# CYCLES ON A MULTISET WITH ONLY EVEN-ODD DROPS 

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#### Abstract

For a finite subset $A$ of $\mathbb{Z}_{>0}$, Lazar and Wachs (2019) conjectured that the number of cycles on $A$ with only even-odd drops is equal to the number of D-cycles on $A$. In this note, we introduce cycles on a multiset with only even-odd drops and prove bijectively a multiset version of their conjecture. As a consequence, the number of cycles on $[2 n]$ with only even-odd drops equals the Genocchi number $g_{n}$. With Laguerre histories as an intermediate structure, we also construct a bijection between a class of permutations of length $2 n-1$ known to be counted by $g_{n}$ invented by Dumont and the cycles on $[2 n]$ with only even-odd drops.


## 1. Introduction

The Genocchi numbers

$$
\left\{g_{n}\right\}_{n \geq 1}=\{1,1,3,17,155,2073,38227,929569, \ldots\}
$$

can be defined by the following exponential generating function formula

$$
\sum_{n \geq 1} g_{n} \frac{x^{2 n}}{(2 n)!}=x \tan \frac{x}{2}
$$

Let $\mathfrak{S}_{n}$ be the set of permutations of $[n]:=\{1,2, \ldots, n\}$. Dumont [2] introduced the class of permutations

$$
\mathfrak{D}_{2 n-1}:=\left\{\sigma \in \mathfrak{S}_{2 n-1}: \forall i \in[2 n-2], \sigma(i)>\sigma(i+1) \text { if and only if } \sigma(i) \text { is even }\right\}
$$

and proved that

$$
\begin{equation*}
\left|\mathfrak{D}_{2 n-1}\right|=g_{n} . \tag{1.1}
\end{equation*}
$$

For instance, $\mathfrak{D}_{5}=\{42135,21435,34215\}$ and so $\left|\mathfrak{D}_{5}\right|=3=g_{3}$. Based on Dumont's interpretation of Genocchi numbers, the objective of this paper is to present two different bijective proofs of a conjecture due to Lazar and Wachs [5, Conjecture 6.4] which asserts that cycles on $[2 n]$ with only even-odd drops are also counted by $g_{n}$. Actually, we will also introduce cycles on a multiset with only even-odd drops and prove bijectively a multiset generalization of another related conjecture of Lazar and Wachs [5, Conjecture 6.5].

Let $M$ be a finite multiset with exactly $m$ elements from $\mathbb{Z}_{>0}$. We introduce cycles on $M$ with only even-odd drops and D-cycles on $M$, generalizing the concepts introduced in [5] for $M$ being a subset of $\mathbb{Z}_{>0}$.

[^0]Key words and phrases. Genocchi numbers; Even-odd drops; D-cycles; Laguerre histories.

Definition 1. A cycle $\left(a_{1}, a_{2}, \ldots, a_{m}\right)$ on $M$ (i.e., as multiset $\left.\left\{a_{i}: 1 \leq i \leq m\right\}=M\right)$ is said to have only even-odd drops if whenever $a_{i}>a_{i+1}$ (as a cycle, we usually assume $a_{m+1}=a_{1}$ ), then $a_{i}$ is even and $a_{i+1}$ is odd; it is a $\boldsymbol{D}$-cycle if $a_{i} \leq a_{i+1}$ when $a_{i}$ is odd and $a_{i} \geq a_{i+1}$ when $a_{i}$ is even for all $1 \leq i \leq m$. For example, as cycles on the multiset $\left\{1^{2}, 2^{2}, 3^{2}, 4^{2}\right\},(1,2,1,2,4,3,3,4)$ has only even-odd drops but is not a D-cycle, while $(1,2,1,4,3,3,4,2)$ is a D-cycle that does not have only even-odd drops. Let $\mathcal{E C}_{M}$ be the set of cycles on $M$ with only even-odd drops and let $\mathcal{D} \mathcal{C}_{M}$ be the set of D-cycles on $M$.

Note that in order for $\mathcal{E C}_{M}$ or $\mathcal{D C}_{M}$ to be non empty, the smallest element in $M$ must be odd and the greatest element in $M$ must be even. When $M=[n]$, we simply write $\mathcal{E C}_{n}$ and $\mathcal{D C}_{n}$ for $\mathcal{E C}_{[n]}$ and $\mathcal{D} \mathcal{C}_{[n]}$, respectively. For example,

$$
\begin{aligned}
\mathcal{E C}_{6} & =\{(1,2,3,4,5,6),(1,2,4,3,5,6),(1,2,5,6,3,4)\}, \\
\mathcal{D C}_{6} & =\{(1,3,5,6,4,2),(1,4,3,5,6,2),(1,5,6,3,4,2)\}
\end{aligned}
$$

The next result proves [5, Conjecture 6.4] bijectively ${ }^{1}$ in view of Dumont's result (1.1).
Theorem 2. There exists two bijections, $\Phi$ and $\Psi$, between $\mathcal{E C}_{2 n}$ and $\mathfrak{D}_{2 n-1}$.
The construction of $\Phi$ is based on the classical Françon-Viennot bijection [3] that encodes permutations as Laguerre histories, while $\Psi$ is the composition of the bijection $\psi$ below with a simple transformation.

Theorem 3. For a fixed multiset $M$, there exists a bijection $\psi$ between $\mathcal{E C}_{M}$ and $\mathcal{D C}_{M}$.
For $M$ being a subset of $\mathbb{Z}_{>0}$, the above theorem proves [5, Conjecture 6.5] bijectively. Thus, Theroem 3 is a multiset generalization of [5, Conjecture 6.5]. As an immediate consequence of Theorems 2 and 3 and Dumont's result (1.1), we have

Corollary 4. For $n \geq 1,\left|\mathcal{E C}_{2 n}\right|=\left|\mathcal{D} \mathcal{C}_{2 n}\right|=g_{n}$.
Remark 5. Let $\mathcal{E C}_{2 n}^{(k)}:=\mathcal{E C}_{M}$ with $M=\left\{1^{k}, 2^{k}, \ldots,(2 n)^{k}\right\}$. Since $\left|\mathcal{E C}_{2 n}\right|=g_{n},\left|\mathcal{E C}_{2 n}^{(k)}\right|$ can be considered as a new generalization of the Genocchi numbers. For another generalization of the Genocchi numbers using the model of trees, the reader is referred to [4]. Can the generating function for $\left|\mathcal{E C}_{2 n}^{(k)}\right|$ be calculated? Is there any divisibility property for $\left|\mathcal{E C}_{2 n}^{(k)}\right|$ similar to $g_{n}$ (see [4])?

The rest of this paper is organized as follows. After recalling the Françon-Viennot bijection, we construct $\Phi$ in Section 2. In Section 3, we first present the bijection $\psi$ for Theorem 3 and then use it to construct $\Psi$. Finally, in Section 4, we provide an InclusionExclusion approach to Dumont's result (1.1) for the sake of completeness.

## 2. The construction of $\Phi$

In order to construct $\Phi$, we need to recall the classical Françon-Viennot bijection [3] first. A Motzkin path of length $n$ is a lattice path in the first quadrant starting from ( 0,0 ), ending at ( $n, 0$ ), and using three possible steps:

$$
U=(1,1)(\text { up step }), L=(1,0) \text { (level step) and } D=(1,-1) \text { (down step). }
$$

[^1]A 2-Motzkin path is a Motzkin path in which each level step is further distinguished into two different types of level steps $L_{0}$ or $L_{1}$. A 2 -Motzkin paths will be represented as a word over the alphabet $\left\{U, D, L_{0}, L_{1}\right\}$. A Laguerre history of length $n$ is a pair $(w, \mu)$, where $w=w_{1} \cdots w_{n}$ is a 2 -Motzkin path and $\mu=\left(\mu_{1}, \cdots, \mu_{n}\right)$ is a vector satisfying $0 \leq \mu_{i} \leq h_{i}(w)$, where

$$
h_{i}(w):=\#\left\{j \mid j<i, w_{j}=U\right\}-\#\left\{j \mid j<i, w_{j}=D\right\}
$$

is the height of the starting point of the $i$-th step of $w$. Denote by $\mathfrak{L}_{n}$ the set of all Laguerre histories of length $n$.

For a permutation $\sigma \in \mathfrak{S}_{n}$, a letter $\sigma(i)$ is called a valley (resp. peak, double descent, double ascent) of $\sigma$ if $\sigma(i-1)>\sigma(i)<\sigma(i+1)$ (resp. $\sigma(i-1)<\sigma(i)>\sigma(i+1)$, $\sigma(i-1)>\sigma(i)>\sigma(i+1), \sigma(i-1)<\sigma(i)<\sigma(i+1)$ ), where we use the assumption $\sigma(0)=\sigma(n+1)=0$. For a fixed $1 \leq k \leq n-1$, define

$$
(2-13)_{k} \sigma=\#\{i: i-1>j \text { and } \sigma(i-1)<\sigma(j)=k<\sigma(i)\}
$$

The Françon-Viennot bijection $\phi_{F V}: \mathfrak{S}_{n} \rightarrow \mathfrak{L}_{n-1}$ that we need is the following modified version (see for example [7]) defined as $\phi_{F V}(\sigma)=(w, \mu) \in \mathfrak{L}_{n-1}$, where for each $i \in[n-1]$ :

$$
w_{i}= \begin{cases}U & \text { if } i \text { is a valley of } \sigma \\ D & \text { if } i \text { is a peak of } \sigma \\ L_{0} & \text { if } i \text { is a double ascent of } \sigma \\ L_{1} & \text { if } i \text { is a double descent of } \sigma\end{cases}
$$

and $\mu_{i}=(2-13)_{i}(\sigma)$. For example, if $\sigma=528713649 \in \mathfrak{S}_{9}$, then $\phi_{F V}(\sigma)=(w, \mu)$, where $w=U U L_{0} U D D L_{1} D$ and $\mu=(0,1,0,0,3,1,1,1)$. The reverse algorithm $\phi_{F V}^{-1}$ building a permutation $\sigma$ from a Laguerre history $(w, \mu) \in \mathfrak{L}_{n-1}$ can be described iteratively as:

- Initialization: $\sigma=0$;
- At the $i$-th $(1 \leq i \leq n-1)$ step of the algorithm, replace the $\left(h_{i}(w)+1\right)$-th $\circ$ (from right to left) of $\sigma$ by

$$
\begin{cases}\text { oio } & \text { if } w_{i}=U \\ i \circ & \text { if } w_{i}=L_{0} \\ \text { oi } & \text { if } w_{i}=L_{1} \\ i & \text { if } w_{i}=D\end{cases}
$$

- The final permutation is obtained by replacing the last remaining $\circ$ by $n$.

For example, if $(w, \mu)=\left(U L_{1} U D L_{0} D,(0,1,1,2,0,0)\right) \in \mathfrak{L}_{6}$, then $\sigma$ is built as follows:

$$
\sigma=\circ \rightarrow \circ 1 \circ \rightarrow \circ 21 \circ \rightarrow \circ 3 \circ 21 \circ \rightarrow 43 \circ 21 \circ \rightarrow 43 \circ 215 \circ \rightarrow 43 \circ 2156 \rightarrow 4372156 .
$$

Now we begin to construct $\Phi$ step by step. Let $\mathfrak{S}_{2 n-1}^{o e}$ be the set of permutations $\sigma \in$ $\mathfrak{S}_{2 n-1}$ with only odd-even descents (i.e., whenever $\sigma(i)>\sigma(i+1)$, then $\sigma(i)$ is odd and $\sigma(i+1)$ is even) and whose last entry is odd. For instance, $\mathfrak{S}_{5}^{o e}=\{12345,13245,14523\}$.
Lemma 6. There exists a bijection $\eta: \mathcal{E C}_{2 n} \rightarrow \mathfrak{S}_{2 n-1}^{o e}$.
Proof. For a cycle $\alpha=\left(a_{1}, a_{2}, \ldots, a_{2 n}\right) \in \mathcal{E C}_{2 n}$ with $a_{1}=1$, define $\eta(\alpha)$ to be the permutation $a_{2}-1, a_{3}-1, \ldots, a_{2 n}-1$ (in one line notation), which is clearly in $\mathfrak{S}_{2 n-1}^{o e}$. It is easy to see that $\eta$ sets up an one-to-one correspondence between $\mathcal{E} \mathcal{C}_{2 n}$ and $\mathfrak{S}_{2 n-1}^{o e}$.

Let us consider the subset $M_{2 n}$ of Laguerre histories $(w, \mu) \in \mathfrak{L}_{2 n}$ with the restriction that

$$
w_{i}= \begin{cases}D \text { or } L_{0}, & \text { if } i \text { is odd } \\ U \text { or } L_{0}, & \text { if } i \text { is even }\end{cases}
$$

For instance,

$$
M_{4}=\left\{\left(L_{0} U D L_{0},(0,0,0,0)\right),\left(L_{0} U D L_{0},(0,0,1,0)\right),\left(L_{0} L_{0} L_{0} L_{0},(0,0,0,0)\right)\right\}
$$

Lemma 7. The Françon-Viennot bijection $\phi_{F V}$ restricts to a bijection between $\mathfrak{S}_{2 n-1}^{o e}$ and $M_{2 n-2}$.
Proof. This follows from the observation that $\sigma \in \mathfrak{S}_{2 n-1}$ is a permutation in $\mathfrak{S}_{2 n-1}^{o e}$ if and only if for each $i \in[2 n-2]$, the letter $i$ of $\sigma$ is a double ascent or a peak whenever $i$ is odd and is a double ascent or a valley whenever $i$ is even.

Let us consider another subset $M_{2 n}^{*}$ of Laguerre histories $(w, \mu) \in \mathfrak{L}_{2 n}$ with the restriction that

$$
w_{i}= \begin{cases}U \text { or } L_{0}, & \text { if } i \text { is odd } \\ D \text { or } L_{1}, & \text { if } i \text { is even }\end{cases}
$$

and $1 \leq \mu_{i} \leq h_{i}(w)$ when $i$ is even. For instance,

$$
M_{4}^{*}=\left\{(U D U D,(0,1,0,1)),\left(U L_{1} L_{0} D,(0,1,0,1)\right),\left(U L_{1} L_{0} D,(0,1,1,1)\right)\right\}
$$

Lemma 8. The Françon-Viennot bijection $\phi_{F V}$ restricts to a bijection between $\mathfrak{D}_{2 n-1}$ and $M_{2 n-2}^{*}$.
Proof. Observe that $\sigma \in \mathfrak{S}_{2 n-1}$ is a permutation in $\mathfrak{D}_{2 n-1}$ if and only if (i) $\sigma(2 n-1)=2 n-1$ and (ii) for each $i \in[2 n-2]$, the letter $i$ of $\sigma$ is a double ascent or a valley whenever $i$ is odd and is a double descent or a peak whenever $i$ is even. Thus, if $\sigma \in \mathfrak{D}_{2 n-1}$, then $\phi_{F V}(\sigma) \in M_{2 n-2}^{*}\left(\right.$ as $\sigma(2 n-1)=2 n-1$ forces $\mu_{i} \geq 1$ when $i$ is even). Conversely, if $(w, \mu) \in M_{2 n-2}^{*}$, then as $\mu_{i} \geq 1$ when $i$ is even, it follows from the iterative construction of $\sigma=\phi_{F V}^{-1}(w, \mu)$ that the o at the end of $\sigma$ remains until the last step, i.e., $\sigma(2 n-1)=2 n-1$. Therefore, we have $\phi_{F V}^{-1}(w, \mu) \in \mathfrak{D}_{2 n-1}$ for any $(w, \mu) \in M_{2 n-2}^{*}$.
Lemma 9. There exists a bijection $\rho: M_{2 n} \rightarrow M_{2 n}^{*}$.


Figure 1. The construction of $\rho$ : a dashed level step represents $L_{1}$.
Proof. For a Laguerre history $(w, \mu) \in M_{2 n}$, we construct $\rho(w, \mu)=\left(w^{\prime}, \mu^{\prime}\right)$ by transforming each consecutive two steps $\left(w_{2 i-1}, w_{2 i}\right)$ and their weights $\left(\mu_{2 i-1}, \mu_{2 i}\right)=(x, y)(1 \leq i \leq n)$ according to the following four cases (see Fig. 1):

- If $\left(w_{2 i-1}, w_{2 i}\right)=\left(L_{0}, L_{0}\right)$, then $\left(w_{2 i-1}^{\prime}, w_{2 i}^{\prime}\right)=(U, D)$ and $\left(\mu_{2 i-1}^{\prime}, \mu_{2 i}^{\prime}\right)=(x, y+1)$;
- If $\left(w_{2 i-1}, w_{2 i}\right)=(D, U)$, then $\left(w_{2 i-1}^{\prime}, w_{2 i}^{\prime}\right)=\left(L_{0}, L_{1}\right)$ and $\left(\mu_{2 i-1}^{\prime}, \mu_{2 i}^{\prime}\right)=(x, y+1)$;
- If $\left(w_{2 i-1}, w_{2 i}\right)=\left(L_{0}, U\right)$, then $\left(w_{2 i-1}^{\prime}, w_{2 i}^{\prime}\right)=\left(U, L_{1}\right)$ and $\left(\mu_{2 i-1}^{\prime}, \mu_{2 i}^{\prime}\right)=(x, y+1)$;
- If $\left(w_{2 i-1}, w_{2 i}\right)=\left(D, L_{0}\right)$, then $\left(w_{2 i-1}^{\prime}, w_{2 i}^{\prime}\right)=\left(L_{0}, D\right)$ and $\left(\mu_{2 i-1}^{\prime}, \mu_{2 i}^{\prime}\right)=(x, y+1)$. It is routine to check that $\rho$ sets up an one-to-one correspondence between $M_{2 n}$ and $M_{2 n}^{*}$.

By Lemmas 6, 7, 8 and 9, the composition $\Phi:=\phi_{F V}^{-1} \circ \rho \circ \phi_{F V} \circ \eta$ is a bijection between $\mathcal{E C}_{2 n}$ and $\mathfrak{D}_{2 n-1}$.

## 3. The construction of $\psi$ and $\Psi$

3.1. The construction of $\psi$. It is clear that every cycle $\alpha$ on $M$ can be written uniquely as $\alpha=\left(a_{1}^{l_{1}}, a_{2}^{l_{2}}, \ldots, a_{k}^{l_{k}}\right)$, called the compact form of $\alpha$, where $a_{i} \neq a_{i+1}$ for $1 \leq i \leq k$ (by convention $a_{k+1}=a_{1}$ ) and $l_{i} \geq 1$, that is, all the adjacency letters with the same values are pinched into a bundle. For example, the compact form of the cycle (1, 2, 2, 1, 1, 1, 3, 4, 4, 2, 1) is $\left(1^{2}, 2^{2}, 1^{3}, 3,4^{2}, 2\right)$. A bundle $a_{i}^{l_{i}}(1 \leq i \leq k)$ is called a cyclic double ascent (resp. cyclic double descent) of $\alpha$ if $a_{i-1}<a_{i}<a_{i+1}$ (resp. $a_{i-1}>a_{i}>a_{i+1}$ ). The parity of a bundle $a_{i}^{l_{i}}$ is the parity of $a_{i}$. Now if $\alpha \in \mathcal{E C}_{M}$, then define $\psi(\alpha)$ to be the cycle obtained from $\alpha$ by moving each even cyclic double ascent bundle to the place immediately before the closest (in clockwise direction) bundle with smaller value. For example (see Fig. 2), if

$$
\alpha=\left(1,2^{2}, 4^{3}, 6,5^{2}, 6,1^{2}, 8,1^{2}, 4,5,8,3^{2}, 4\right)
$$

then

$$
\psi(\alpha)=\left(1,6,5^{2}, 6,4^{3}, 2^{2}, 1^{2}, 8,1^{2}, 5,8,4,3^{2}, 4\right)
$$



Figure 2. An example of $\psi$.
Two key observations about $\psi$ are:

- the resulting cycle $\psi(\alpha)$ is independent of the order of the movings;
- if the bundle $a_{i}^{l_{i}}$ is an even cyclic double ascent of $\alpha$, then $a_{i}^{l_{i}}$ becomes an even cyclic double descent bundle of $\psi(\alpha)$.
Moreover, it is routine to check that $\psi(\alpha) \in \mathcal{D C}_{M}$. To see that $\psi$ is a bijection between $\mathcal{E C}_{M}$ and $\mathcal{D C} \mathcal{C}_{M}$, we define its inverse explicitly. For a given cycle $\alpha \in \mathcal{D} \mathcal{C}_{M}$, define $\psi^{-1}(\alpha)$ to be the cycle obtained from $\alpha$ by moving each even cyclic double descent bundle to the place immediately before the closest (in anti-clockwise direction) bundle with smaller value. It is routine to check that $\psi$ and $\psi^{-1}$ are inverse of each other and thus $\psi$ is indeed a bijection.
3.2. The construction of $\Psi$. Suppose that $\alpha=\left(a_{1}, a_{2}, \ldots, a_{2 n}\right) \in \mathcal{D} \mathcal{C}_{2 n}$ with $a_{1}=1$ and $a_{k}=2 n$ for some $k$, then define $\vartheta(\alpha)$ to be the permutation

$$
a_{k+1}, a_{k+2}, \ldots, a_{2 n}, a_{1}, a_{2}, \ldots, a_{k-1} \text { (in one line notation), }
$$

which is clearly in $\mathfrak{D}_{2 n-1}$. For example, if $\alpha=(1,5,6,3,4,2) \in \mathcal{D C}_{6}$, then $\vartheta(\alpha)=34215 \in$ $\mathfrak{D}_{5}$. Thus, $\vartheta$ sets up an one-to-one correspondence between $\mathcal{D} \mathcal{C}_{2 n}$ and $\mathfrak{D}_{2 n-1}$. Now define $\Psi$ to be the composition $\vartheta \circ \psi$, which is another bijection between $\mathcal{E} \mathcal{C}_{2 n}$ and $\mathfrak{D}_{2 n-1}$ in view of Theorem 3.

## 4. An Inclusion-Exclusion approach to Dumont's result (1.1)

For the sake of completeness, this section is devoted to an Inclusion-Exclusion approach to Dumont's result (1.1). Our starting point is the following expression for Genocchi numbers deduced by Dumont [2, Proposition 1]:

$$
\begin{equation*}
g_{n+1}=\sum(-1)^{n-u_{n}}\left(u_{1} u_{2} \cdots u_{n}\right)^{2} \tag{4.1}
\end{equation*}
$$

summed over all $\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ such that $u_{1}=1$ and $u_{i}$ equals $u_{i-1}$ or $u_{i-1}+1$ for $2 \leq i \leq n$. For example, $g_{4}=(1 \cdot 1 \cdot 1)^{2}-(1 \cdot 1 \cdot 2)^{2}-(1 \cdot 2 \cdot 2)^{2}+(1 \cdot 2 \cdot 3)^{2}=17$.

For a permutation $\sigma \in \mathfrak{S}_{n}$, a letter $\sigma(i)$ is called a descent top of $\sigma$ if $i \in[n-1]$ and $\sigma(i)>\sigma(i+1)$. Denote by $\operatorname{DT}(\sigma)$ the set of all descent tops of $\sigma$. For example, $\mathrm{DT}(34215)=\{2,4\}$. For any $S \subseteq[2, n]:=\{2,3, \ldots, n\}$, let us introduce

$$
\begin{aligned}
& \mathrm{DT}_{=}(S, n):=\left\{\sigma \in \mathfrak{S}_{n}: \operatorname{DT}(\sigma)=S\right\} \\
& \operatorname{DT}_{\leq}(S, n):=\left\{\sigma \in \mathfrak{S}_{n}: \operatorname{DT}(\sigma) \subseteq S\right\}
\end{aligned}
$$

Let $f_{=}(S, n)=\left|\mathrm{DT}_{=}(S, n)\right|$ and $f_{\leq}(S, n)=\left|\mathrm{DT}_{\leq}(S, n)\right|$. Then, it follows from the principle of Inclusion-Exclusion (see [8, Sec. 2.1]) that

$$
\begin{equation*}
f_{=}(S, n)=\sum_{T \subseteq S}(-1)^{|S-T|} f_{\leq}(T, n) \tag{4.2}
\end{equation*}
$$

Suppose that $S=\left\{s_{1}, s_{2}, \ldots, s_{k}\right\} \subseteq[2, n]$ with $s_{1}>s_{2}>\cdots>s_{k}>1$. Let $d_{i}(S)=s_{i}-s_{i+1}$ for $i \in[k-1]$ and $d_{k}(S)=s_{k}-1$. We have the following product formula for $f_{\leq}(T, n)$.

Lemma 10. Let $T \subseteq S$ and let $u_{i}=u_{i}(T):=1+\left|\left\{t \in T: t \geq s_{i}\right\}\right|$. Then

$$
\begin{equation*}
f_{\leq}(T, n)=\prod_{i=1}^{k} u_{i}^{d_{i}(S)} \tag{4.3}
\end{equation*}
$$

Proof. For any letter $\ell, s_{i+1} \leq \ell<s_{i}$, and any partial permutation $p$ of $\{n, n-1, \ldots, \ell+1\}$ whose descent top includes in $T$, there are exactly $u_{i}$ positions to insert the letter $\ell$ into $p$ to obtain a partial permutation of $\{n, n-1, \ldots, \ell\}$ with descent top includes in $T$. These $u_{i}$ positions are the leftmost space of $p$ plus the spaces immediately after each letter from $\left\{t \in T: t \geq s_{i}\right\}$. The desired product formula for $f_{\leq}(T)$ then follows.

Combining (4.2) and (4.3) we have the following formula for $f=(S, n)$ that was obtained by Chang, Ma and Yeh [1, Theorem 1.1] via different approach.

Theorem 11 (Chang, Ma and Yeh [1]). For any $S \subseteq[2, n]$ with $|S|=k$, we have

$$
\begin{equation*}
f_{=}(S, n)=\sum(-1)^{k+1-u_{k}} \prod_{i=1}^{k} u_{i}^{d_{i}(S)} \tag{4.4}
\end{equation*}
$$

summed over all $\left(u_{0}, u_{1}, u_{2}, \ldots, u_{k}\right)$ such that $u_{0}=1$ and $u_{i}$ equals $u_{i-1}$ or $u_{i-1}+1$ for $1 \leq i \leq k$.

Since $\mathfrak{D}_{2 n+1}=\mathrm{DT}_{=}(\{2,4, \ldots, 2 n\}, 2 n+1)$, it follows from Theorem 11 that

$$
\left|\mathfrak{D}_{2 n+1}\right|=\sum(-1)^{n+1-u_{n}}\left(u_{1} u_{2} \cdots u_{n-1}\right)^{2} u_{n}
$$

summed over all $\left(u_{0}, u_{1}, u_{2}, \ldots, u_{n}\right)$ such that $u_{0}=1$ and $u_{i}$ equals $u_{i-1}$ or $u_{i-1}+1$ for $1 \leq i \leq n$. As $u_{n}=u_{n-1}$ or $u_{n}=u_{n-1}+1$, the above summation is simplified to the right-hand side of (4.1), which proves $\left|\mathfrak{D}_{2 n+1}\right|=g_{n+1}$.

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[^0]:    Date: August 10, 2021.

[^1]:    ${ }^{1}$ We learnt that Qiongqiong Pan and Jiang Zeng [6] have also proved [5, Conjecture 6.4] independently using continued fractions.

