 6 + 1$, 3m - 3 = 27and 4m - 6 = 34. Then we assume that $A \ge 23$. So by Lemma 4.16 and 4.17, we find that Case (2), (5) and (6) need to be discussed. Moreover, Case (2) and (6) are the most involved parts. Under Case (2) and (6), there exist partitions which are hard to decompose. They will be dealt with the help of computer.

	Cases	$B = 23$ and $A_3 = 21, A_2 = 2$	$B = 20$ and $A_3 = 24$, $A_2 = 2$
(1)	$b_1, b_2, b_3 > 0$	By (3) of Lemma 4.16	By (3) of Lemma 4.16
(2)	$b_1 = 0$ and $b_2, b_3 > 0$	Hard	Hard
(3)	$b_2 = 0$ and $b_1, b_3 > 0$	By (5) of Lemma 4.16	By (5) of Lemma 4.16
(4)	$b_3 = 0$ and $b_1, b_2 > 0$	By (2) of Lemma 4.16	By (2) of Lemma 4.16
(5)	$b_1 = b_2 = 0$ and $b_3 > 0$	Ø	Ø
(6)	(6) $b_1 = b_3 = 0$ and $b_2 > 0$	Ø	μ is 3-decomposable
(0)			for $v = (9, 9, 9)$
(7)	$b_2 = b_3 = 0$ and $b_1 > 0$	By (1) of Lemma 4.16	By (1) of Lemma 4.16

TABLE 2. Decomposability of μ' for $\mu \in S(10, 3)$, where " \emptyset " means impossible.

Case (2): Under Case (2), we have $A = A_2 + A_3 \ge 23$. The discussion will be given in the following conditions:

(i) $A_3 \ge 19$ and $A_2 = 2$; (ii) $A_3 \ge 19$ and $A_2 \ge 4$; (iii) $A_2 \ge 19$; (iv) $A_2, A_3 < 19$. (i): Suppose that μ satisfies condition (i). Since $A = A_3 + A_2 \ge 23$, we have $A_3 \ge 23 - A_2 = 21$. So we have $A_3 \in \{21, 24, 27, ...\}$. If $A_3 \ge 27 = 3m - 3$, then by (1) of Lemma 4.17 we have μ is 3-decomposable.

Now suppose that $A_3 \in \{21, 24\}$ and $A_2 = 2$. Then we have $B \in \{23, 20\}$. In both conditions, we will mainly discuss the decomposability of μ' whose arm weight is *B*. There are 7 cases on *B* which are the same as (4.2). The discussion is summarized in Table 2. In the following, we give an explanation of Table 2. We will find that there are 4 partitions which are hard to decompose.

In Table 2, under Case (1), (3), (4) and (7), the decomposability of μ' can be obtained from Lemma 4.16. For Case (2), if B = 23, that is, $b_1 = 0$, b_2 , $b_3 > 0$, then there are exactly 4 pairs (b_2 , b_3) such that $23 = 2b_2 + 3b_3$, which are {(10, 1), (7, 3), (4, 5), (1, 7)}. Suppose that (b_2 , b_3)=(10, 1). We have $2m-1 = 19 = 2x_2+3x_3$ where (x_2 , x_3) = (8, 1). Let (x_1, x_2, x_3) = (0, 8, 1) be the *S*-vector for $\nu = (x_3 + x_2, x_3 + x_2, x_3) = (9, 9, 1)$. By Lemma 4.9 we have μ' is 2-decomposable for $\nu = (9, 9, 1)$. Similarly, if (b_2 , b_3)=(7, 3), we can set (x_1, x_2, x_3) = (0, 5, 3). If (b_2, b_3)=(4, 5), we can set (x_1, x_2, x_3) = (0, 2, 5). However, when (b_2, b_3)=(1, 7) we cannot find $0 \le x_2 \le b_2$, $0 \le x_3 \le b_3$ such that $2m - 1 = 19 = 2x_2 + 3x_3$. When (b_2, b_3)=(1, 7), we can see that the corresponding partition is (11, 11, 10, 3⁷, 2).

On the other hand, suppose that B = 20 for Case (2). Then there are only 3 pairs (b_2, b_3) such that $20 = 2b_2 + 3b_3$, which are $\{(1, 6), (4, 4), (7, 2)\}$. We can see that for these 3 pairs, we cannot find $0 \le x_2 \le b_2$, $0 \le x_3 \le b_3$ such that $2m - 1 = 19 = 2x_2 + 3x_3$. Moreover, when $(b_2, b_3)=(1, 6)$, (4, 4) and (7, 2), the corresponding partitions are $(12, 12, 11, 3^6, 2)$, $(12, 12, 11, 3^4, 2^4)$ and $(12, 12, 11, 3^2, 2^7)$, respectively.

So by discussions above, in Table 2 when B = 23 and 20 satisfy Case (2), we find 4 partitions which are hard to decompose: $\mu_1 = (11, 11, 10, 3^7, 2), \mu_2 = (12, 12, 11, 3^6, 2), \mu_3 = (12, 12, 11, 3^4, 2^4)$ and $\mu_4 = (12, 12, 11, 3^2, 2^7)$. Let $\tau_7 = (7, 7, 7), v_1 = (4^2, 3^8, 2), v_2 = (5^2, 4, 3^6, 2), v_3 = (5^2, 4, 3^4, 2^4)$ and $v_4 = (5^2, 4, 3^2, 2^7)$. Then we have $\mu_i = \tau_7 + v_i$ $(1 \le i \le 4)$. By computer, we can verify that $(\sigma_{10}^4, v_i) \in K$ for $1 \le i \le 4$. Thus μ_i is 4-decomposable for v_i .

Suppose that $B \in \{23, 20\}$. Since 23 and 20 cannot be divided by 3, Case (5) cannot happen which is denoted by " \emptyset " in Table 2. Since 23 cannot be divided by 2, Case (6) cannot happen for B = 23. For B = 20 and $A_3 = 24$, $A_2 = 2$, if B satisfies Case (6), that is, $b_1 = b_3 = 0$ and $b_2 > 0$, then only one partition satisfies this condition, which is $\mu = (12^2, 11, 2^{10})$. Under this assumption, we discuss the decomposability of μ . By direct computation, we have that μ is 3-decomposable for $\nu = (9, 9, 9)$.

(ii): Suppose that μ satisfies condition (ii). Since $A_3 \ge 19$ and $A_2 \ge 4$, we can extract 5 columns in \mathcal{A}_3 and 2 columns in \mathcal{A}_2 to form $\upsilon = (7, 7, 5) \vdash 19 = 2m - 1$ with S-vector (0, 2, 5). By Lemma 4.9 we have μ is 2-decomposable.

(iii): Suppose that μ satisfies condition (iii). Since $A_2 \ge 19$ and $A_3 > 0$, we can extract 1 column in \mathcal{A}_3 and 8 columns in \mathcal{A}_2 to form $\nu = (9, 9, 1) \vdash 19 = 2m - 1$ with S-vector (0, 8, 1). By Lemma 4.9 we have μ is 2-decomposable.

(iv): Suppose that μ satisfies condition (iv). For $A_2 + A_3 \ge 23$, if $A_3, A_2 < 19$, we have $4 < A_2, A_3 < 19$. More precisely, $A_2 \in \{6, 8, 10, 12, 14, 16, 18\}$ and $A_3 \in \{6, 9, 12, 15, 18\}$. The decomposability of μ is given below.

(iv-1): If $A_3 = 6$, from $A_2 + A_3 \ge 23$ we should have $A_2 = 18$. So we can extract 1 column in \mathcal{A}_3 and 8 columns in \mathcal{A}_2 to form $\upsilon = (9, 9, 1) \vdash 19 = 2m - 1$ with S-vector (0, 8, 1). By Lemma 4.9 we have μ is 2-decomposable for υ .

(iv-2): If $A_3 = 9$, from $A_2 + A_3 \ge 23$ we should have $A_2 \in \{14, 16, 18\}$. So we can extract 3 columns in \mathcal{A}_3 and 5 columns in \mathcal{A}_2 to form $\upsilon = (8, 8, 3) \vdash 19 = 2m - 1$ with S-vector (0, 5, 3). By Lemma 4.9 we have μ is 2-decomposable for υ .

(iv-3): If $A_3 = 12$, from $A_2 + A_3 \ge 23$ we should have $A_2 \in \{12, 14, 16, 18\}$. The discussion is the same as (iv-2). That is, μ is also 2-decomposable for $\nu = (8, 8, 3)$.

(iv-4): If $A_3 = 15$, from $A_2 + A_3 \ge 23$ we should have $A_2 \in \{8, 10, 12, 14, 16, 18\}$. So we can extract 5 columns in \mathcal{A}_3 and 2 columns in \mathcal{A}_2 to form $\upsilon = (7, 7, 5) \vdash 19 = 2m - 1$ with *S*-vector (0, 2, 5). By Lemma 4.9 we have μ is 2-decomposable for υ .

(iv-5): If $A_3 = 18$, from $A_2 + A_3 \ge 23$ we should have $A_2 \in \{6, 8, 10, 12, 14, 16, 18\}$. The discussion is the same as (iv-4). That is, μ is also 2-decomposable for v = (7, 7, 5).

Case (5): Suppose that μ satisfies Case (5). Then $A = A_3 \ge 23$ which means that $A_3 \in \{24, 27, 30...\}$. If $A_3 \ge 27 = 3m - 3$, then by (1) of Lemma 4.17 we have that μ is 3-decomposable for v = (9, 9, 9). If $A = A_3 = 24$, then $B = 46 - A_3 = 22$. Under this assumption, we mainly discuss the decomposability of μ' whose arm weight is *B*. The discussion is given under 7 cases as in (4.2).

(b-1): Suppose that $b_1, b_2, b_3 > 0$. Since $B = 22 \ge 19 = 2m - 1$, by (3) of Lemma 4.16 we have μ' is 2-decomposable.

(b-2): Suppose that $b_1 = 0$ and $b_2, b_3 > 0$. There are 3 pairs (b_2, b_3) such that $22 = 2b_2 + 3b_3$, which are {(2, 6), (5, 4), (8, 2)}. If $(b_2, b_3)=(2, 6)$, we have $2m - 1 = 19 = 2x_2 + 3x_3$ where $(x_2, x_3) = (2, 5)$. So by Lemma 4.9 we have μ' is 2-decomposable for $\nu = (x_3 + x_2, x_3 + x_2, x_3) = (7, 7, 5)$, whose *S*-vector is $(x_1, x_2, x_3) = (0, 2, 5)$. Similarly, if $(b_2, b_3)=(5, 4)$, we can set $(x_1, x_2, x_3) = (0, 5, 3)$. If $(b_2, b_3)=(8, 2)$, we can set $(x_1, x_2, x_3) = (0, 8, 1)$.

(**b-3**): Suppose that $b_2 = 0$ and $b_1, b_3 > 0$. Since $B = 22 \ge 19 = 2m-1$ and $19 = 3 \times 6+1$, by (5) of Lemma 4.16 we have μ' is 2-decomposable.

(b-4): Suppose that $b_3 = 0$ and $b_1, b_2 > 0$. Since $B = 22 \ge 19 = 2m - 1$, by (2) of Lemma 4.16 we have μ' is 2-decomposable.

(**b-5**): Suppose that $b_1 = b_2 = 0$ and $b_3 > 0$. Since 22 cannot be divided by 3, this case cannot happen.

(**b-6**): Suppose that $b_1 = b_3 = 0$ and $b_2 > 0$. Since $A = A_3 = 24$, there is only one partition satisfying this case, which is $\mu = (11, 11, 11, 2^{11})$. By direct computation, we have μ is 3-decomposable for $\nu = (9, 9, 9)$.

(b-7): Suppose that $b_2 = b_3 = 0$ and $b_1 > 0$. Since $B = 22 \ge 19 = 2m - 1$, by (1) of Lemma 4.16 we have μ' is 2-decomposable.

Case (6): Suppose that μ satisfies Case (6). Then $A = A_2 \ge 23$. Firstly, if $A_2 \ge 34 = 4m-6$, then by (2) of Lemma 4.11 we have μ is 4-decomposable for $\nu = (17, 17)$. Secondly, suppose that $23 \le A_2 < 34$. Then we have $A_2 \in \{24, 26, 28, 30, 32\}$. Let

$$C_6 = \{\mu \in S(10,3) | \mu \text{ satisfies Case (6) and } 23 \le A_2 < 34\}$$

denote the set of these partitions. Generally, by previous method it is hard to verify whether these partitions are *i*-decomposable or not. So in this case, we don't discuss the decomposability of μ . With the help of computer, we can show that $(\rho_{10}, \mu) \in K$ for $\mu \in C_6$. In fact, let $\eta_6 = (6, 6, 6, 6, 6, 5) \vdash 35$. Then we have $\rho_{10} = (\eta_6 \cup \rho_4) + \rho_4$. For $\mu \in C_6$, we can see that $\nu = (\mu - \tau_5) - \tau_5$ is still a partition, where $\nu \vdash 35$ and $\tau_5 = (5, 5)$. So we have $\mu = \nu + \tau_5 + \tau_5$. By computer we can check that $(\eta_6, \nu) \in K$ for all $\nu \vdash 35$. For ρ_4 , we have $(\rho_4, \tau_5) \in K$. Then by Lemma 4.4 we have

$$(\eta_6 \cup \rho_4, \nu + \tau_5) \in K.$$

And then we have

$((\eta_6 \cup \rho_4) + \rho_4, \nu + \tau_5 + \tau_5) = (\rho_{10}, \mu) \in K.$

By now, we can see that when $11 \le m \le 14$ and $\mu \in S(m, 3)$, there exists some $i \in \{2, 3, 4\}$ such that μ or μ' is *i*-decomposable. However, there are some exceptions when $\mu \in S(10, 3)$. Let C_6 denote the set of these partitions. With the help of computer, we checked that $(\rho_{10}, \mu) \in K$ for $\mu \in C_6$. Thus, the upper bound 14 in Proposition 4.19 can be reduced to 9. In [13, Sec. 7], using a computer to implement the semigroup property in conjunction with Theorem 2.1 of [11] on dominance ordering, Luo and Sellke verified the Saxl Conjecture up to ρ_9 . Thus, from discussions above we have the main theorem of this paper.

Theorem 4.22. For each $m \in \mathbb{N}$, if $\mu \in S(m, 3)$, we have $(\rho_m, \mu) \in K$.

4.3. Applications to chopped square and caret shapes.

In this part, we apply our technique for staircase shapes to chopped square and caret shapes, which were defined in [16]. Similar results are obtained for the occurrences of hooks and double-hooks.

Let

$$\eta_k = (k^{k-1}, k-1) \vdash n$$

where $n = k^2 - 1$. Let

$$\gamma_k = (3k - 1, 3k - 3, ..., k + 3, k + 1, k, k - 1, k - 1, k - 2, k - 2, ..., 2, 2, 1, 1) \vdash n$$

where $n = 3k^2$. In [16], η_k and γ_k are called *the chopped square shape of order k* and *the caret shape of order k*, respectively. It has been shown that hooks and two row shapes appear in $[\eta_k] \otimes [\eta_k]$ and $[\gamma_k] \otimes [\gamma_k]$ for sufficient large k [16].

In the following, we discuss the occurrences of hooks in $[\eta_k] \otimes [\eta_k]$ and $[\gamma_k] \otimes [\gamma_k]$.

Proposition 4.23. For every $v \vdash k^2 - 1$, if d(v) = 1 (i.e. v is a hook), then we have $(\eta_k, v) \in K$.

Proof. Suppose that $v \vdash k^2 - 1$ and d(v) = 1. Then by the notations in Definition 4.1, we have

$$k^{2} - 1 = |v| = 1 + A + B = 1 + a_{1} + b_{1}.$$

Suppose that $(\eta_i, \mu) \in K$ for all $d(\mu) = 1$ and $1 \le i \le k - 1$. We can show that $(\eta_k, \nu) \in K$ by induction.

If $A \ge 2k - 1$, then $\nu - (2k - 1)$ is still a partition. Suppose that $(\eta_{k-1}, \nu - (2k - 1)) \in K$. Then by $((1^{k-1}), (k-1)) \in K$ we have

$$(\eta_{k-1} + (1^{k-1}), \nu - (2k-1) + (k-1)) \in K.$$

Let $\theta_{k-1} = \eta_{k-1} + (1^{k-1})$ and $\nu_{k-1} = \nu - (2k-1) + (k-1)$. Then by $((k), (k)) \in K$ and Lemma 4.4 we have

$$(\theta_{k-1} \cup (k), \nu_{k-1} + (k)) = (\eta_k, \nu) \in K.$$

Similarly, if $B \ge 2k - 1$, from $g(\eta_k, \eta_k, \nu') = g(\eta_k, \eta_k, \nu)$ we have $(\eta_k, \nu) \in K$. Suppose that both *A* and *B* are less than 2k - 1. Then we have

$$k^{2} - 1 = 1 + A + B$$

$$\leq 1 + 2k - 2 + 2k - 2$$

$$= 4k - 3,$$

which implies that $k \le 3$. It has been verified that $(\eta_i, \nu) \in K$ for all $\nu \vdash i^2 - 1$ with $d(\nu) = 1$ and $i \in \{1, 2, 3\}$ [16, Sect. 3.4].

Proposition 4.24. For every $v \vdash 3k^2$, if d(v) = 1, then we have $(\gamma_k, v) \in K$.

Proof. The proof is similar to Proposition 4.23. Suppose that $v \vdash 3k^2$ and d(v) = 1. Then by the notations in Definition 4.1, we have

$$3k^2 = |v| = 1 + A + B = 1 + a_1 + b_1$$

Note that $\gamma_k = (\gamma_{k-1} + (1^{3k-2})) \cup (3k-1)$. So if $A \ge 6k-3$, then $(\gamma_{k-1}, \nu - (6k-3)) \in K$ implies $(\gamma_k, \nu) \in K$. Similar result is hold for $B \ge 6k-3$.

If both A and B are less than 6k - 3, we have

$$3k^{2} = 1 + A + B$$

$$\leq 1 + 6k - 4 + 6k - 4$$

$$= 12k - 7.$$

which implies that $k \le 3$. By computer, we can easily check that $(\gamma_i, \nu) \in K$ for all $\nu \vdash 3i^2$ with $d(\nu) = 1$ and $i \in \{1, 2, 3\}$.

In the following, we discuss the positivity of $g(\eta_k, \eta_k, \mu)$ and $g(\gamma_k, \gamma_k, \nu)$, where $\mu \in D(k^2 - 1, 2)$ and $\nu \in D(3k^2, 2)$.

Lemma 4.25. For $k \in \mathbb{N}$, if k is even, we have $((k^2), (k^2)) \in K$. If k is odd, we have $g((k^2), (k^2), (k^2)) = 0$. However, if k is odd, we have $((k^4), (2k, 2k)) \in K$.

Proof. By Lemma 1.6 of [24], we have that if k is even, then $g((k^2), (k^2), (k^2)) = 1 > 0$ and if k is odd, then $g((k^2), (k^2), (k^2)) = 0$.

If k is odd, write k as 2s+1 for some s, then we have k = 2(s-1)+3 and 2k = 4(s-1)+6. So we have

$$(k^4) = (k, k, k, k) = (2(s-1) + 3, 2(s-1) + 3, 2(s-1) + 3, 2(s-1) + 3)$$

= (s-1)(2, 2, 2, 2) + (3, 3, 3, 3) = (s-1)(2^4) + (3^4),

and

$$(2k, 2k) = (4(s-1) + 6, 4(s-1) + 6) = (s-1)(4, 4) + (6, 6).$$

Since $(2^4)' = (4, 4)$ and $((4, 4), (4, 4)) \in K$, we have

$$((2^4), (4, 4)) \in K$$

and therefore

$$((s-1)(2^4), (s-1)(4,4)) \in K.$$

By computer we can verify that

$$((3^4), (6, 6)) \in K$$
.

So by semigroup property we have

$$((k^4), (2k, 2k)) = ((s-1)(2^4) + (3^4), (s-1)(4, 4) + (6, 6)) \in K.$$

Lemma 4.26. Suppose that $k \ge 12$ is even and $\mu \in D(k^2 - 1, 2)$. Then deciding $(\eta_k, \mu) \in K$ can be reduced to η_{k-1} or η_{k-2} .

Proof. For $\mu \in D(k^2 - 1, 2)$, with notations in Definition 4.1 we have

$$k^2 - 1 = |\mu| = 4 + A + B = 4 + A_1 + A_2 + B_1 + B_2.$$

If $k \ge 12$, then one of the following conditions must occur:

$$A_2 \ge 4k - 4, \ B_2 \ge 4k - 4, \ A_1 \ge 2k - 1 \text{ and } B_1 \ge 2k - 1.$$
 (4.3)

Otherwise, we have

$$k^{2} - 1 = 4 + A_{1} + A_{2} + B_{1} + B_{2}$$

$$\leq 4 + 2k - 2 + 4k - 5 + 2k - 2 + 4k - 5$$

$$= 12k - 10$$

which contradicts $k \ge 12$.

Suppose that *k* is even. If $\mu \in D(k^2 - 1, 2)$ satisfies one of the four conditions in (4.3), then deciding $(\eta_k, \mu) \in K$ can be reduced to η_{k-2} or η_{k-1} . More precisely, if $A_1 \ge 2k - 1$ or $B_1 \ge 2k - 1$, then from the proof of Proposition 4.23 we know that $(\eta_{k-1}, \mu - (2k - 1)) \in K$ or $(\eta_{k-1}, \mu' - (2k - 1)) \in K$ implies $(\eta_k, \mu) \in K$. Suppose that $A_2 \ge 4k - 4$ and let $\nu = \mu - (2k - 2, 2k - 2) \vdash (k - 2)^2 - 1$. If *k* is even, by Lemma 4.25 we have

$$((k, k), (k, k))$$
 and $((k - 2, k - 2), (k - 2, k - 2)) \in K$

Since $(k - 2, k - 2)' = (2^{k-2})$, we have

$$((2^{k-2}), (k-2, k-2)) \in K.$$

Observe that

$$\eta_k = \left(\eta_{k-2} + (2^{k-2})\right) \cup (k,k).$$

If we let $(\eta_{k-2}, \nu) \in K$, then

$$(\eta_{k-2} + (2^{k-2}), \nu + (k-2, k-2)) \in K,$$

and therefore

$$\left(\left(\eta_{k-2} + (2^{k-2})\right) \cup (k,k), \nu + (k-2,k-2) + (k,k)\right) = (\eta_k,\mu) \in K.$$

Similarly, if $B_2 \ge 4k - 4$, then $(\eta_{k-2}, \mu' - (2k - 2, 2k - 2)) \in K$ implies $(\eta_k, \mu) \in K$.

Suppose that k is odd. By Lemma 4.25 we have g((k, k), (k, k), (k, k)) = 0. So the argument in Lemma 4.26 is not suitable for the odd number. Interestingly, by Lemma 4.25 we have $((k^4), (2k, 2k)) \in K$ when k is odd. By similar argument in Lemma 4.26, in the following lemma we will show that if $k \ge 19$ is odd, then deciding $(\eta_k, \mu) \in K$ can be reduced to η_{k-4} or η_{k-1} . Since the discussions are similar, we just outline the proof here.

Lemma 4.27. Suppose that $k \ge 19$ is odd and $\mu \in D(k^2 - 1, 2)$. Then deciding $(\eta_k, \mu) \in K$ can be reduced to η_{k-1} or η_{k-4} .

Proof. If $k \ge 19$, then one of the following conditions must occur:

$$A_2 \ge 8k - 16, B_2 \ge 8k - 16, A_1 \ge 2k - 1 \text{ and } B_1 \ge 2k - 1.$$
 (4.4)

Otherwise, we have

$$k^{2} - 1 = 4 + A_{1} + A_{2} + B_{1} + B_{2}$$

$$\leq 4 + 2k - 2 + 8k - 17 + 2k - 2$$

$$= 20k - 34,$$

which contradicts $k \ge 19$.

If $\mu \in D(k^2 - 1, 2)$ satisfies one of the four conditions in (4.4), then deciding $(\eta_k, \mu) \in K$ can be reduced to η_{k-4} or η_{k-1} . We just discuss the condition when $A_2 \ge 8k - 16$ here.

Suppose that $A_2 \ge 8k - 16$ and let $v = \mu - (4k - 8, 4k - 8) \vdash (k - 4)^2 - 1$. If k is odd, by Lemma 4.25 we have

$$((k^4), (2k, 2k)) \in K$$
 and $(((k-4)^4), (2k-8, 2k-8)) \in K$

and from $((k-4)^4)' = (4^{k-4})$ we have

$$((4^{k-4}), (2k-8, 2k-8)) \in K.$$

Observe that

$$\eta_k = \left(\eta_{k-4} + (4^{k-4})\right) \cup (k^4).$$

So if we let $(\eta_{k-4}, \nu) \in K$, then $(\eta_k, \mu) \in K$.

Thus, by Lemma 4.26 and 4.27, for $\mu \in D(k^2 - 1, 2)$ we have the following proposition.

Proposition 4.28. Suppose that $(\eta_k, \mu) \in K$ for all k such that $1 \le k \le 18$ and all $\mu \in D(k^2 - 1, 2)$. Then for all $k \ge 19$ and $\mu \in D(k^2 - 1, 2)$, we also have $(\eta_k, \mu) \in K$.

Lemma 4.29. For $n \in \mathbb{N}$, if n is odd, we have $((n, n - 2), (n - 1, n - 1)) \in K$. If n is even, we have g((n, n - 2), (n, n - 2), (n - 1, n - 1)) = 0. However, if n is even, we have $((n, n - 2, n - 4, n - 6), (2n - 6, 2n - 6)) \in K$.

Proof. If *n* is odd, by Theorem 1.7 of [24] we have

$$g((n, n-2), (n, n-2), (n-1, n-1)) = g((n, n-2), (n-1, n-1), (n, n-2)) = 1 > 0.$$

Moreover, if *n* is even, we have g((n, n-2), (n, n-2), (n-1, n-1)) = 0. Since $(n, n-2, n-4, n-6) \leq (2n-6, 2n-6)$, by Lemma 4.8 we have

$$((n, n-2, n-4, n-6), (2n-6, 2n-6)) \in K.$$

Lemma 4.30. Suppose that $\mu \in D(3k^2, 2)$, where $k \ge 12$ and 3k - 1 is odd. Then deciding $(\gamma_k, \mu) \in K$ can be reduced to γ_{k-1} or γ_{k-2} .

Proof. For $\mu \in D(3k^2, 2)$, with notations in Definition 4.1 we have

$$3k^2 = |\mu| = 4 + A + B = 4 + A_1 + A_2 + B_1 + B_2.$$

If $k \ge 12$, then one of the following conditions must occur:

$$A_2 \ge 12k - 12, \ B_2 \ge 12k - 12, \ A_1 \ge 6k - 3 \text{ and } B_1 \ge 6k - 3.$$
 (4.5)

Otherwise, we have

$$3k^{2} = 4 + A_{1} + A_{2} + B_{1} + B_{2}$$

$$\leq 4 + 12k - 13 + 6k - 4 + 12k - 13 + 6k - 4$$

$$= 36k - 30,$$

which contradicts $k \ge 12$.

Suppose that 3k - 1 is odd. If $\mu \in D(3k^2, 2)$ satisfies one of the four conditions in (4.5), then deciding $(\gamma_k, \mu) \in K$ can be reduced to γ_{k-2} or γ_{k-1} . More precisely, if $A_1 \ge 6k - 3$ or $B_1 \ge 6k - 3$, then from the proof of Proposition 4.24 we know that $(\gamma_{k-1}, \mu - (6k - 3)) \in K$ or $(\gamma_{k-1}, \mu' - (6k - 3)) \in K$ implies $(\gamma_k, \mu) \in K$. Suppose that $A_2 \ge 12k - 12$ and let $\nu = \mu - (6k - 6, 6k - 6) \vdash 3(k - 2)^2$. If 3k - 1 is odd, by Lemma 4.29 we have

$$((3k-1, 3k-3), (3k-2, 3k-2)) \in K$$
 and $((3k-3, 3k-5), (3k-4, 3k-4)) \in K$.

and from $(3k - 3, 3k - 5)' = (2^{3k-5}, 1, 1)$ we have

$$((2^{3k-5}, 1, 1), (3k-4, 3k-4)) \in K.$$

Observe that

$$\gamma_k = \left(\gamma_{k-2} + (2^{3k-5}, 1, 1)\right) \cup (3k-1, 3k-3).$$

If we let $(\gamma_{k-2}, \nu) \in K$, then

$$(\gamma_{k-2} + (2^{3k-5}, 1, 1), \nu + (3k-4, 3k-4)) \in K.$$

Denote $\overline{\gamma_{k-2}} = \gamma_{k-2} + (2^{3k-5}, 1, 1)$. Then we have

$$(\overline{\gamma_{k-2}} \cup (3k-1, 3k-3), \nu + (3k-4, 3k-4) + (3k-2, 3k-2)) = (\gamma_k, \mu) \in K.$$

Similarly, if $B_2 \ge 12k - 12$, then $(\gamma_{k-2}, \mu' - (6k - 6, 6k - 6)) \in K$ implies $(\gamma_k, \mu) \in K$. \Box

Suppose that 3k - 1 is even. By Lemma 4.29 we have g((3k - 1, 3k - 3), (3k - 1, 3k - 3), (3k - 2, 3k - 2)) = 0. So the argument in Lemma 4.30 is not suitable for the even number 3k - 1. However, we have $((3k - 1, 3k - 3, 3k - 5, 3k - 7), (6k - 8, 6k - 8)) \in K$. So we can also obtain result that is similar to Lemma 4.30. Since the discussions are similar, we just outline the proof here.

Lemma 4.31. Suppose that $\mu \in D(3k^2, 2)$, where $k \ge 19$ and 3k - 1 is even. Then deciding $(\gamma_k, \mu) \in K$ can be reduced to γ_{k-1} or γ_{k-4} .

Proof. If $k \ge 19$, then one of the following conditions must occur:

$$A_2 \ge 24k - 48, \ B_2 \ge 24k - 48, \ A_1 \ge 6k - 3 \text{ and } B_1 \ge 6k - 3.$$
 (4.6)

Otherwise, we have

$$3k^{2} = 4 + A_{1} + A_{2} + B_{1} + B_{2}$$

$$\leq 4 + 24k - 49 + 6k - 4 + 24k - 49 + 6k - 4$$

$$= 60k - 102,$$

which contradicts $k \ge 19$.

Suppose that 3k - 1 is even. If $\mu \in D(3k^2, 2)$ satisfies one of the four conditions in (4.6), then deciding $(\gamma_k, \mu) \in K$ can be reduced to γ_{k-4} or γ_{k-1} . We just discuss the condition when $A_2 \ge 24k - 48$. Suppose that $A_2 \ge 24k - 48$ and let $\nu = \mu - (12k - 24, 12k - 24) \vdash 3(k-4)^2$. If 3k - 1 is even, by Lemma 4.29 we have

$$((3k-1, 3k-3, 3k-5, 3k-7), (6k-8, 6k-8)) \in K,$$

and

$$((3k - 5, 3k - 7, 3k - 9, 3k - 11), (6k - 16, 6k - 16)) \in K$$

So by $(3k - 5, 3k - 7, 3k - 9, 3k - 11)' = (4^{3k-11}, 3, 3, 2, 2, 1, 1)$, we have
 $((4^{3k-11}, 3, 3, 2, 2, 1, 1), (6k - 16, 6k - 16)) \in K.$

Observe that

$$\gamma_k = \left(\gamma_{k-4} + (4^{3k-11}, 3, 3, 2, 2, 1, 1)\right) \cup (3k-1, 3k-3, 3k-5, 3k-7).$$

So if we let $(\gamma_{k-4}, \nu) \in K$, then $(\gamma_k, \mu) \in K$.

Thus, by Lemma 4.30 and 4.31, for $\mu \in D(3k^2, 2)$ we have the following proposition.

Proposition 4.32. Suppose that $(\gamma_k, \mu) \in K$ for all k such that $1 \le k \le 18$ and all $\mu \in D(3k^2, 2)$. Then for all $k \ge 19$ and $\mu \in D(3k^2, 2)$, we also have $(\gamma_k, \mu) \in K$.

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5. FINAL REMARKS AND PROBLEMS

5.1. The distribution of Durfee sizes. In [6], the authors found an asymptotic formula for |D(n, k)|. By Corollary 1 there, we can see that for each fixed k the proportion |D(n, k)|/|P(n)| tends to zero if $n \to \infty$. Moreover, the authors showed that the sequence $\{|D(n, k)|\}, 0 \le k \le \lfloor \sqrt{n} \rfloor$, is asymptotically normal, unimodal, and log concave. The most likely size of the Durfee square for a partition in P(n) is asymptotic to $(\sqrt{6} \log 2/\pi)(\sqrt{n})$.

5.2. **Relation with Permutohedron.** By definition $\Lambda(\mu) = \{v | v \leq \mu, v \in P(n)\}$. In literature (see e.g. [22, Sec. 3.1]), $\Lambda(\mu)$ (resp. $V(\mu)$) is called the *principal order ideal* generated by μ (resp. *principal dual order ideal* generated by μ). Suppose that $\mu = (\mu_1, \mu_2, \dots, \mu_n) \vdash n$. If $\ell(\mu) < n$, we let $\mu_i = 0$ for $i > \ell(\mu)$. For $k \geq \ell(\mu)$, μ can be viewed as a vector in \mathbb{R}^k . For each $k \geq \ell(\mu)$, define the *k-th permutohedron* $P_k(\mu) \subseteq \mathbb{R}^k$ by

 $P_k(\mu) := \text{ConvexHull}\{(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \dots, \mu_{\sigma(k)}) | \sigma \in S_k\}.$

It is well known that $\Lambda(\mu) \subseteq P_n(\mu)$, the *n*-th permutohedron $P_n(\mu)$ associated to μ (see e. g. [7, 20]).

By the results in [20], can we give an estimate of $\Lambda(\mu)$? It is nontrival, since when $\mu = (n)$, we have $\Lambda((n)) = |P(n)|$ whose asymptotic estimation was given by Hardy and Ramanujan [1]. $P_m(\rho_{m-1})$ is called *the regular permutohedron* in [20]. It was shown that $P_m(\rho_{m-1})$ is a *graphical zonotope*. It would be interesting to decide whether $P_{\frac{m(m-1)}{2}}(\rho_{m-1})$ is a graphical zonotope or not.

5.3. Irreducible characters vanishing on ρ_m and $\widehat{\rho_m}$. If $\chi^{\lambda}(\rho_m)$ and $\chi^{\lambda}(\widehat{\rho_m}) = 0$, by Pak and Bessenrodt's criteria we can't decide whether $[\lambda]$ appears in $[\rho_m] \otimes [\rho_m]$ or not [3, 16]. Thus Pak and Bessenrodt's criteria lead us to study the non-zero character values on a fixed conjugacy class which correspond to the non-zero elements on columns of the character table. It can also be seen to connect to work on nonvanishing conjugacy classes, that is, conjugacy classes on which no irreducible character vanishes. A partition is a nonvanishing partition if it labels a nonvanishing conjugacy class of a symmetric group.

Let $N(\mu)$ denote the number of irreducible characters χ^{λ} such that $\chi^{\lambda}(\mu) = 0$. It has been shown that any nonvanishing partition should be of the form $(3^a, 2^b, 1^c)$ for some $a, b, c \ge 0$ [9, 15]. Thus we should have $N(\rho_m)$, $N(\widehat{\rho_m}) > 0$ for $m \ge 4$. By known results (see e. g. Remark 4.5 in [3]), we can see that irreducible characters vanishing on $\widehat{\rho_m}$ may be more than ρ_m . Thus we have the following problem.

Problem 5.1. Does it hold that $N(\rho_m) < N(\widehat{\rho_m})$ for $m \ge 3$?

The following problem gives an upper bound of non-zero elements on columns of the character table of S_n . It was mentioned in [15] and raised by A. Evseev.

Problem 5.2. [15] Let $\pi \in S_n$. Does it always hold that the number of irreducible characters of S_n not vanishing on π is at most equal to the number of irreducible characters of $C_{S_n}(\pi)$?

5.4. Some complexity observations. It was shown in [17, Thm 7.1] that for any λ , ν deciding whether $\chi^{\lambda}(\nu) = 0$ is NP-hard. It can be reduced to the classical NP-complete Knapsack problem. Expanding the power sum function p_{μ} into Schur functions s_{λ} (see [22, Cor. 7.17.4]) we have $p_{\mu} = \sum_{\lambda \in P(n)} \chi^{\lambda}(\mu) s_{\lambda}$. Hence, the irreducible character values on μ are the coefficients in the expansion of p_{μ} . In [2, Cor. 4.3], the authors showed that there exist probabilistic polynomial time algorithms for computing an expansion of a given power sum p_{μ} . That is, there exist probabilistic polynomial time algorithms for computing

the set $\{\chi^{\lambda}(\mu) \mid \lambda \in P(n)\}$. Similarly, in [10] it was shown that deciding whether $\chi^{\lambda}(\mu) = 0$ can be done in probabilistic polynomial time.

5.5. **The Staircase-like partition.** For each *n*, we consider self-conjugate partitions that are close to staircase partitions as follows. For each $n \in \mathbb{N}$, there exist *m*, *k* such that $n = \frac{m(m+1)}{2} + k$ where $0 \le k \le m$. It is not hard to verify the following conditions.

(5.5.1) If *m* is even, then for each *k* there are self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+1}$.

(5.5.2) If *m* is odd and *k* is even, then for each *k* there are self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+1}$.

(5.5.3) If *m* and *k* are odd, then no self-conjugate partitions of *n* lie between ρ_m and ρ_{m+1} . But we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_{m-1} \subseteq \lambda \subseteq \rho_{m+2}$.

In fact, if k = 1 we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_{m-1} \subseteq \lambda \subseteq \rho_{m+1}$. For example, if n = 7, then m = 3 and k = 1. We let $\lambda = (4, 1, 1, 1)$. If k = 3, 5, 7... we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+2}$. For example, if n = 18, then m = 5 and k = 3. We let $\lambda = (5, 4, 4, 4, 1)$.

Definition 5.3. We will call it a *staircase-like partition* if a self-conjugate partition satisfies one of three conditions in (5.5.1), (5.5.2) and (5.5.3) above.

There exist staircase-like partitions for each $n \ge 3$. If *n* is a triangular number, i.e. of the form m(m + 1)/2, then the corresponding staircase-like partition is just ρ_m . Comparing with Conjecture 1.1 of [16], we propose a generalised Saxl Conjecture as follows.

Conjecture 1 (*Generalised Saxl Conjecture*). For any λ of size *n* different from 2, 4, 9, if λ is a staircase-like partition, then $[\lambda] \otimes [\lambda]$ contains all irreducible representations of S_n as constituents.

With the help of computer, we can easily verify Conjecture 1 for $n \le 35$. It is interesting to see that Bessenrodt et al. also proposed a generalised Saxl Conjecture which is related to *p*-cores [4].

If λ is self-conjugate, then in several ways we can add 1 or 2 boxes on λ to make it become another self-conjugate partition. For example, adding a box on $\rho_4 = (4, 3, 2, 1)$ we get (4, 3, 3, 1) which is self-conjugate. Adding 2 boxes on $\rho_4 = (4, 3, 2, 1)$ we get a self-conjugate partition (5, 3, 2, 1, 1). We would like to know the growth behavior of Kronecker coefficient as the growth of partitions. We raise the following problem. Related discussions can be found in [5].

Problem 5.4. For $\lambda \vdash n$, suppose that $[\lambda] \otimes [\lambda]$ contains all irreducible representations of S_n as constituents. By adding at most 2 boxes on λ we get another self-conjugate partition μ (not uniquely) such that $\lambda \subseteq \mu$ and $|\mu \setminus \lambda| = 2$ (or 1). Does there always exist some μ such that $[\mu] \otimes [\mu]$ also contains all irreducible representations of S_{n+2} (or S_{n+1}) as constituents?

If Conjecture 1 is true, then it can be viewed as a special case of Problem 5.4. In fact, we can get ρ_m by adding 1 or 2 boxes on ρ_{m-1} step by step. In each step, we add 1 or 2 boxes on ρ_{m-1} such that the partition is staircase-like. For example, when m = 5, one typical process can be illustrated by Young diagrams as follows.



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