## CYCLES ON A MULTISET WITH ONLY EVEN-ODD DROPS

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ABSTRACT. For a finite subset A of  $\mathbb{Z}_{>0}$ , Lazar and Wachs (2019) conjectured that the number of cycles on A with only even-odd drops is equal to the number of D-cycles on A. In this note, we introduce cycles on a multiset with only even-odd drops and prove bijectively a multiset version of their conjecture. As a consequence, the number of cycles on [2n] with only even-odd drops equals the Genocchi number  $g_n$ . With Laguerre histories as an intermediate structure, we also construct a bijection between a class of permutations of length 2n - 1 known to be counted by  $g_n$  invented by Dumont and the cycles on [2n] with only even-odd drops.

### 1. INTRODUCTION

The Genocchi numbers

$${g_n}_{n>1} = {1, 1, 3, 17, 155, 2073, 38227, 929569, \ldots}$$

can be defined by the following exponential generating function formula

$$\sum_{n \ge 1} g_n \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}.$$

Let  $\mathfrak{S}_n$  be the set of permutations of  $[n] := \{1, 2, \dots, n\}$ . Dumont [2] introduced the class of permutations

$$\mathfrak{D}_{2n-1} := \{ \sigma \in \mathfrak{S}_{2n-1} : \forall i \in [2n-2], \ \sigma(i) > \sigma(i+1) \text{ if and only if } \sigma(i) \text{ is even} \}$$

and proved that

$$|\mathfrak{D}_{2n-1}| = g_n.$$

For instance,  $\mathfrak{D}_5 = \{42135, 21435, 34215\}$  and so  $|\mathfrak{D}_5| = 3 = g_3$ . Based on Dumont's interpretation of Genocchi numbers, the objective of this paper is to present two different bijective proofs of a conjecture due to Lazar and Wachs [5, Conjecture 6.4] which asserts that cycles on [2n] with only even-odd drops are also counted by  $g_n$ . Actually, we will also introduce cycles on a multiset with only even-odd drops and prove bijectively a multiset generalization of another related conjecture of Lazar and Wachs [5, Conjecture 6.5].

Let M be a finite multiset with exactly m elements from  $\mathbb{Z}_{>0}$ . We introduce cycles on M with only even-odd drops and D-cycles on M, generalizing the concepts introduced in [5] for M being a subset of  $\mathbb{Z}_{>0}$ .

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**Definition 1.** A cycle  $(a_1, a_2, \ldots, a_m)$  on M (i.e., as multiset  $\{a_i : 1 \le i \le m\} = M$ ) is said to have only even-odd drops if whenever  $a_i > a_{i+1}$  (as a cycle, we usually assume  $a_{m+1} = a_1$ ), then  $a_i$  is even and  $a_{i+1}$  is odd; it is a **D-cycle** if  $a_i \le a_{i+1}$  when  $a_i$  is odd and  $a_i \ge a_{i+1}$  when  $a_i$  is even for all  $1 \le i \le m$ . For example, as cycles on the multiset  $\{1^2, 2^2, 3^2, 4^2\}$ , (1, 2, 1, 2, 4, 3, 3, 4) has only even-odd drops but is not a D-cycle, while (1, 2, 1, 4, 3, 3, 4, 2) is a D-cycle that does not have only even-odd drops. Let  $\mathcal{EC}_M$  be the set of cycles on M with only even-odd drops and let  $\mathcal{DC}_M$  be the set of D-cycles on M.

Note that in order for  $\mathcal{EC}_M$  or  $\mathcal{DC}_M$  to be non empty, the smallest element in M must be odd and the greatest element in M must be even. When M = [n], we simply write  $\mathcal{EC}_n$ and  $\mathcal{DC}_n$  for  $\mathcal{EC}_{[n]}$  and  $\mathcal{DC}_{[n]}$ , respectively. For example,

$$\mathcal{EC}_6 = \{ (1, 2, 3, 4, 5, 6), (1, 2, 4, 3, 5, 6), (1, 2, 5, 6, 3, 4) \},\$$
  
$$\mathcal{DC}_6 = \{ (1, 3, 5, 6, 4, 2), (1, 4, 3, 5, 6, 2), (1, 5, 6, 3, 4, 2) \}.$$

The next result proves [5, Conjecture 6.4] bijectively<sup>1</sup> in view of Dumont's result (1.1).

**Theorem 2.** There exists two bijections,  $\Phi$  and  $\Psi$ , between  $\mathcal{EC}_{2n}$  and  $\mathfrak{D}_{2n-1}$ .

The construction of  $\Phi$  is based on the classical *Françon–Viennot bijection* [3] that encodes permutations as Laguerre histories, while  $\Psi$  is the composition of the bijection  $\psi$  below with a simple transformation.

**Theorem 3.** For a fixed multiset M, there exists a bijection  $\psi$  between  $\mathcal{EC}_M$  and  $\mathcal{DC}_M$ .

For M being a subset of  $\mathbb{Z}_{>0}$ , the above theorem proves [5, Conjecture 6.5] bijectively. Thus, Theorem 3 is a multiset generalization of [5, Conjecture 6.5]. As an immediate consequence of Theorems 2 and 3 and Dumont's result (1.1), we have

# Corollary 4. For $n \geq 1$ , $|\mathcal{EC}_{2n}| = |\mathcal{DC}_{2n}| = g_n$ .

**Remark 5.** Let  $\mathcal{EC}_{2n}^{(k)} := \mathcal{EC}_M$  with  $M = \{1^k, 2^k, \dots, (2n)^k\}$ . Since  $|\mathcal{EC}_{2n}| = g_n$ ,  $|\mathcal{EC}_{2n}^{(k)}|$  can be considered as a new generalization of the Genocchi numbers. For another generalization of the Genocchi numbers using the model of trees, the reader is referred to [4]. Can the generating function for  $|\mathcal{EC}_{2n}^{(k)}|$  be calculated? Is there any divisibility property for  $|\mathcal{EC}_{2n}^{(k)}|$  similar to  $g_n$  (see [4])?

The rest of this paper is organized as follows. After recalling the Françon–Viennot bijection, we construct  $\Phi$  in Section 2. In Section 3, we first present the bijection  $\psi$  for Theorem 3 and then use it to construct  $\Psi$ . Finally, in Section 4, we provide an Inclusion-Exclusion approach to Dumont's result (1.1) for the sake of completeness.

## 2. The construction of $\Phi$

In order to construct  $\Phi$ , we need to recall the classical Françon–Viennot bijection [3] first. A *Motzkin path* of length n is a lattice path in the first quadrant starting from (0, 0), ending at (n, 0), and using three possible steps:

U = (1, 1) (up step), L = (1, 0) (level step) and D = (1, -1) (down step).

<sup>&</sup>lt;sup>1</sup>We learnt that Qiongqiong Pan and Jiang Zeng [6] have also proved [5, Conjecture 6.4] independently using continued fractions.

A 2-Motzkin path is a Motzkin path in which each level step is further distinguished into two different types of level steps  $L_0$  or  $L_1$ . A 2-Motzkin paths will be represented as a word over the alphabet  $\{U, D, L_0, L_1\}$ . A Laguerre history of length n is a pair  $(w, \mu)$ , where  $w = w_1 \cdots w_n$  is a 2-Motzkin path and  $\mu = (\mu_1, \cdots, \mu_n)$  is a vector satisfying  $0 \le \mu_i \le h_i(w)$ , where

$$h_i(w) := \#\{j \mid j < i, w_j = U\} - \#\{j \mid j < i, w_j = D\}$$

is the *height* of the starting point of the *i*-th step of w. Denote by  $\mathfrak{L}_n$  the set of all Laguerre histories of length n.

For a permutation  $\sigma \in \mathfrak{S}_n$ , a letter  $\sigma(i)$  is called a *valley* (resp. *peak*, *double descent*, *double ascent*) of  $\sigma$  if  $\sigma(i-1) > \sigma(i) < \sigma(i+1)$  (resp.  $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ ,  $\sigma(i-1) > \sigma(i) > \sigma(i+1)$ ,  $\sigma(i-1) < \sigma(i) < \sigma(i+1)$ ), where we use the assumption  $\sigma(0) = \sigma(n+1) = 0$ . For a fixed  $1 \le k \le n-1$ , define

$$(2-13)_k \sigma = \#\{i : i-1 > j \text{ and } \sigma(i-1) < \sigma(j) = k < \sigma(i)\}.$$

The Françon–Viennot bijection  $\phi_{FV} : \mathfrak{S}_n \to \mathfrak{L}_{n-1}$  that we need is the following modified version (see for example [7]) defined as  $\phi_{FV}(\sigma) = (w, \mu) \in \mathfrak{L}_{n-1}$ , where for each  $i \in [n-1]$ :

$$w_i = \begin{cases} U & \text{if } i \text{ is a valley of } \sigma, \\ D & \text{if } i \text{ is a peak of } \sigma, \\ L_0 & \text{if } i \text{ is a double ascent of } \sigma, \\ L_1 & \text{if } i \text{ is a double descent of } \sigma, \end{cases}$$

and  $\mu_i = (2-13)_i(\sigma)$ . For example, if  $\sigma = 528713649 \in \mathfrak{S}_9$ , then  $\phi_{FV}(\sigma) = (w, \mu)$ , where  $w = UUL_0UDDL_1D$  and  $\mu = (0, 1, 0, 0, 3, 1, 1, 1)$ . The reverse algorithm  $\phi_{FV}^{-1}$  building a permutation  $\sigma$  from a Laguerre history  $(w, \mu) \in \mathfrak{L}_{n-1}$  can be described iteratively as:

- Initialization:  $\sigma = \circ$ ;
- At the *i*-th  $(1 \le i \le n-1)$  step of the algorithm, replace the  $(h_i(w)+1)$ -th  $\circ$  (from right to left) of  $\sigma$  by

$$\begin{cases} \circ i \circ & \text{if } w_i = U, \\ i \circ & \text{if } w_i = L_0, \\ \circ i & \text{if } w_i = L_1, \\ i & \text{if } w_i = D; \end{cases}$$

• The final permutation is obtained by replacing the last remaining  $\circ$  by n.

For example, if  $(w, \mu) = (UL_1UDL_0D, (0, 1, 1, 2, 0, 0)) \in \mathfrak{L}_6$ , then  $\sigma$  is built as follows:

 $\sigma = \circ \to \circ 1 \circ \to \circ 21 \circ \to \circ 3 \circ 21 \circ \to 43 \circ 21 \circ \to 43 \circ 215 \circ \to 43 \circ 2156 \to 4372156.$ 

Now we begin to construct  $\Phi$  step by step. Let  $\mathfrak{S}_{2n-1}^{oe}$  be the set of permutations  $\sigma \in \mathfrak{S}_{2n-1}$  with only *odd-even descents* (i.e., whenever  $\sigma(i) > \sigma(i+1)$ , then  $\sigma(i)$  is odd and  $\sigma(i+1)$  is even) and whose last entry is odd. For instance,  $\mathfrak{S}_5^{oe} = \{12345, 13245, 14523\}$ .

**Lemma 6.** There exists a bijection  $\eta : \mathcal{EC}_{2n} \to \mathfrak{S}_{2n-1}^{oe}$ 

Proof. For a cycle  $\alpha = (a_1, a_2, \ldots, a_{2n}) \in \mathcal{EC}_{2n}$  with  $a_1 = 1$ , define  $\eta(\alpha)$  to be the permutation  $a_2 - 1, a_3 - 1, \ldots, a_{2n} - 1$  (in one line notation), which is clearly in  $\mathfrak{S}_{2n-1}^{oe}$ . It is easy to see that  $\eta$  sets up an one-to-one correspondence between  $\mathcal{EC}_{2n}$  and  $\mathfrak{S}_{2n-1}^{oe}$ .

Let us consider the subset  $M_{2n}$  of Laguerre histories  $(w, \mu) \in \mathfrak{L}_{2n}$  with the restriction that

$$w_i = \begin{cases} D \text{ or } L_0, & \text{if } i \text{ is odd,} \\ U \text{ or } L_0, & \text{if } i \text{ is even.} \end{cases}$$

For instance,

$$M_4 = \{ (L_0 U D L_0, (0, 0, 0, 0)), (L_0 U D L_0, (0, 0, 1, 0)), (L_0 L_0 L_0 L_0, (0, 0, 0, 0)) \}$$

**Lemma 7.** The Françon–Viennot bijection  $\phi_{FV}$  restricts to a bijection between  $\mathfrak{S}_{2n-1}^{oe}$  and  $M_{2n-2}$ .

*Proof.* This follows from the observation that  $\sigma \in \mathfrak{S}_{2n-1}$  is a permutation in  $\mathfrak{S}_{2n-1}^{oe}$  if and only if for each  $i \in [2n-2]$ , the letter i of  $\sigma$  is a double ascent or a peak whenever i is odd and is a double ascent or a valley whenever i is even.

Let us consider another subset  $M_{2n}^*$  of Laguerre histories  $(w, \mu) \in \mathfrak{L}_{2n}$  with the restriction that

$$w_i = \begin{cases} U \text{ or } L_0, & \text{if } i \text{ is odd,} \\ D \text{ or } L_1, & \text{if } i \text{ is even,} \end{cases}$$

and  $1 \leq \mu_i \leq h_i(w)$  when *i* is even. For instance,

$$M_4^* = \{ (UDUD, (0, 1, 0, 1)), (UL_1L_0D, (0, 1, 0, 1)), (UL_1L_0D, (0, 1, 1, 1)) \}$$

**Lemma 8.** The Françon-Viennot bijection  $\phi_{FV}$  restricts to a bijection between  $\mathfrak{D}_{2n-1}$  and  $M^*_{2n-2}$ .

Proof. Observe that  $\sigma \in \mathfrak{S}_{2n-1}$  is a permutation in  $\mathfrak{D}_{2n-1}$  if and only if (i)  $\sigma(2n-1) = 2n-1$ and (ii) for each  $i \in [2n-2]$ , the letter i of  $\sigma$  is a double ascent or a valley whenever iis odd and is a double descent or a peak whenever i is even. Thus, if  $\sigma \in \mathfrak{D}_{2n-1}$ , then  $\phi_{FV}(\sigma) \in M^*_{2n-2}$  (as  $\sigma(2n-1) = 2n-1$  forces  $\mu_i \geq 1$  when i is even). Conversely, if  $(w,\mu) \in M^*_{2n-2}$ , then as  $\mu_i \geq 1$  when i is even, it follows from the iterative construction of  $\sigma = \phi_{FV}^{-1}(w,\mu)$  that the  $\circ$  at the end of  $\sigma$  remains until the last step, i.e.,  $\sigma(2n-1) = 2n-1$ . Therefore, we have  $\phi_{FV}^{-1}(w,\mu) \in \mathfrak{D}_{2n-1}$  for any  $(w,\mu) \in M^*_{2n-2}$ .

**Lemma 9.** There exists a bijection  $\rho: M_{2n} \to M_{2n}^*$ .

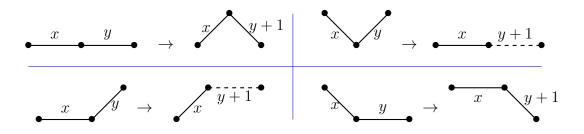


FIGURE 1. The construction of  $\rho$ : a dashed level step represents  $L_1$ .

*Proof.* For a Laguerre history  $(w, \mu) \in M_{2n}$ , we construct  $\rho(w, \mu) = (w', \mu')$  by transforming each consecutive two steps  $(w_{2i-1}, w_{2i})$  and their weights  $(\mu_{2i-1}, \mu_{2i}) = (x, y)$   $(1 \le i \le n)$  according to the following four cases (see Fig. 1):

- If  $(w_{2i-1}, w_{2i}) = (L_0, L_0)$ , then  $(w'_{2i-1}, w'_{2i}) = (U, D)$  and  $(\mu'_{2i-1}, \mu'_{2i}) = (x, y+1)$ ;
- If  $(w_{2i-1}, w_{2i}) = (D, U)$ , then  $(w'_{2i-1}, w'_{2i}) = (L_0, L_1)$  and  $(\mu'_{2i-1}, \mu'_{2i}) = (x, y + 1)$ ; If  $(w_{2i-1}, w_{2i}) = (L_0, U)$ , then  $(w'_{2i-1}, w'_{2i}) = (U, L_1)$  and  $(\mu'_{2i-1}, \mu'_{2i}) = (x, y + 1)$ ; If  $(w_{2i-1}, w_{2i}) = (D, L_0)$ , then  $(w'_{2i-1}, w'_{2i}) = (L_0, D)$  and  $(\mu'_{2i-1}, \mu'_{2i}) = (x, y + 1)$ .

It is routine to check that  $\rho$  sets up an one-to-one correspondence between  $M_{2n}$  and  $M_{2n}^*$ .  $\Box$ 

By Lemmas 6, 7, 8 and 9, the composition  $\Phi := \phi_{FV}^{-1} \circ \rho \circ \phi_{FV} \circ \eta$  is a bijection between  $\mathcal{EC}_{2n}$  and  $\mathfrak{D}_{2n-1}$ .

## 3. The construction of $\psi$ and $\Psi$

3.1. The construction of  $\psi$ . It is clear that every cycle  $\alpha$  on M can be written uniquely as  $\alpha = (a_1^{l_1}, a_2^{l_2}, \dots, a_k^{l_k})$ , called the *compact form* of  $\alpha$ , where  $a_i \neq a_{i+1}$  for  $1 \leq i \leq k$  (by convention  $a_{k+1} = a_1$  and  $l_i \ge 1$ , that is, all the adjacency letters with the same values are pinched into a bundle. For example, the compact form of the cycle (1, 2, 2, 1, 1, 1, 3, 4, 4, 2, 1)is  $(1^2, 2^2, 1^3, 3, 4^2, 2)$ . A bundle  $a_i^{l_i}$   $(1 \le i \le k)$  is called a cyclic double ascent (resp. cyclic double descent) of  $\alpha$  if  $a_{i-1} < a_i < a_{i+1}$  (resp.  $a_{i-1} > a_i > a_{i+1}$ ). The parity of a bundle  $a_i^{l_i}$ is the parity of  $a_i$ . Now if  $\alpha \in \mathcal{EC}_M$ , then define  $\psi(\alpha)$  to be the cycle obtained from  $\alpha$  by moving each even cyclic double ascent bundle to the place immediately before the closest (in clockwise direction) bundle with smaller value. For example (see Fig. 2), if

$$\alpha = (1, 2^2, 4^3, 6, 5^2, 6, 1^2, 8, 1^2, 4, 5, 8, 3^2, 4),$$

then

$$\psi(\alpha) = (1, 6, 5^2, 6, 4^3, 2^2, 1^2, 8, 1^2, 5, 8, 4, 3^2, 4).$$

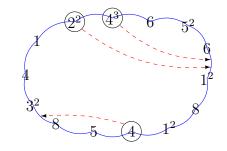


FIGURE 2. An example of  $\psi$ .

Two key observations about  $\psi$  are:

- the resulting cycle  $\psi(\alpha)$  is independent of the order of the movings;
- if the bundle  $a_i^{l_i}$  is an even cyclic double ascent of  $\alpha$ , then  $a_i^{l_i}$  becomes an even cyclic double descent bundle of  $\psi(\alpha)$ .

Moreover, it is routine to check that  $\psi(\alpha) \in \mathcal{DC}_M$ . To see that  $\psi$  is a bijection between  $\mathcal{EC}_M$  and  $\mathcal{DC}_M$ , we define its inverse explicitly. For a given cycle  $\alpha \in \mathcal{DC}_M$ , define  $\psi^{-1}(\alpha)$ to be the cycle obtained from  $\alpha$  by moving each even cyclic double descent bundle to the place immediately before the closest (in anti-clockwise direction) bundle with smaller value. It is routine to check that  $\psi$  and  $\psi^{-1}$  are inverse of each other and thus  $\psi$  is indeed a bijection.

3.2. The construction of  $\Psi$ . Suppose that  $\alpha = (a_1, a_2, \ldots, a_{2n}) \in \mathcal{DC}_{2n}$  with  $a_1 = 1$  and  $a_k = 2n$  for some k, then define  $\vartheta(\alpha)$  to be the permutation

$$a_{k+1}, a_{k+2}, \ldots, a_{2n}, a_1, a_2, \ldots, a_{k-1}$$
 (in one line notation),

which is clearly in  $\mathfrak{D}_{2n-1}$ . For example, if  $\alpha = (1, 5, 6, 3, 4, 2) \in \mathcal{DC}_6$ , then  $\vartheta(\alpha) = 34215 \in \mathfrak{D}_5$ . Thus,  $\vartheta$  sets up an one-to-one correspondence between  $\mathcal{DC}_{2n}$  and  $\mathfrak{D}_{2n-1}$ . Now define  $\Psi$  to be the composition  $\vartheta \circ \psi$ , which is another bijection between  $\mathcal{EC}_{2n}$  and  $\mathfrak{D}_{2n-1}$  in view of Theorem 3.

#### 4. An Inclusion-Exclusion approach to Dumont's result (1.1)

For the sake of completeness, this section is devoted to an Inclusion-Exclusion approach to Dumont's result (1.1). Our starting point is the following expression for Genocchi numbers deduced by Dumont [2, Proposition 1]:

(4.1) 
$$g_{n+1} = \sum (-1)^{n-u_n} (u_1 u_2 \cdots u_n)^2,$$

summed over all  $(u_1, u_2, \dots, u_n)$  such that  $u_1 = 1$  and  $u_i$  equals  $u_{i-1}$  or  $u_{i-1} + 1$  for  $2 \le i \le n$ . For example,  $g_4 = (1 \cdot 1 \cdot 1)^2 - (1 \cdot 1 \cdot 2)^2 - (1 \cdot 2 \cdot 2)^2 + (1 \cdot 2 \cdot 3)^2 = 17$ .

For a permutation  $\sigma \in \mathfrak{S}_n$ , a letter  $\sigma(i)$  is called a *descent top* of  $\sigma$  if  $i \in [n-1]$ and  $\sigma(i) > \sigma(i+1)$ . Denote by  $DT(\sigma)$  the set of all descent tops of  $\sigma$ . For example,  $DT(34215) = \{2, 4\}$ . For any  $S \subseteq [2, n] := \{2, 3, \ldots, n\}$ , let us introduce

$$DT_{=}(S,n) := \{ \sigma \in \mathfrak{S}_n : DT(\sigma) = S \}, \\ DT_{\leq}(S,n) := \{ \sigma \in \mathfrak{S}_n : DT(\sigma) \subseteq S \}.$$

Let  $f_{=}(S, n) = |DT_{=}(S, n)|$  and  $f_{\leq}(S, n) = |DT_{\leq}(S, n)|$ . Then, it follows from the principle of Inclusion-Exclusion (see [8, Sec. 2.1]) that

(4.2) 
$$f_{=}(S,n) = \sum_{T \subseteq S} (-1)^{|S-T|} f_{\leq}(T,n).$$

Suppose that  $S = \{s_1, s_2, \ldots, s_k\} \subseteq [2, n]$  with  $s_1 > s_2 > \cdots > s_k > 1$ . Let  $d_i(S) = s_i - s_{i+1}$  for  $i \in [k-1]$  and  $d_k(S) = s_k - 1$ . We have the following product formula for  $f_{\leq}(T, n)$ .

**Lemma 10.** Let  $T \subseteq S$  and let  $u_i = u_i(T) := 1 + |\{t \in T : t \ge s_i\}|$ . Then

(4.3) 
$$f_{\leq}(T,n) = \prod_{i=1}^{k} u_i^{d_i(S)}$$

Proof. For any letter  $\ell$ ,  $s_{i+1} \leq \ell < s_i$ , and any partial permutation p of  $\{n, n-1, \ldots, \ell+1\}$  whose descent top includes in T, there are exactly  $u_i$  positions to insert the letter  $\ell$  into p to obtain a partial permutation of  $\{n, n-1, \ldots, \ell\}$  with descent top includes in T. These  $u_i$  positions are the leftmost space of p plus the spaces immediately after each letter from  $\{t \in T : t \geq s_i\}$ . The desired product formula for  $f_{\leq}(T)$  then follows.

Combining (4.2) and (4.3) we have the following formula for  $f_{=}(S, n)$  that was obtained by Chang, Ma and Yeh [1, Theorem 1.1] via different approach. **Theorem 11** (Chang, Ma and Yeh [1]). For any  $S \subseteq [2, n]$  with |S| = k, we have

(4.4) 
$$f_{=}(S,n) = \sum (-1)^{k+1-u_k} \prod_{i=1}^{k} u_i^{d_i(S)}$$

summed over all  $(u_0, u_1, u_2, ..., u_k)$  such that  $u_0 = 1$  and  $u_i$  equals  $u_{i-1}$  or  $u_{i-1} + 1$  for  $1 \le i \le k$ .

Since 
$$\mathfrak{D}_{2n+1} = \mathrm{DT}_{=}(\{2, 4, \dots, 2n\}, 2n+1)$$
, it follows from Theorem 11 that

$$|\mathfrak{D}_{2n+1}| = \sum (-1)^{n+1-u_n} (u_1 u_2 \cdots u_{n-1})^2 u_n$$

summed over all  $(u_0, u_1, u_2, \ldots, u_n)$  such that  $u_0 = 1$  and  $u_i$  equals  $u_{i-1}$  or  $u_{i-1} + 1$  for  $1 \leq i \leq n$ . As  $u_n = u_{n-1}$  or  $u_n = u_{n-1} + 1$ , the above summation is simplified to the right-hand side of (4.1), which proves  $|\mathfrak{D}_{2n+1}| = g_{n+1}$ .

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