Steiner 3-diameter, maximum degree and size of a graph *

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Abstract

The Steiner k-diameter $sdiam_k(G)$ of a graph G, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural generalization of the concept of classical diameter. When k=2, $sdiam_2(G)=diam(G)$ is the classical diameter. The problem of determining the minimum size of a graph of order n whose diameter is at most d and whose maximum is ℓ was first introduced by Erdös and Rényi. In this paper, we generalize the above problem for Steiner k-diameter, and study the problem of determining the minimum size of a graph of order n whose Steiner 3-diameter is at most d and whose maximum degree is at most ℓ .

Keywords: Diameter; Steiner diameter; maximum degree.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to [7] for graph theoretical notation and terminology not described here. For a graph G, let V(G), E(G) and e(G) denote the set of vertices, the set of edges and the size of G, respectively. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

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1.1 Distance and its generalizations

Distance is one of the most basic concepts of graph-theoretic subjects. If G is a connected graph and $u, v \in V(G)$, then the distance d(u, v) between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the eccentricity e(v) of v is defined by $e(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the radius rad(G) and $diameter\ diam(G)$ of G are defined by $rad(G) = \min\{e(v) \mid v \in V(G)\}$ and $diam(G) = \max\{e(v) \mid v \in V(G)\}$. These last two concepts are related by the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$. The center C(G) of a connected graph G is the subgraph induced by the vertices u of G with e(u) = rad(G). Recently, Goddard and Oellermann gave a survey paper on this subject, see [29].

The distance between two vertices u and v in a connected graph G also equals the minimum size of a connected subgraph of G containing both u and v. This observation suggests a generalization of distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph G(V, E) and a set $S \subseteq V(G)$ of at least two vertices, an S-Steiner tree or a Steiner tree connecting S (or simply, an S-tree) is a such subgraph T(V', E') of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the Steiner distance $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. Note that if S is a connected subgraph of S such that $S \subseteq V(H)$ and $S \subseteq V(H) = S \subseteq V(H)$, then $S \subseteq V(H) = S \subseteq V(H)$ is nothing new but the classical distance between S and S set S when there is no S-Steiner tree in S.

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when there is no S-Steiner tree in G.

Observation 1.1 Let G be a graph of order n and k be an integer with $2 \le k \le n$. If $S \subseteq V(G)$ and |S| = k, then $d_G(S) \ge k - 1$.

Let n and k be two integers with $2 \le k \le n$. The Steiner k-eccentricity $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The Steiner k-radius of G is $srad_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, while the Steiner k-diameter of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note for every connected graph G that $e_2(v) = e(v)$ for all vertices v of G and that $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$.

Let G be a k-connected graph and u, v be a pair of vertices of G. Let $P_k(u,v) = \{P_1, P_2, \dots, P_k\}$ be a family of k internally vertex-disjoint paths between u and v and $l(P_k(u,v))$ be the length of the longest path in $P_k(u,v)$. Then the k-distance $d_k(u,v)$ between vertices u and v is the smallest $l(P_k(u,v))$ among all $P_k(u,v)$'s and the k-diameter $d_k(G)$ of G is the maximum k-distance $d_k(u,v)$ over all pairs u,v of vertices of G. The concept of k-diameter has its origin in the analysis of routings in networks as described by Chung [12], Du, Lyuu and Hsu [16], Hsu [34, 35], Meyer and Pradhan [42].

1.2 Application backgrounds

Perhaps the most famous Steiner type problem is the Steiner tree problem. The original Steiner tree problem was stated for the Euclidean plane: Given a set of points on the plane, the goal is to connect these points, and possibly additional points, by line segments between some pairs of these points such that the total length of these line segments is minimized. The graph theoretical version [30, 36] is as follows: Given a graph and a set of vertices S, find a connected subgraph with minimum number of edges that contains S. This is, in general, an NP-hard problem [33]. There is also a corresponding weighted version. Obviously, this has applications in computer science and electrical engineering. For example, a graph can be a computer network with vertices being computers and edges being links between them. Here the Steiner tree problem is to find a subnetwork containing these computers with the least number of links. We can replace processors by electrical stations for applications in electrical networks.

Li et al. [24] gave such a concept. They defined the k-center Steiner Wiener index $SW_k(G)$ of the graph G to be

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \ |S| = k}} d(S).$$

For k = 2, it coincides with the ordinary Wiener index. One usually considers SW_k for $2 \le k \le n-1$. However, the above definition can be extended to k = 1 and k = n as well where

 $SW_1(G) = 0$ and $SW_n(G) = n - 1$. There are other related concepts such as the Steiner Harary index. Both indices have chemical applications [20, 23]. In addition, the Steiner degree distance by Gutman [22], Steiner Harary index by Furtula, Gutman, Katanić [20], Steiner Gutman index by Mao and Das [39], Steiner hyper-Wiener index by Tratnik [44] was introduced and studied. We refer the readers to [20, 23, 22, 24, 24, 39, 40, 41] for details.

1.3 Recent progress of Steiner distance

In [14], Dankelmann, Swart and Oellermann obtained a bound on $sdiam_k(G)$ for a graph G in terms of the order of G and the minimum degree of G, that is, $sdiam_k(G) \leq \frac{3p}{\delta+1} + 3n$. Later, Ali, Dankelmann, Mukwembi [2] improved the bound of $sdiam_k(G)$ and showed that $sdiam_k(G) \leq \frac{3p}{\delta+1} + 2n - 5$ for all connected graphs G. Moreover, they constructed graphs to show that the bounds are asymptotically best possible.

Arunandhi, Cheng and Melekian [3] studied the Steiner k-diameters of the tensor product of complete graphs.

The following observation is easily seen.

Observation 1.2 Let k be an integer with $2 \le k \le n$.

- (1) If H is a spanning subgraph of G, then $sdiam_k(G) \leq sdiam_k(H)$.
- (2) For a connected graph G, $sdiam_k(G) \leq sdiam_{k+1}(G)$.

In [10], Chartrand, Okamoto, Zhang obtained the following results.

Theorem 1.1 [10] Let G be a connected graph of order n. Then

$$k-1 \le sdiam_k(G) \le n-1.$$

Moreover, the bounds are sharp.

1.4 Classical extremal problem and our generalization

What is the minimal size of a graph of order n and diameter d? What is the maximal size of a graph of order n and diameter d? It is not surprising that these questions can be answered without the slightest effort (see [5]) just as the similar questions concerning the connectivity or the chromatic number of a graph. The class of maximal graphs of order n and diameter d is easy to describe and reduce every question concerning maximal graphs to a not necessarily easy question about binomial coefficient, as in [31, 32, 43, 45]. Therefore, the authors study the minimal size of a graph of order n and under various additional conditions.

Erdös and Rényi [18] introduced the following problem. Let d, ℓ and n be natural numbers, d < n and $\ell < n$. Denote by $\mathcal{H}(n, \ell, d)$ the set of all graphs of order n with maximum degree ℓ and diameter at most d. Put

$$e(n,\ell,d) = \min\{e(G) : G \in \mathcal{H}(n,\ell,d)\}.$$

If $\mathcal{H}(n,\ell,d)$ is empty, then, following the usual convention, we shall write $e(n,\ell,d) = \infty$. For more details on this problem, we refer to [5, 6, 18, 19].

We now consider the generalization of the above problem. Let d, ℓ and n be natural numbers, d < n and $\ell < n$. Denote by $\mathscr{H}_k(n, \ell, d)$ the set of all graphs of order n with maximum degree ℓ and $sdiam_k(G) \leq d$. Put

$$e_k(n,\ell,d) = \min\{e(G) : G \in \mathcal{H}_k(n,\ell,d)\}.$$

If $\mathscr{H}_k(n,\ell,d)$ is empty, then, following the usual convention, we shall write $e_k(n,\ell,d) = \infty$. From Theorem 1.1, we have $k-1 \le d \le n-1$.

In this paper, we focus our attention on the case k = 3, and study the exact value of $e_3(n, \ell, d)$ for d = n - 1, n - 2, n - 3, 2, 3. For general d, we give an upper bound of $e_3(n, \ell, d)$.

2 Preliminaries

The following observation is immediate.

Observation 2.1 [37] (1) For a cycle C_n , $sdiam_k(C_n) = \left\lfloor \frac{n(k-1)}{k} \right\rfloor$; (2) For a complete graph K_n , $sdiam_k(K_n) = k-1$.

The following result can be easily proved, which will be used later.

Theorem 2.1 For $2 \le \ell \le n-1$ and $3 \le k \le n$,

$$e_k(n, \ell, n-1) = n-1.$$

Proof. For $\ell = 2$, let G be a path of order n. Since $sdiam_k(G) \leq n-1$, $\Delta(G) = 2$ and e(G) = n-1, it follows that $e_k(n, \ell, n-1) \leq n-1$. On the other hand, since we only consider connected graphs, it follows that $e(G) \geq n-1$ for a connected graph G of order n. So $e_k(n, \ell, n-1) = n-1$, as desired.

Suppose $3 \le \ell \le n-1$. Let G be a graph obtained from a star S_ℓ and a path $P_{n-\ell+1}$ by identifying the center of the star and one endpoint of the path. Since $\Delta(G) = \ell$, $sdiam_k(G) \le n-1$ and e(G) = n-1, it follows that $e_k(n,\ell,n-1) \le n-1$. On the other hand, since we only consider the connected graph, it follows that $e(G) \ge n-1$ for a connected graph G is of order n. So $e_k(n,\ell,n-1) = n-1$. The proof is now complete.

Lemma 2.1 Let T be a tree of order n with r leaves in T. If $r \geq 4$, then $sdiam_3(T) \leq n - r + 2$.

Proof. Let v_1, v_2, \dots, v_r be all the leaves of T. For any $S \subseteq V(T)$ and |S| = 3, there are at least r-3 leaves in T not belonging to S. Pick up r-3 of these vertices of degree 1 and then delete them. The resulting graph is also a tree of order n-(r-3)=n-r+3, say T'. By our choosing, it is clear that $S \subseteq V(T')$, that is, the tree T' is an S-Steiner tree. Therefore, $d_G(S) \le e(T') = n-r+3-1 = n-r+2$. From the arbitrariness of S, we have $sdiam_3(T) \le n-r+2$. The proof is now complete.

Lemma 2.2 Let n, d, ℓ be three integers with $2 \le d \le n-2$ and $n-d+2 \le \ell \le n-2$. Then

$$e_3(n,\ell,d) = n - 1.$$

Proof. Let G be a graph obtained from a star S_{ℓ} and a path $P_{n-\ell+1}$ by identifying the center of the star and one end of the path. Clearly, $\Delta(G) = \ell$ and G has ℓ leaves. Note that $\ell \geq n-d+2 \geq 4$. From Lemma 2.1, $sdiam_3(G) \leq n-\ell+2 \leq n-(n-d+2)+2=d$. Therefore, this graph shows that $e_3(n,\ell,d) \leq n-1$. On the other hand, we only consider connected graphs, which implies $e_3(n,\ell,d) \geq n-1$. So $e_3(n,\ell,d) = n-1$.

The following result is from [37].

Lemma 2.3 [37] Let G be a connected graph of order $n \ (n \ge 3)$. Then $sdiam_3(G) = n-1$ if and only if G satisfies one of the following conditions.

- (i) $G = T_{a,b,c}$, where $T_{a,b,c}$ is a tree with a vertex v of degree 3 such that $T_{a,b,c} v = P_a \cup P_b \cup P_c$, where $0 \le a \le b \le c$.
- (ii) $G = C_3(a, b, c)$, where $C_3(a, b, c)$ is a graph containing a triangle K_3 and satisfying $C_3(a, b, c) V(K_3) = P_a \cup P_b \cup P_c$, where $0 \le a \le b \le c$.

Corollary 2.1 Let G be a connected graph of order n. If $sdiam_3(G) \leq n-2$ and $\Delta(G) = 2$, then G is cycle of order n.

Lemma 2.4 [37] Let G be a connected graph of order n. Then $sdiam_3(G) = 2$ if and only if $0 \le \Delta(\overline{G}) \le 1$ if and only if $n - 2 \le \delta(G) \le n - 1$.

3 For small d

From Theorem 1.1, we have $2 \le d \le n-1$. In this section, we discuss the cases d=2 and d=3.

3.1 The case d = 2

If $sdiam_3(G) = 2$, then it follows from Lemma 2.4 that

$$n - 2 \le \delta(G) \le n - 1,\tag{4.1}$$

and hence $n-2 \le \Delta(G) \le n-1$. So we assume that $n-2 \le \ell \le n-1$.

Theorem 3.1 (1) For $\ell = n - 1$, $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n-1}{2}$ for n odd; $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n-2}{2}$ for n even.

(2) For
$$\ell = n - 2$$
, $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n}{2}$ for n even; $e_3(n, \ell, 2) = \infty$ for n odd.

Proof. (1) For n odd, we let G be a graph obtained from a complete graph K_n by deleting a maximum matching M. Clearly, $|M| = \frac{n-1}{2}$ and $\Delta(G) = n-1$. From Lemma 2.4, we have $sdiam_3(G) = 2$, and hence $e_3(n, n-1, 2) \leq \binom{n}{2} - \frac{n-1}{2}$ for n odd. We claim that $e_3(n, n-1, 2) = \binom{n}{2} - \frac{n-1}{2}$. Assume, to the contrary, that $e_3(n, n-1, 2) \leq \binom{n}{2} - \frac{n-1}{2} - 1$. Then there exists a graph G such that $sdiam_3(G) = 2$, $\Delta(G) = n-1$ and $e(G) \leq \binom{n}{2} - \frac{n-1}{2} - 1$. Clearly, $\Delta(\overline{G}) \geq 2$ and hence $\delta(G) = n-1 - \Delta(\overline{G}) \leq n-3$, which contradicts to (4.1). So $e_3(n, n-1, 2) = \binom{n}{2} - \frac{n-1}{2}$.

For n even, let G be a graph obtained from a complete graph K_n by deleting a matching M such that $|M| = \frac{n-2}{2}$. Obviously, $\Delta(G) = n-1$. From Lemma 2.4, we have $sdiam_3(G) = 2$, and hence $e_3(n, n-2, 2) \leq \binom{n}{2} - \frac{n-2}{2}$ for n even. We claim that $e_3(n, n-2, 2) = \binom{n}{2} - \frac{n-2}{2}$. Assume, to the contrary, that $e_3(n, n-1, 2) \leq \binom{n}{2} - \frac{n-2}{2} - 1$. Then there exists a graph G such that $sdiam_3(G) = 2$, $\Delta(G) = n-1$ and $e(G) \leq \binom{n}{2} - \frac{n-2}{2} - 1$. Clearly, $\Delta(\overline{G}) \geq 1$, $\delta(G) = n-1 - \Delta(\overline{G}) \leq n-2$, and hence $\delta(G) = n-2$ by (4.1). Since $e(G) \leq \binom{n}{2} - \frac{n}{2}$ and n is even, it follows that G is a graph obtained from a complete graph K_n by deleting a perfect matching, which implies $\Delta(G) = n-2$, a contradiction. So $e_3(n, n-1, 2) = \binom{n}{2} - \frac{n-1}{2}$.

(2) For n even, let G be a graph obtained from a complete graph K_n by deleting a perfect matching M. Obviously, $|M| = \frac{n}{2}$, $\Delta(G) = n - 2$ and $sdiam_3(G) = 2$. Therefore, $e_3(n, n - 2, 2) \leq \binom{n}{2} - \frac{n}{2}$ for n even. We claim that $e_3(n, n - 2, 2) = \binom{n}{2} - \frac{n}{2}$. Assume, to the contrary, that $e_3(n, n - 2, 2) \leq \binom{n}{2} - \frac{n}{2} - 1$. Then there exists a graph G such that $sdiam_3(G) = 2$, $\Delta(G) = n - 2$ and $e(G) \leq \binom{n}{2} - \frac{n}{2} - 1$. Clearly, $\Delta(\overline{G}) \geq 2$, $\delta(G) = n - 1 - \Delta(\overline{G}) \leq n - 3$, a contradiction. So $e_3(n, n - 1, 2) = \binom{n}{2} - \frac{n}{2}$.

For n odd, let G be a graph such that $sdiam_3(G)=2$ and $\Delta(G)=n-2$. Since $sdiam_3(G)=2$, it follows that $n-2 \leq \delta(G) \leq n-1$. Since $\Delta(G)=n-2$, it follows that $\delta(G)=\Delta(G)=n-2$, and hence G is (n-2)-regular graph of order n, which is impossible. So $e_3(n,n-2,2)=\infty$ for n odd.

3.2 The case d = 3

The following proposition and lemma are immediate.

Proposition 3.1 Let K_{n_1,n_2,\dots,n_r} be a complete r-partite graph with $n_1 \leq n_2 \leq \dots \leq n_r$. Then

$$\operatorname{sdiam}_{k}(K_{n_{1},n_{2},\cdots,n_{r}}) = \begin{cases} k-1, & \text{if } k > n_{r} \\ k, & \text{if } k \leq n_{r}. \end{cases}$$

Proof. Set $G=K_{n_1,n_2,\cdots,n_r}$. Let V_1,V_2,\cdots,V_r be the parts of complete r-partite graph G, and set $|V_i|=n_i$ $(1\leq i\leq r)$. Suppose $k>n_r$. Since $n_1\leq n_2\leq\cdots\leq n_r$, it follows that for any $S\subseteq V(G)$ and |S|=k, there exist two parts V_i,V_j such that $S\cap V_i\neq\varnothing$ and $S\cap V_j\neq\varnothing$. Set $S=\{v_1,v_2,\cdots,v_k\}$. Without loss of generality, let $S\cap V_i=\{v_1,v_2,\cdots,v_s\}$ and $S\cap V_j=\{v_{s+1},v_{s+2},\cdots,v_t\}$. Then the tree induced by the edges in

$$\{v_1v_i \mid s+1 \le i \le k\} \cup \{v_{s+1}v_i \mid 2 \le i \le s\}$$

is an S-Steiner tree, and hence $d_G(S) \leq k-1$, and hence $\operatorname{sdiam}_k(G) \leq k-1$. From Theorem 1.1, we have $\operatorname{sdiam}_k(G) = k-1$.

Suppose $k \leq n_r$. Choose $S \subseteq V(G)$ and |S| = k such that $S \subseteq V_r$. Observe that any S-Steiner tree must use at least k edges. Therefore, $d_G(S) \geq k$, and hence $\operatorname{sdiam}_k(G) \geq k$. One can easily check that $\operatorname{sdiam}_k(G) \leq k$. So $\operatorname{sdiam}_k(G) = k$, as desired.

Lemma 3.1 Let T be a tree of order $n \ (n \ge 5)$. Then $sdiam_3(T) = 3$ if and only if T is a star.

Proof. If T is a star, then $sdiam_3(T)=3$. Conversely, we suppose $sdiam_3(T)=3$. Suppose that T contains a path P as its subgraph such that $|V(P)| \geq 3$ and each vertex in P is an internal vertex of T. Let u, v be two endpoints of P. Then there exit two leaves, say x, y, such that $xu \in E(T)$ and $yw \in E(T)$. Choose $S = \{x, u, y\}$. Then $d_G(S) \geq 4$ and hence $sdiam_3(T) \geq 4$, a contradiction. Suppose that T has exactly two internal vertices of T. Since $n \geq 5$, it follows that T has at least three leaves. Choose three of them as S. Then $d_G(S) \geq 4$ and hence $sdiam_3(T) \geq 4$, a contradiction. We conclude that T has exactly one internal vertex and hence T is a star.

We first give an upper bound of our parameter for general ℓ and d=3.

Lemma 3.2 For $\frac{n}{2} \le \ell \le n - 4$,

$$n \le e_3(n, \ell, 3) \le \ell(n - \ell).$$

Proof. Let $K_{\ell,n-\ell}$ be a complete bipartite graph. Since $\ell \geq \frac{n}{2}$, it follows that $\ell \geq n-\ell$, and hence $\Delta(G) = \ell$. From Proposition 3.1, we have $sdiam_3(G) = 3$. So $e_3(n,\ell,3) \leq \ell(n-\ell)$.

Let G be a graph with $sdiam_3(G) \leq 3$ and $\frac{n}{2} \leq \Delta(G) = \ell \leq n-4$. If $sdiam_3(G) = 2$, then it follows from Lemma 2.4 that $n-3 \leq \delta(G) \leq n-2$, which contradicts $\Delta(G) = \ell \leq n-4$. Suppose $sdiam_3(G) = 3$. From Lemma 3.1, if G is tree, then G is a star, and hence $\Delta(G) = n-1$, a contradiction. So $e_3(n,\ell,3) \geq n$.

Lemma 3.3 For $\ell = n - 1$, $e_3(n, \ell, 3) = n - 1$.

Proof. For $\ell = n - 1$, let G be a star of order n. Then $sdiam_3(G) = 3$ and $\Delta(G) = n - 1$, and hence $e_3(n, n - 1, 3) \le n - 1$. Since we only consider the connected graph, it follows that $e_3(n, n - 1, 3) \ge n - 1$. So $e_3(n, n - 1, 3) = n - 1$.

Lemma 3.4 For $\ell = n - 2$, $e_3(n, \ell, 3) = 2n - 5$.

Proof. For $\ell=n-2$, let $G=K_{2,n-2}^-$ be a graph obtained from a complete bipartite graph $K_{2,n-2}$ by deleting an edge; see Figure 3.1 (a). Let u, v, x_{n-2} be the vertices of degree n-1, n-2, 1 in $K_{2,n-2}^-$, respectively. Set $X = V(G) - \{u, v, x_{n-2}\} = \{x_1, x_2, \dots, x_{n-3}\}.$ Now, we show $sdiam_3(G) \leq 3$. It suffices to prove that $d_G(S) \leq 3$ for any $S \subseteq V(G)$ and |S|=3. If $|S\cap X|=3$, without loss of generality, let $S=\{x_1,x_2,x_3\}$, then the tree T induced by the edges in $\{ux_1, ux_2, ux_3\}$ is an S-Steiner tree and hence $d_G(S) \leq$ 3. If $|S \cap X| = 0$, then $S = \{u, v, x_{n-2}\}$, then the tree T induced by the edges in $\{ux_1, vx_1, ux_{n-2}\}\$ is an S-Steiner tree and hence $d_G(S) \leq 3$. Suppose $|S \cap X| = 2$. Then $|S \cap \{u, v, x_{n-2}\}| = 1$. Without loss of generality, let $S = \{x_1, x_2, u\}$ or $S = \{x_1, x_2, v\}$ or $S = \{x_1, x_2, x_{n-2}\}$. If $S = \{x_1, x_2, u\}$, then the tree T_1 induced by the edges in $\{ux_1, ux_2\}$ is a Steiner tree connecting $\{x_1, x_2, u\}$. If $S = \{x_1, x_2, v\}$, then the tree T_2 induced by the edges in $\{vx_1, vx_2\}$ is a Steiner tree connecting $\{x_1, x_2, v\}$. If $\{x_1, x_2, x_{n-2}\}$, then the tree T_3 induced by the edges in $\{ux_1, ux_2, ux_{n-2}\}$ is a Steiner tree connecting $\{x_1, x_2, x_{n-2}\}$. Therefore, $d_G(S) \leq 3$. Suppose $|S \cap X| = 1$. Then $|S \cap \{u, v, x_{n-2}\}| = 2$. Without loss of generality, let $S = \{x_1, u, v\}$ or $S = \{x_1, u, x_{n-2}\}$ or $S = \{x_1, v, x_{n-2}\}$. If $S = \{x_1, u, v\}$, then the tree T_1 induced by the edges in $\{ux_1, vx_1\}$ is a Steiner tree connecting $\{x_1, u, v\}$. If $\{x_1, u, x_{n-2}\}$, then the tree T_2 induced by the edges in $\{vx_1, ux_1, ux_{n-2}\}$ is a Steiner tree connecting $\{x_1, u, x_{n-2}\}$. If $\{x_1, v, x_{n-2}\}$, then the tree T_2 induced by the edges in $\{ux_1, ux_{n-2}\}\$ is a Steiner tree connecting $\{x_1, v, x_{n-2}\}\$. Therefore, $d_G(S) \leq 3$. From the above argument, we conclude that $d_G(S) \leq 3$ for any $S \subseteq V(G)$ and |S| = 3. Since

We now show $e_3(n, n-2, 3) \geq 2n-5$. Let G be a graph with $\Delta(G) = n-2$ and $sdiam(G) \leq 3$. Since $\Delta(G) = \ell = n-2$, there exists a vertex u such that $d_G(u) = n-2$. Then $|N_G(u)| = n-2$ and $\{u\} \cup N_G(u) \subseteq V(G)$. Clearly, there is a vertex $v \in V(G)$ such that $uv \notin E(G)$. Let $X' = V(G) \setminus \{u, v\} = \{x_1, x_2, \dots, x_{n-2}\}$. Clearly, $|X'| \geq 3$. Since G is connected, it follows that there exists a vertex, say $x_j \in X$ such that $vx_j \in E(G)$.

 $sdiam_3(G) \leq 3$ and $\Delta(G) = n - 2$, we have $e_3(n, n - 2, 3) \leq 2n - 5$.

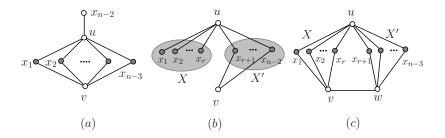


Figure 3.1 Graphs for Lemma 3.4.

Without loss of generality, let $vx_{n-2} \in E(G)$. Observe that for each x_i $(1 \le i \le n-3)$ we have $vx_i \in E(G)$ or $vx_i \notin E(G)$. Without loss of generality, let $vx_1, vx_2, \cdots, vx_r \notin E(G)$ and $vx_{r+1}, vx_{r+2}, \cdots, vx_{n-3} \in E(G)$, where $0 \le r \le n-3$. Set $X = \{x_1, x_2, \cdots, x_r\}$. For each two vertices $x_i, x_j \in X$, we choose $S = \{x_i, x_j, v\}$. Since $sdiam_3(G) \le 3$, it follows that any S-Steiner tree contains at most 3 edges. Therefore, there exists a vertex $x_k \in \{x_{r+1}, x_{r+2}, \cdots, x_{n-3}\}$ such that $x_i x_k, x_j x_k \in E(G)$ or $x_i x_j, x_j x_k \in E(G)$ or $x_i x_j, x_j x_k \in E(G)$. For r even, $e(G) \ge (n-2) + (n-2-r) + r = 2n-4$. For r odd, $e(G) \ge (n-2) + (n-2-r) + (r-1) = 2n-5$. So $e_3(n, n-2, 3) \ge 2n-5$.

From the above argument, we conclude that $e_3(n, n-2, 3) = 2n - 5$.

Lemma 3.5 For $\ell = n - 3$, $e_3(n, n - 3, 3) = 2n - 5$.

Proof. Let G be a graph defined as follows; see Figure 3.1 (c).

$$V(G) = \{u, v, w\} \cup \{x_i \mid 1 \le i \le n - 3\}$$

$$E(G) = \{ux_i \mid 1 \le i \le n - 3\} \cup \{vx_i \mid 1 \le i \le r\}$$

$$\cup \{wx_i \mid r + 1 \le i \le n - 3\} \cup \{vw\}.$$

One can check that $sdiam_3(G) = 3$ and $\Delta(G) = n - 3$. Therefore, $e_3(n, n - 3, 3) \le 2n - 5$.

We only need to show $e_3(n, n-3, 3) \geq 2n-5$. Let G be a connected graph with $\Delta(G) = n-3$ and $sdiam_3(G) = 3$. It suffices to show that $e(G) \geq 2n-5$. Since $\Delta(G) = \ell = n-3$, there exists a vertex u such that $d_G(u) = n-3$, and hence $|N_G(u)| = n-3$ and $\{u\} \cup N_G(u) \subseteq V(G)$. Clearly, there exist two vertices $v, w \in V(G)$ such that $v, w \notin N_G(u)$. Let $V(G) \setminus \{u, v, w\} = \{x_1, x_2, \dots, x_{n-3}\}$. We have the following two cases to consider.

Case 1. $vw \in E(G)$

Since G is connected, it follows that there exist two vertices x_j, x_k such that $vx_j \in E(G)$ and $vx_k \in E(G)$ (note that x_j, x_k are not necessarily different). Without loss of generality, let $vx_1, wx_1 \in E(G)$ or $vx_1, wx_2 \in E(G)$. Suppose $vx_1, wx_1 \in E(G)$. For any x_i ($2 \le i \le n-3$), we choose $S = \{x_i, v, w\}$. Since $sdiam_3(G) = 3$, it follows that $x_1x_i \in E(G)$ or

 $vx_i \in E(G)$ or $wx_i \in E(G)$, and hence $e(G) \ge (n-3)+2+(n-4)=2n-5$, as desired. Suppose $vx_1, wx_2 \in E(G)$. For x_3 , we choose $S = \{x_3, v, w\}$. Since $sdiam_3(G) = 3$, it follows that $vx_3, wx_3 \in E(G)$ or $x_3x_1, wx_1 \in E(G)$ or $x_3x_2, vx_2 \in E(G)$. For any x_i $(2 \le i \le n-3)$, we choose $S = \{x_i, v, w\}$. Since $sdiam_3(G) = 3$, it follows that $x_1x_i \in E(G)$ or $x_2x_i \in E(G)$ or $x_3x_i \in E(G)$ or $vx_i \in E(G)$ or $vx_i \in E(G)$, and hence $e(G) \ge (n-3)+4+(n-6)=2n-5$, as desired.

Case 2. $vw \notin E(G)$

Since G is connected, it follows that there exists a vertex x_j such that $vx_j \in E(G)$ or $wx_j \in E(G)$. Without loss of generality, let $vx_1 \in E(G)$. For any x_i $(2 \le i \le n-3)$, we choose $S = \{x_i, v, w\}$. Since $sdiam_3(G) = 3$, it follows that $x_1x_i \in E(G)$ or $vx_i \in E(G)$ or $wx_i \in E(G)$, and hence $e(G) \ge (n-3)+1+1+(n-4)=2n-5$, as desired.

From the argument, we conclude that $e(G) \ge 2n-5$, and hence $e_3(n, n-3, 3) \ge 2n-5$.

We now conclude our results for d = 3.

Theorem 3.2 (1) For $\ell = n - 1$, $e_3(n, n - 1, 3) = n - 1$;

- (2) For $\ell = n-2$, $e_3(n, n-2, 3) = 2n-5$:
- (3) For $\ell = n 3$, $e_3(n, n 3, 3) = 2n 5$;
- (4) For $\ell = 2$, $e_3(n, 2, 3) = 3$ for n = 4; $e_3(n, 2, 3) = 5$ for n = 5; $e_3(n, 2, 3) = \infty$ for $n \ge 6$.
 - (5) For $\frac{n}{2} \le \ell \le n 4$, $n \le e_3(n, \ell, 3) \le \ell(n \ell)$.

Proof. The results in (1)-(3) follow from Lemmas 3.3, 3.4 and 3.5. Let G be a connected graph with $\Delta(G) = 2$ and $sdiam_3(G) = 3$. Then $G = P_n$ or $G = C_n$. If $G = P_n$, then it follows from Lemma 2.3 that $3 = sdiam_3(G) = sdiam_3(P_n) = n - 1$, and hence n = 4. If $G = C_n$, then it follows from Observation 2.1 that $3 = sdiam_3(G) = sdiam_3(C_n) = \lfloor \frac{2n}{3} \rfloor$, and hence n = 5. Furthermore, $e_3(n, 2, 3) = \infty$ for $n \geq 6$. The result in (5) follow from Lemma 3.2.

4 For large d

For d = n - 1 and $4 \le \ell \le n - 1$, we have proved that $e_3(n, \ell, n - 1) = n - 1$. We study the cases k = n - 2, n - 3, n - 4 in this section.

Let T(a,b,c) be a tree obtained from two stars $K_{1,a}, K_{1,c}$ and a path $P = v_1v_2 \dots v_b$ by identifying u and v_1 , w and v_b , where a+b+c=n, u is the center of $K_{1,a}$, and w is the center of $K_{1,c}$. Let $T^*(x,y)$ be a tree obtained from three stars $K_{1,3}, K_{1,3}, K_{1,3}$, and v path $P = v_1v_2 \dots v_b$ by identifying u, v, v_1 and v, v_b , where $v_1v_2 \dots v_b$ by identifying $v_1v_2 \dots v_b$, where $v_1v_2 \dots v_b$ is a leaf of $v_1v_2 \dots v_b$ by identifying $v_1v_2 \dots v_b$, where $v_1v_2 \dots v_b$ is a leaf of another $v_1v_2 \dots v_b$, and $v_1v_3 \dots v_b$ is a leaf of another $v_1v_2 \dots v_b$.

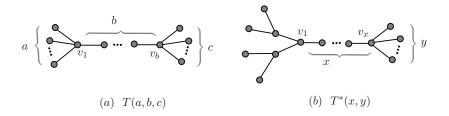


Figure 4.1 Graphs for $n \geq 8$ in Lemma 4.2.

4.1 The case d = n - 2

In this subsection, we study the case d = n - 2.

Theorem 4.1 (1) For $n \ge 4$, $e_3(n, 2, n - 2) = n$.

(2) For $n \geq 4$,

$$e_3(n,3,n-2) = \begin{cases} n+1 & \text{if } n=4, \\ n & \text{if } n=5, \\ n-1 & \text{if } n \ge 6. \end{cases}$$

(3) For $n \ge 5$ and $4 \le \ell \le n - 1$, $e_3(n, \ell, n - 2) = n - 1$.

Proof. (1) For $n \geq 4$, let $G = C_n$ be a cycle of order n. Then $\ell = \Delta(G) = 2$. From Observation 2.1, we have $sdiam_3(G) \leq n-2$ and hence this graph shows $e_3(n,2,n-2) \leq n$. It suffices to show that $e_3(n,2,n-2) \geq n$. Let G be a graph with $\Delta(G) = 2$ and $sdiam_3(G) \leq n-2$. Since $\Delta(G) = \ell = 2$, it follows that $G = P_n$ or $G = C_n$. From Lemma 2.3, we have $sdiam_3(P_n) = n-1$ and $sdiam_3(C_n) \leq n-2$. So $e_3(n,2,n-2) \geq n$ and hence $e_3(n,2,n-2) = n$.

(2) For $n \geq 6$, let G be a graph obtained from three paths $P_1 = u_1u_2u_3$, $P_2 = w_1w_2w_3$ and $P_3 = v_1v_2\cdots v_{n-4}$ by identifying the vertices u_2 and v_1 , and then identifying the vertices w_2 and v_{n-4} . Clearly, $\ell = \Delta(G) = 3$. From Lemma 2.3, we have $sdiam_3(G) \leq n-2$. Therefore, $e_3(n,3,n-2) \leq n-1$ and hence $e_3(n,3,n-2) = n-1$. For n=4, the graph $G = K_4^-$ shows that $e_3(4,3,2) \leq 5$. It suffices to show that $e_3(4,3,2) \geq 5$. Let G be a graph with $\Delta(G) = 2$ and $sdiam_3(G) \leq n-2 = 2$. Since $\ell = \Delta(G) = 3$, there exists a vertex of degree 3, say u_1 . Let $u_1u_2, u_1u_3, u_1u_4 \in E(G)$. Choose $S = \{u_2, u_3, u_4\}$. Since $sdiam_3(G) = 2$, it follows that $u_2u_3, u_3u_4 \in E(G)$. Therefore, $e(G) \geq 5$ and hence $e_3(4,3,2) \geq 5$. So $e_3(4,3,2) = 5$. For n=5, let G be a graph obtained from a cycle C_4 by adding a pendent edge at one vertex of C_4 . One can check that $\Delta(G) = 3$ and $sdiam_3(G) \leq 3$. Therefore, $e_3(5,3,3) \leq 5$. We need to show $e_3(5,3,3) \geq 5$. Let G be a graph with $\Delta(G) = 3$ and $sdiam_3(G) \leq n-2 = 3$. Since $\ell = \Delta(G) = 3$, there exists a vertex of degree 3, say u_1 . Let $u_1u_2, u_1u_3, u_1u_4 \in E(G)$. Since n=5, it follows that there

exists a vertex $u_5 \in V(G)$. Furthermore, there exists some vertex u_j $(2 \le j \le 4)$ such that $u_j u_5 \in E(G)$. Without loss of generality, let $u_2 u_5 \in E(G)$. Choose $S = \{u_3, u_4, u_5\}$. Since $sdiam_3(G) \le 3$, it follows that $u_3 u_5 \in E(G)$, or $u_4 u_5 \in E(G)$, or $u_2 u_3, u_2 u_4 \in E(G)$. Therefore, $e(G) \ge 5$ and hence $e_3(5,3,3) \ge 5$, as desired. So $e_3(5,3,3) = 5$.

(3) For $n \geq 5$ and $4 \leq \ell \leq n-1$, let G be a graph obtained from a star S_{ℓ} and a path $P_{n-\ell+1}$ by identifying the center of the star and one end of the path. Clearly, $\Delta(G) = \ell$. From Lemma 2.3, we have $sdiam_3(G) \leq n-2$ and hence this graph shows $e_3(n,\ell,n-2) \leq n-1$. Since we only consider connected graphs, we have $e_3(n,\ell,n-2) \geq n-1$. So $e_3(n,\ell,n-2) = n-1$.

4.2 The case d = n - 3

Let us now turn to the case d = n - 3.

Lemma 4.1 For $n \geq 5$,

$$e_3(n, 2, n - 3) = \begin{cases} \infty & \text{if } n = 5, 6, \\ n & \text{if } n \ge 7. \end{cases}$$

Proof. For $n \geq 7$ and $\ell = 2$, let $G = C_n$ be a cycle of order n. From Observation 2.1, we have $sdiam_3(G) = \lfloor \frac{2n}{3} \rfloor \leq n-3$ and hence this graph shows $e_3(n,2,n-3) \leq n$. It suffices to prove $e_3(n,2,n-3) \geq n$. Let G be a connected graph with $\Delta(G) = 2$ and $sdiam_3(G) \leq n-3$. From Corollary 2.1, we have $G = C_n$ and hence $e_3(n,2,n-3) \geq n$. So $e_3(n,2,n-3) = n$. For n = 5, 6, we have $G = C_n$ by Corollary 2.1. From Observation 2.1, we have $sdiam_3(G) = \lfloor \frac{2n}{3} \rfloor > n-3$. So $e_3(n,2,n-3) = \infty$ for n = 5, 6.

Lemma 4.2 For $n \geq 5$,

$$e_3(n,3,n-3) = \begin{cases} \infty & \text{if } n = 5, \\ n+1 & \text{if } n = 6,7, \\ n-1 & \text{if } n > 8. \end{cases}$$

Proof. For $n \geq 8$, let $G = T^*(n-7,1)$. Then G has exactly 5 leaves and $\Delta(T') = 3$. From Lemma 2.1, $sdiam_3(T') \leq n-3$. This tree shows that $e_3(n,3,n-3) \leq n-1$ and hence $e_3(n,3,n-3) = n-1$. For n=5, let G be a graph with $\Delta(G) = 3$ and $sdiam_3(G) \leq n-3 = 2$. Furthermore, $sdiam_3(G) = 2$. From Lemma 2.4, we have $3 \leq \delta(G) \leq 4$ and hence $3 \leq \delta(G) \leq \Delta(G) = 3$, which implies that G is 3-regular. The degree sum of graph G is exactly 15, a contradiction. So $e_3(5,3,2) = \infty$.

For n = 6, we let $G = A_4$; see Figure 4.2 (d). One can check that $\Delta(G) = 3$ and $sdiam_3(G) \le n - 3 = 3$. Then $e_3(6,3,3) \le 7$. It suffices to show $e_3(6,3,3) \ge 7$. Let G be

a graph such that $\Delta(G) = 3$ and $sdiam_3(G) \leq 3$. If G is a tree, then $G = A_1$ or $G = A_2$ or $G = A_3$; see Figure 4.2 (a), (b), (c). Clearly, if we choose vertex set S consisting of three black vertices, then $d_G(S) > 3$, which results in $sdiam_3(G) > 3$, a contradiction. So G contains at least one cycle. Furthermore, we have the following claim.

Claim 1. G contains at least two cycles.

Proof of Claim 1. Assume, to the contrary, that G is a unicyclic graph. Let c(G) be the circumference of graph G. Clearly, $3 \le c(G) \le 6$. If c(G) = 6, then $G = C_6$, which contradicts to $\Delta(G) = 3$. We may assume that $3 \le c(G) \le 5$. If c(G) = 5, then $G = A_5$; see Figure 4.2 (e). One can also check that $sdiam_3(G) > 3$, also a contradiction. If c(G) = 4, then $G = A_6$ or $G = A_7$ or $G = A_8$; see Figure 4.2 (f), (g), (h). One can check that $sdiam_3(G) > 3$, a contradiction. For c(G) = 3, one can also check that $sdiam_3(G) > 3$, also a contradiction.

From Claim 1, G contains at least two cycles and hence $e(G) \geq 7$. So $e_3(6,3,3) \geq 7$, as desired.

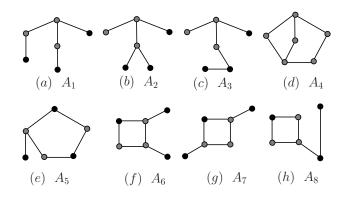


Figure 4.2 Graphs for $n \geq 8$ in Lemma 4.2.

For n = 7, we let $G = B_1$; see Figure 4.3 (a). Then $\Delta(G) = 3$ and $sdiam_3(G) \le n - 3 = 4$. This graph shows that $e_3(7,3,4) \le 8$. It suffices to show that $e_3(7,3,4) \ge 8$. Let G be connected graph of order n with $sdiam_3(G) \le 4$ and $\Delta(G) = 3$.

Claim 2. G contains at least two cycles.

Proof of Claim 2. Assume, to the contrary, that G is a tree or has exactly one cycle. If G is a tree, then $G \in \{B_i \mid 10 \le i \le 14\}$, and hence $sdiam_3(G) > 5$ by choosing the three black vertices as S, a contradiction. We now suppose that G contains exactly one cycle. Let c(G) denote the circumference of G. Clearly, $3 \le c(G) \le 7$. If c(G) = 7, then $G = C_7$, which contradicts to $\Delta(G) = 3$. If c(G) = 6, then $G = B_2$; see Figure 4.3 (b). By choosing the three black vertices as S, we can see that $sdiam_3(G) > 5$, a contradiction. If c(G) = 5, then $G = B_3$ or $G = B_4$ or $G = B_5$; see Figure 4.3 (b). By choosing the

three black vertices as S, we can see that $sdiam_3(G) > 5$, a contradiction. If c(G) = 4, then $G \in \{B_i \mid 6 \le i \le 9\}$; see Figure 4.3 (b). By choosing the three black vertices as S, we can see that $sdiam_3(G) > 5$, a contradiction. For c(G) = 3, one can also check that $sdiam_3(G) > 5$, also a contradiction.

From Claim 2, G contains at least two cycles, and hence $e(G) \geq 8$. So $e_3(n,3,4) = e(G) \geq 8$, as desired.

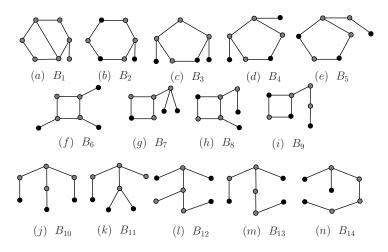


Figure 4.3 Graphs for n = 7 in Lemma 3.2.

Lemma 4.3 For $n \geq 5$,

$$e_3(n,4,n-3) = \begin{cases} \binom{n}{2} - 2 & \text{if } n = 5, \\ n+1 & \text{if } n = 6, \\ n-1 & \text{if } n \ge 7. \end{cases}$$

Proof. For $n \geq 7$, let G = T(3,2,2). The the tree T(3,2,2) have exactly 5 leaves. Clearly, $\Delta(T(3,2,2)) = 4$. From Lemma 2.1, $sdiam_3(T(3,2,2)) \leq n-3$. This tree shows that $e_3(n,4,n-3) \leq n-1$, and hence $e_3(n,4,n-3) = n-1$. For n=5, let G be the graph obtained from a complete graph K_5 by deleting a maximum matching. It is clear that $\Delta(G) = 4$ and $sdiam_3(G) = 2 = n-3$. This graph shows that $e_3(5,4,2) \leq {5 \choose 2} - 2 = 8$. We need to show that $e_3(5,4,2) \geq 8$. Let G be a graph with $\Delta(G) = 4$ and $sdiam_3(G) \leq n-3=2$. From Lemma 2.4, we have $3 \leq \delta(G) \leq 4$. From this together with $\Delta(G) = 4$ and $sdiam_3(G) = 2$, it follows that $e(G) \geq 8$ and hence $e_3(5,4,2) \geq 8$, as desired. So $e_3(5,4,2) = 8$.

For n = 6, the graph D_1 shown in Figure 4.4 (a) satisfies $\Delta(H_1) = 4$ and $sdiam_3(D_1) \le 3 = n - 3$. This graph shows that $e_3(6, 4, 3) \le 7$. It suffices to prove that $e_3(6, 4, 3) \ge 7$.

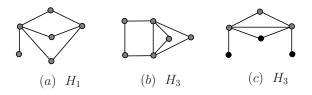


Figure 4.4 Graphs for Lemma 4.3.

Let G be a graph with $\Delta(G)=4$ and $sdiam_3(G)\leq 3$. Since $\Delta(G)=4$, it follows that there exists a vertex of degree 4, say u. Set $N_G(u)=\{x_1,x_2,x_3,x_4\}$. Since n=6, it follows that there is a remaining vertex in G, say v. Because G is connected, the vertex v is adjacent to one of $\{x_1,x_2,x_3,x_4\}$. Without loss of generality, let $vx_4\in E(G)$. Choose $S=\{x_2,x_3,v\}$. Since $sdiam_3(G)\leq 3$, it follows that $x_2v\in E(G)$, or $x_3v\in E(G)$, or $x_2x_4,x_3x_4\in E(G)$. If $x_2v\in E(G)$, then we choose $S=\{x_1,x_3,v\}$ and hence $x_1v\in E(G)$, or $x_3v\in E(G)$, or $x_1x_2,x_3x_2\in E(G)$, or $x_1x_4,x_3x_4\in E(G)$. By symmetry, we only need to consider $x_1v\in E(G)$ or $x_1x_2,x_3x_2\in E(G)$. For the former, we have $x_1v\in E(G)$ and hence $G=D_1$; see Figure 4.4 (b). If $x_1x_2,x_3x_2\in E(G)$, then $G=D_2$; see Figure 4.4 (c). The case $x_3v\in E(G)$ is just the same as $x_2v\in E(G)$, and so we omit its discussion. We may assume that $x_2x_4,x_3x_4\in E(G)$. Clearly, $G=D_3$; see Figure 4.4 (d). Choose the three black vertices to form vertex set S. Then $d_G(S)\geq 4$. So $e_3(6,4,3)=7$ if and only if $G\in \{D_1,D_2\}$.

From Lemmas 3.1, 3.2, 3.3 and 2.2, we have the following result.

Theorem 4.2 (1) For $n \ge 5$,

$$e_3(n, 2, n - 3) = \begin{cases} \infty & \text{if } n = 5, 6, \\ n & \text{if } n \ge 7. \end{cases}$$

(2) For $n \geq 5$,

$$e_3(n,3,n-3) = \begin{cases} \infty, & \text{if } n = 5, \\ n+1 & \text{if } n = 6, \\ n & \text{if } n = 7, \\ n-1 & \text{if } n \ge 8. \end{cases}$$

(3) For $n \ge 5$,

$$e_3(n,4,n-3) = \begin{cases} \binom{n}{2} - 2 & \text{if } n = 5, \\ n+1 & \text{if } n = 6, \\ n-1 & \text{if } n \ge 7. \end{cases}$$

(4) For $n \ge 6$ and $5 \le \ell \le n-1$, $e_3(n, \ell, n-3) = n-1$.

4.3 The case d = n - 4

In this subsection, we consider the case d = n - 4.

Lemma 4.4 For $n \geq 5$,

$$e_3(n, 2, n - 4) = \begin{cases} \infty & \text{if } 5 \le n \le 9, \\ n & \text{if } n \ge 10. \end{cases}$$

Proof. For $n \geq 10$ and $\ell = 2$, let $G = C_n$ be a cycle of order n. From Observation 2.1, we have $sdiam_3(G) = \lfloor \frac{2n}{3} \rfloor \leq n-4$ and hence this graph shows $e_3(n,2,n-4) \leq n$. It suffices to prove $e_3(n,2,n-3) \geq n$. Let G be a connected graph with $\Delta(G) = 2$ and $sdiam_3(G) \leq n-4$. From Corollary 2.1, we have $G = C_n$ and hence $e_3(n,2,n-4) \geq n$. So $e_3(n,2,n-4) = n$. For $5 \leq n \leq 9$, we have $G = C_n$ by Corollary 2.1. From Observation 2.1, we have $sdiam_3(G) = \lfloor \frac{2n}{3} \rfloor > n-4$. So $e_3(n,2,n-4) = \infty$ for $5 \leq n \leq 9$.

Lemma 4.5 For $n \geq 6$,

$$e_3(n,3,n-4) = \begin{cases} \infty & \text{if } n = 6, \\ n+3 & \text{if } n = 7, \\ n+2 & \text{if } n = 8, \\ n+1 & \text{if } n = 9, \\ n-1 & \text{if } n \ge 10. \end{cases}$$

Proof. For $n \geq 10$, let T_3 be a tree of maximum degree 3 with exactly 6 leaves. Clearly, $\Delta(T'') = 3$. From Lemma 2.1, we have $sdiam_3(T_3) \leq n - 4$. This tree shows that $e_3(n,3,n-4) \leq n-1$ and hence $e_3(n,3,n-4) = n-1$. For n=6, let G be a graph with $\Delta(G) = 3$ and $sdiam_3(G) \leq n-4=2$. From Lemma 2.4, we have $4 \leq \delta(G) \leq 5$, which contradicts to the fact $\Delta(G) = 3$. So $e_3(6,3,2) = \infty$.

Suppose $7 \le n \le 9$. For n = 9, let G be a graph obtained from a cycle $C = v_1v_2 \dots v_9$ by adding a new edge v_1v_5 . One can easily check that $sdiam_3(G) \le 5 = n - 4$ and hence this graph shows $e_3(9,3,5) \le 10 = n + 1$. It suffices to show that $e_3(9,3,5) \ge 10 = n + 1$. Let G be a graph with $\Delta(G) = 3$ and $sdiam_3(G) \le 5$. Suppose G is a tree. We claim that G has at most 5 leaves. Assume, to the contrary, that G has t ($6 \le t \le 8$) leaves. Since $\Delta(G) = 3$, it follows that $16 = 2e(G) = \sum_{v \in V(G)} d(v) \le t + 3(9 - t) = 27 - 2t \le 15$, a contradiction. Since $\Delta(G) = 3$, it follows that G has at least 3 leaves and at most 5 leaves. If G is a tree with f (f is a tree with f (f is a tree with f (f is not a tree, and hence f has cycles. Furthermore, we have the following claim.

Claim 1. G contains at least two cycles.

Proof of Claim 1. Assume, to the contrary, that G has exactly one cycle. Let x be the number of vertices of degree 1 in G, and y be the number of vertices of degree 2 in G. We claim that $0 \le x \le 4$. Assume, to the contrary, that $x \ge 5$. Since $\Delta(G) = 3$, it follows that $18 = 2e(G) = \sum_{v \in V(G)} d(v) \le x + 3(9 - x) = 27 - 2x \le 17$, a contradiction. Furthermore, if x = 4, then we claim that y = 1. Assume, to the contrary, that $y \ge 2$. Then $18 = 2e(G) = \sum_{v \in V(G)} d(v) \le x + 2y + 3(9 - x - y) = 27 - 2x - y = 19 - y \le 17$, a contradiction. Similarly, if x = 3, then $y \le 3$.

Suppose x=4 and y=1. Then G is a unicyclic graph obtained by a cycle $C=v_1v_2\dots v_5v_1$ by adding four edges $v_2v_6, v_3v_7, v_4v_8, v_5v_9$. Choose $S=\{v_6, v_7, v_9\}$. Then $d_G(S)\geq 6$, which contradicts $sdiam_3(G)\leq 5$. If x=0, then $G=C_9$, and hence $sdiam_3(G)=6$ by Observation 2.1, a contradiction. If x=1, then G is a unicyclic graph obtained by a cycle $C_r=v_1v_2\dots v_rv_1$ and a path $P_{9-r}=wv_{r+1}\dots v_9$ by identifying the vertex v_1 in C_r and the endvertex w in P_{9-r} . Choose $S=\{v_{\lfloor \frac{r}{3}\rfloor},v_{\lfloor \frac{2r}{3}\rfloor},v_9\}$. Since $3\leq r\leq 8$, it follows that $d_G(S)\geq \lfloor \frac{2r}{3}\rfloor+(9-r)\geq 6$, which contradicts $sdiam_3(G)\leq 5$.

Suppose x=3 and $y \leq 3$. Then $3 \leq r \leq 6$. Let $C_r(a,b,c)$ be a graph obtained from a cycle C_r and three paths P_a, P_b, P_c by adding three edges z_1u_a, z_2u_b, z_3u_c , where $0 \leq a \leq b \leq c$, 9=r+a+b+c-3, z_1, z_2, z_3 are three distinct vertices in C_r, u_a, u_b, u_c are leaves of P_a, P_b, P_c , respectively. If r=3, then $G=C_3(2,2,2)$ or $G=C_3(1,2,3)$ or $G=C_3(1,1,4)$. Choose S consisting of all the three vertices of degree 1 in $G=C_3(2,2,2)$ or $G=C_3(1,2,3)$ or $G=C_3(1,1,4)$. Then $d_G(S) \geq 6$, which contradicts $sdiam_3(G) \leq 5$. If r=4, then $G=C_4(1,2,2)$ or $G=C_4(1,1,3)$. Choose S consisting of all the three vertices of degree 1 in $G=C_4(1,2,2)$ or $G=C_4(1,1,3)$. Then $d_G(S) \geq 6$, which contradicts $sdiam_3(G) \leq 5$. If r=5, then $G=C_5(1,1,2)$. Choose S consisting of all the three vertices of degree 1 in $G=C_5(1,1,2)$. Then $d_G(S) \geq 6$, which contradicts $sdiam_3(G) \leq 5$. If r=6, then $G=C_6(1,1,1)$. One can easily check that $sdiam_3(G) \geq 6$, a contradiction.

Suppose x = 2. We define two graph classes as follows.

- Let \mathcal{G}_9^1 be a graph class, each graph G of which is a unicyclic graph obtained by a cycle $C_r = z_1 z_2 \dots z_r v_1$ and two paths $P_s = u_1 u_2 \dots u_s$, $P_t = v_1 v_2 \dots v_t$ by identifying a vertex z_i of C_r and the endvertex u_1 of P_s , and then identifying the other vertex z_j of C_r and the endvertex v_1 of P_t , where 0 = r + s + t 2, $0 \le r \le t$ and $0 \le t \le t$.
- Let \mathcal{G}_9^2 be a graph class, each graph G of which is a unicyclic graph obtained by a cycle $C_r = z_1 z_2 \dots z_r z_1$ and $T_{a,b,c}$ (see Lemma 2.3) by identifying a vertex of C_r and a leaf of $T_{a,b,c}$, where 9 = r + a + b + c 3 and $3 \le r \le 6$.

For $G \in \mathcal{G}_9^1$, z_i and z_j divide the cycle C_r into two paths $Q_r^1 = z_i z_{i+1} \dots z_j$ and $Q_r^2 = z_j z_{j+1} \dots z_r \dots z_i$. Without loss of generality, let $e(Q_r^1) \geq e(Q_r^2)$. Then $e(Q_r^1) \geq \lceil \frac{r}{2} \rceil$. Choose $S = \{u_s, v_t, z_k\}$ where z_k is an internal vertex of Q_r^1 . Since $1 \leq r \leq r$, it follows

that $d_G(S) \geq \lceil \frac{r}{2} \rceil + (9-r) \geq 6$, which contradicts $sdiam_3(G) \leq 5$.

For $G \in \mathcal{G}_9^2$, let u, v, w be the three leaves in $T_{a,b,c}$ and w be the identifying vertex in C_r . Without loss of generality, let $w = v_1$. Choose $S = \{u, v, v_{\lfloor \frac{r}{2} \rfloor}\}$. Then $d_G(S) \ge \lfloor \frac{r}{2} \rfloor + (9-r) \ge 6$, which contradicts $sdiam_3(G) \le 5$.

From Claim 1, we conclude that $e_3(9,3,5) = 10 = n + 1$.

For n = 7, let G be a graph obtained from a cycle $C = v_1v_2...v_7$ by adding three new edges v_1v_4, v_2v_5, v_3v_6 . One can check that $sdiam_3(G) \leq 3 = n - 4$ and hence this graph shows $e_3(7,3,3) \leq 10 = n + 3$. Similarly to the proof of n = 9, we can prove that $e_3(7,3,3) \geq 10 = n + 3$. So $e_3(7,3,3) = 10 = n + 3$. For n = 8, let G be a graph obtained from a cycle $C = v_1v_2...v_8$ by adding two new edges v_1v_5, v_3v_7 . One can check that $sdiam_3(G) \leq 4 = n - 4$ and hence this graph shows $e_3(8,3,4) \leq 10 = n + 2$. Similarly to the proof of n = 9, we can prove that $e_3(8,3,4) \geq 10 = n + 2$. So $e_3(8,3,4) = 10 = n + 2$.

Lemma 4.6 For $n \geq 6$,

$$e_3(n,4,n-4) = \begin{cases} \binom{n}{2} - 3 & \text{if } n = 6, \\ n+2 & \text{if } n = 7, \\ n-1 & \text{if } n \ge 8. \end{cases}$$

Proof. For $n \geq 8$, let G = T(3, n-6, 3). Then G has exactly 6 leaves and $\Delta(G) = 4$. From Lemma 2.1, $sdiam_3(G) \leq n-4$. This tree shows that $e_3(n, 4, n-4) \leq n-1$ and hence $e_3(n, 4, n-4) = n-1$. For n = 7, from (3) of Theorem 3.2, we have $e_3(7, 4, 3) = 9$. For n = 6, let G be a graph with $\Delta(G) = 4$ and $sdiam_3(G) \leq n-4=2$. From Lemma 2.4, we have $4 \leq \delta(G) \leq 5$, and hence G is 4-regular. Clearly, G is a graph obtained from K_6 by deleting a perfect matching. Therefore, $e(G) = \binom{6}{2} - 3 = 12$ and $e_3(6,3,2) = 12 = \binom{n}{2} - 3$, as desired.

Lemma 4.7 For $n \geq 6$,

$$e_3(n,5,n-4) = \begin{cases} 2n+1 & \text{if } n = 6, \\ n+2 & \text{if } n = 7, \\ n-1 & \text{if } n \ge 8. \end{cases}$$

Proof. For $n \geq 8$, let G = T(4, n-6, 2). Then G has exactly 6 leaves and $\Delta(T_5) = 5$. From Lemma 2.1, $sdiam_3(G) \leq n-4$. This tree shows that $e_3(n, 5, n-4) \leq n-1$ and hence $e_3(n, 5, n-4) = n-1$. For n = 7, from (2) of Theorem 3.2, we have $e_3(7, 5, 3) = 9$. For n = 6, we let G be the graph obtained from a complete graph K_6 by deleting a matching of size 2. It is clear that $\Delta(G) = 5$ and $sdiam_3(G) = 2 = n-4$. This graph shows that $e_3(6,5,2) \leq {6 \choose 2} - 2 = 13$. We need to show that $e_3(6,5,2) \geq 13$. Let G be a graph with $\Delta(G) = 5$ and $sdiam_3(G) \leq n-4 = 2$. From Lemma 2.4, we have $4 \leq \delta(G) \leq 5$. From

this together with $\Delta(G) = 5$ and $sdiam_3(G) = 2$, it follows that $e(G) \geq 13$ and hence $e_3(6,5,2) \geq 13$. So $e_3(6,5,2) = 13$, as desired.

Theorem 4.3 (1) For $n \ge 5$,

$$e_3(n, 2, n - 4) = \begin{cases} \infty & \text{if } 5 \le n \le 9, \\ n & \text{if } n \ge 10. \end{cases}$$

(2) For $n \ge 6$,

$$e_3(n,3,n-4) = \begin{cases} \infty & \text{if } n = 6, \\ n+3 & \text{if } n = 7, \\ n+2 & \text{if } n = 8, \\ n+1 & \text{if } n = 9, \\ n-1 & \text{if } n \ge 10. \end{cases}$$

(3) For $n \ge 6$,

$$e_3(n,4,n-4) = \begin{cases} 2n & \text{if } n = 6, \\ n+2 & \text{if } n = 7, \\ n-1 & \text{if } n \ge 8. \end{cases}$$

(4) For $n \ge 6$,

$$e_3(n,5,n-4) = \begin{cases} 2n+1 & \text{if } n = 6, \\ n+2 & \text{if } n = 7, \\ n-1 & \text{if } n \ge 8. \end{cases}$$

(5) For $n \ge 7$ and $6 \le \ell \le n-1$, $e_3(n, \ell, n-4) = n-1$.

5 For general d

We now construct a graph and give an upper bound of $e_3(n, \ell, d)$ for general ℓ and d.

Proposition 5.1 For $4 \le d \le n-1$ and $2 \le \ell \le n-1$,

$$e_3(n,\ell,d) \le \frac{(n-d+1)(n-d+2)}{2} + d - 3.$$

Proof. Let U_p, W_q be two cliques of order p, q, respectively, where $p \geq q$ and p + q = n - d + 1. Set $V(U_p) = \{u_1, u_2, \dots, u_p\}$ and $V(W_q) = \{w_1, w_2, \dots, w_q\}$. Let G be a graph defined as follow.

$$\begin{split} V(G) &= V(U_p) \cup V(W_q) \cup \{v_i \mid 1 \leq i \leq d-1\} \\ E(G) &= E(U_p) \cup E(W_q) \cup \{v_1 u_i : 1 \leq i \leq p\} \cup \{v_2 w_i : 1 \leq i \leq p\} \\ &\cup \{u_i w_j : 1 \leq i \leq p, \ 1 \leq j \leq q\} \cup \{v_i v_{i+1} : 2 \leq i \leq d-2\} \end{split}$$

It is clear that |V(G)| = n, $\Delta(G) = p + q = n - d + 1$ and

$$|E(G)| = {p \choose 2} + {q \choose 2} + p + q + pq + d - 3$$

$$= {(p+q)^2 \over 2} + {p+q \over 2} + d - 3$$

$$= {(n-d+1)(n-d+2) \over 2} + d - 3.$$

We need to show that $sdiam_3(G) \leq d$. It suffices to prove that $d_G(S) \leq d$ for any $S \subseteq V(G)$ and |S| = 3. Set $X = \{v_1, v_2, \dots, v_{d-1}\}$. If $S \subseteq X$, the tree induced by the edges in $\{v_1u_1, u_1w_1, u_2w_1\} \cup \{v_iv_{i+1} \mid 2 \leq i \leq d-2\}$ is an S-Steiner tree, and hence $d_G(S) \leq d$. If $|S \cap X| = 2$, then $|S \cap V(U_p)| = 1$ or $|S \cap V(W_q)| = 1$. Without loss of generality, let $S \cap V(U_p) = \{u_i\}$. Then the tree induced by the edges in $\{v_1u_i, u_iw_1, u_2w_1\} \cup$ $\{v_i v_{i+1} \mid 2 \leq i \leq d-2\}$ is an S-Steiner tree, and hence $d_G(S) \leq d$. Suppose $|S \cap X| = 1$. Then $|S \cap V(U_p)| = |S \cap V(W_q)| = 1$ or $|S \cap V(U_p)| = 2$ or $|S \cap V(W_q)| = 2$. We first consider the case $|S \cap V(U_p)| = |S \cap V(W_q)| = 1$. Without loss of generality, let $S \cap V(U_p) = \{u_j\}$ and $S \cap V(W_q) = \{w_k\}$. If $S = \{u_j, w_k, v_1\}$, then the tree induced by the edges in $\{u_iv_1, u_iw_k\}$ is an S-Steiner tree, and hence $d_G(S) \leq 2 < d$. If $v_1 \notin S$, then the tree induced by the edges in $\{u_i w_k, w_k u_2\} \cup \{v_i v_{i+1} \mid 2 \le i \le d-2\}$ is an S-Steiner tree, and hence $d_G(S) < d$. Next, we consider the case $|S \cap V(U_p)| = 2$. Without loss of generality, let $S \cap V(U_p) = \{u_j, u_k\}$. If $S = \{u_j, u_k, v_1\}$, then induced by the edges in $\{u_iv_1, u_kv_1\}$ is an S-Steiner tree, and hence $d_G(S) \leq 2 < d$. If $v_1 \notin S$, then the tree induced by the edges in $\{u_k w_1, u_j w_1, w_1 u_2\} \cup \{v_i v_{i+1} \mid 2 \le i \le d-2\}$ is an S-Steiner tree, and hence $d_G(S) \leq d$. In the end, we consider the case $|S \cap V(W_q)| = 2$. Without loss of generality, let $S \cap V(W_q) = \{w_j, w_k\}$. If $S = \{u_j, u_k, v_1\}$, then induced by the edges in $\{w_j u_1, w_k u_1, v_1 u_1\}$ is an S-Steiner tree, and hence $d_G(S) \leq 3 < d$. If $v_1 \notin S$, then the tree induced by the edges in $\{w_k v_2, w_j v_2\} \cup \{v_i v_{i+1} \mid 2 \leq i \leq d-2\}$ is an S-Steiner tree, and hence $d_G(S) < d$. We conclude that $e_3(n, \ell, d) \le \frac{(n-d+1)(n-d+2)}{2} + d - 3$.

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