# Weight hierarchies of a family of linear codes associated with degenerate quadratic forms 

Fei Li, Xiumei Li

Received: date / Accepted: date


#### Abstract

We restrict a degenerate quadratic form $f$ over a finite field of odd characteristic to subspaces. Thus, a quotient space related to $f$ is introduced. Then we get a non-degenerate quadratic form induced by $f$ over the quotient space. Some related results on the subspaces and quotient space are obtained. Based on this, we solve the weight hierarchies of a family of linear codes related to $f$.


Keywords Weight hierarchy • Generalized Hamming weight • Linear code •
Quadratic form • Quotient space
Mathematics Subject Classification (2010) 94B05 • 11E04 • 11T71

## 1 Introduction

Weight hierarchies of linear codes have been an interesting topic for their important value in theory and applications to cryptography. In 1991, Wei in

[^0]the paper [25] presented his wonderful results about weight hierarchies. It has been shown that the weight hierarchy of a linear code completely characterizes the performance of the code on the type II wire-tap channel. Readers can refer to 22 for a detailed survey on the results up to 1995 about weight hierarchies. The interest towards the knowledge of the weight hierarchy of a linear code has been continually increasing. Many authors devoted themselves to weight hierarchies of particular classes of codes [1,2,3, 7, 10, 13, 14, 26, 27]. In general, it is hard to settle the weight hierarchy of a linear code.

Let $p$ be an odd prime and $\mathbb{F}_{p^{m}}$ be the finite field with $p^{m}$ elements. Denote by $C \subset \mathbb{F}_{p}^{n}$ an $[n, k, d] p$-ary linear code with minimum Hamming distance $d$ [12]. Let $[C, r]_{p}$ be the set of all $r$-dimensional subspaces of $C$. For $V \in[C, r]_{p}$, define

$$
\operatorname{Supp}(V)=\left\{i: x_{i} \neq 0 \text { for some } x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in V\right\} .
$$

Then we define the $r$-th $(1 \leq r \leq k)$ generalized Hamming weight $d_{r}(C)$ of linear code $C$ by

$$
d_{r}(C)=\min \left\{|\operatorname{Supp}(V)|: V \in[C, r]_{p}\right\}
$$

In particular, $d_{1}(C)=d$. The weight hierarchy of $C$ is defined as the set $\left\{d_{i}(C): 1 \leq i \leq k\right\}$ (see [11, 12,15]).

Denote by $\mathbb{F}_{p^{m}}^{*}$ the set of nonzero elements in the finite field $\mathbb{F}_{p^{m}}$. A generic construction of linear code was proposed by Ding et al.([4, 6]). It is as follows. Let $\operatorname{Tr}$ denote the trace function from $\mathbb{F}_{p^{m}}$ to $\mathbb{F}_{p}$ and $D=\left\{d_{1}, d_{2}, \ldots, d_{n}\right\} \subset$ $\mathbb{F}_{p^{m}}^{*}$. Define a linear code $C_{D}$ with length $n$ as follows:

$$
\begin{equation*}
C_{D}=\left\{\left(\operatorname{Tr}\left(x d_{1}\right), \operatorname{Tr}\left(x d_{2}\right), \ldots, \operatorname{Tr}\left(x d_{n}\right)\right): x \in \mathbb{F}_{p^{m}}\right\} \tag{1}
\end{equation*}
$$

and $D$ is called the defining set. Many classes of linear codes with a few weights were obtained by choosing properly defining sets [9, 18, 28, 30, 21.

In this paper, we discuss the generalized Hamming weights of a class of linear codes $C_{D}$, whose defining set is chosen to be

$$
\begin{equation*}
D=D_{f}^{a}=\left\{x \in \mathbb{F}_{p^{m}}: f(x)=a\right\}, a \in \mathbb{F}_{p}^{*} . \tag{2}
\end{equation*}
$$

Here $f$ is a degenerate quadratic form over $\mathbb{F}_{p^{m}}$ with values in $\mathbb{F}_{p}$.

In the paper, we settle the weight hierarchy of $C_{D_{f}^{a}}, a \in \mathbb{F}_{p}^{*}$. In our previous work [20], the weight hierarchy of $C_{D_{f}^{a}}\left(a \in \mathbb{F}_{p}^{*}\right)$ relating to non-degenerate quadratic forms was solved. In [23,24, Z. Wan and X. Wu calculated the weight hierarchies of the projective codes from quadrics by the theory of finite geometry. In the case $a=0$, the weight hierarchy of $C_{D_{f}^{a}}$ can be deduced from Theorem 18 in [24].

The weight distributions of $C_{D_{f}^{a}}$ have been settled. In reference [5], K. Ding and C. Ding constructed the linear codes $C_{D_{f}^{a}}$ in the case $a=0$ relating to the special quadratic form $\operatorname{Tr}\left(X^{2}\right)$ and determined their weight distributions. In [8, 30, [29], the authors calculated the weight distributions of $C_{D_{f}^{a}}$ for general quadratic forms. In these articles, it was shown that the linear codes $C_{D_{f}^{a}}$ have a few weights and can be used to get association schemes, authentication codes, secret sharing schemes with interesting access structures.

Also, by these results, we know that $C_{D_{f}^{a}}$ is an $m$-dimensional linear code. So we can employ a general formula for calculating the generalized Hamming weights of linear codes defined in (1). It is given as follows.

Lemma 1.(Theorem 1, [19]) For each $r(1 \leq r \leq m)$, if the dimension of $C_{D}$ is $m$, then $d_{r}\left(C_{D}\right)=n-\max \left\{|D \bigcap H|: H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}$.

The rest of this paper is organized as follows: in Sect. 2, we present some basic definitions and results of quadratic forms restricted to subspaces and of induced quadratic forms over quotient spaces of finite fields; in Sect. 3, using the results in Sect. 2, we give all the generalized Hamming weights of linear codes defined in (2).

## 2 Quadratic Form, Dual Space and Quotient Space

### 2.1 Restricting Quadratic Forms to Subspaces

The finite field $\mathbb{F}_{p^{m}}$ can be viewed as an $m$-dimensional vector space over $\mathbb{F}_{p}$. Fix a basis $v_{\mathbf{1}}, v_{\mathbf{2}}, \ldots, v_{\mathbf{m}} \in \mathbb{F}_{p^{m}}$ and express each vector $X \in \mathbb{F}_{p^{m}}$ in the unique form $X=x_{1} v_{\mathbf{1}}+x_{2} v_{\mathbf{2}}+x_{m} v_{\mathbf{m}}$, with $x_{1}, x_{2}, \ldots, x_{m} \in \mathbb{F}_{p}$. We can write $X=\left(x_{1}, x_{2}, \ldots, x_{m}\right)^{T}$, where $T$ represents the transpose of a matrix.

Let $f: \mathbb{F}_{p^{m}} \rightarrow \mathbb{F}_{p}$ be a quadratic form over $\mathbb{F}_{p^{m}}$ with values in $\mathbb{F}_{p}$ [16. Set

$$
F(X, Y)=\frac{1}{2}[f(X+Y)-f(X)-f(Y)]
$$

We can write $f(X)=X^{T} A X$, where $A$ is the symmetric matrix $\left(a_{i j}\right)_{1 \leq i, j \leq m}$ and $a_{i i}=f\left(v_{\mathbf{i}}\right), a_{i j}=F\left(v_{\mathbf{i}}, v_{\mathbf{j}}\right)$.

The rank $R_{f}$ of quadratic form $f$ is defined to be the rank of matrix $A$. We say that $f$ is non-degenerate if $R_{f}=m$ and degenerate, otherwise. We can find a invertible matrix $M$ such that $M^{T} A M$ is a diagonal matrix $\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{R_{f}}, 0, \ldots, 0\right)$. Let $\Delta_{f}=\lambda_{1} \cdot \lambda_{2} \cdots \lambda_{R_{f}}$. When $R_{f}=0$, we define $\Delta_{f}=1$. Let $\eta$ be the quadratic character of $\mathbb{F}_{p}$, i.e., $\eta(a)=a^{\frac{p-1}{2}}$ for $a \in \mathbb{F}_{p}^{*}$. In the paper, $\eta(0)$ is defined to be zero. Under the congruent transformation of $A \rightarrow M^{T} A M, \eta\left(\Delta_{f}\right)$ is an invariant. We called $\eta\left(\Delta_{f}\right)$, denoted by $\epsilon_{f}$, the sign of the quadratic form $f$.

For a subspace $H \subseteq \mathbb{F}_{p^{m}}$, define

$$
H^{\perp}=\left\{x \in \mathbb{F}_{p^{m}}: F(x, y)=0 \text { for each } y \in H\right\}
$$

Then $H^{\perp}$ is called the dual space of $H$. And $R_{f}$ can also be defined as the codimension of $\mathbb{F}_{p^{m}}^{\perp}$. Namely, $R_{f}+\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)=m$.

Let $H$ be a $d$-dimensional subspace of $\mathbb{F}_{p^{m}}$. Restricting the quadratic form $f$ to $H$, we get a quadratic form over $H$ in $d$ variables. It is denoted by $\left.f\right|_{H}$. Similarly, we define the dual space $H_{\left.f\right|_{H}}^{\perp}$ of $H$ under $\left.f\right|_{H}$ in itself by

$$
H_{\left.f\right|_{H}}^{\perp}=\{x \in H: f(x+y)=f(x)+f(y) \text { for each } y \in H\} .
$$

Let $R_{H}, \epsilon_{H}$ be the rank and sign of $\left.f\right|_{H}$ over $H$, respectively. Obviously, $H_{\left.f\right|_{H}}^{\perp}=$ $H \bigcap H^{\perp}$ and $R_{H}=d-\operatorname{dim}\left(H_{\left.f\right|_{H}}^{\perp}\right)$.

Example 1 Let $f(X)=x_{1}^{2}-2 x_{1} x_{2}+x_{2}^{2}$ with $X=\left(x_{1}, x_{2}\right)$. It is a degenerate quadratic form over $\mathbb{F}_{p}^{2}$. After simple calculation, we have $F(X, Y)=x_{1} y_{1}-$ $x_{1} y_{2}-x_{2} y_{1}+x_{2} y_{2}$, where $Y=\left(y_{1}, y_{2}\right)$. Let $H=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{F}_{p}^{2}: x_{1}=x_{2}\right\}$. It is not hard to get $\mathbb{F} p^{\perp}=H, R_{f}=1,\left.f\right|_{H}=0, H_{\left.f\right|_{H}}^{\perp}=H, R_{H}=0$ and $\epsilon_{H}=1$.

For $a \in \mathbb{F}_{p}$, the following lemma tells us the number of solutions in $H$ of the equation $f(x)=a$.

Lemma 2.(Proposition 1, [20]) Let $f$ be a quadratic form over $\mathbb{F}_{p^{m}}$, $a \in \mathbb{F}_{p}$ and $H$ be a $d$-dimensional $(d>0)$ subspace of $\mathbb{F}_{p^{m}}$, then the number of solutions of $f(X)=a$ in $H$ is

$$
\left|H \bigcap D_{f}^{a}\right|=\left\{\begin{array}{l}
p^{d-1}+v(a) \eta\left((-1)^{\frac{R_{H}}{2}}\right) \epsilon_{H} p^{d-\frac{R_{H}+2}{2}}, \text { if } R_{H} \equiv 0(\bmod 2) \\
p^{d-1}+\eta\left((-1)^{\frac{R_{H}-1}{2}} a\right) \epsilon_{H} p^{d-\frac{R_{H}+1}{2}}, \text { if } R_{H} \equiv 1(\bmod 2)
\end{array}\right.
$$

where $v(a)=p-1$ if $a=0$, otherwise $v(a)=-1$.
Remark In Proposition 1 of [20], $f$ is set to be non-degenerate. In fact, we know from the proof that this condition is unnecessary. Namely, $f$ can be any quadratic form.

For later use, we need two results in the case that $f$ is a non-degenerate quadratic form. They are listed as follows.

Lemma 3. (Proposition 2, [20]) Let $f$ be a non-degenerate quadratic form over $\mathbb{F}_{p^{m}}$. For each $r$ with $0<2 r<m$, there exist an $r$-dimensional subspace $H \subseteq \mathbb{F}_{p^{m}}(m>2)$ such that $H \subseteq H^{\perp}$.

For $k$ elements $\beta_{1}, \beta_{2}, \ldots, \beta_{k} \in \mathbb{F}_{p^{m}}$, the matrix $M\left(\beta_{1}, \beta_{2}, \ldots, \beta_{k}\right)$ of them is defined as the $k \times k$ square matrix $\left(F\left(\beta_{i}, \beta_{j}\right)\right)_{1 \leq i, j \leq k}$. The discriminant $\Delta\left(\beta_{1}, \beta_{2}, \ldots, \beta_{k}\right)$ of them is defined to be $\operatorname{det}\left(M\left(\beta_{1}, \beta_{2}, \ldots, \beta_{k}\right)\right)$. We denote by $\left\langle\beta_{1}, \beta_{2}, \ldots, \beta_{k}\right\rangle$ the subspace spanned by $\beta_{1}, \beta_{2}, \ldots, \beta_{k}$.

Proposition 1. Let $f$ be a non-degenerate quadratic form over $\mathbb{F}_{p^{m}}$ and $H \subset \mathbb{F}_{p^{m}}$ a subspace with $\operatorname{dim}\left(H \bigcap H^{\perp}\right)=e$. Then, $\epsilon_{H^{\prime}} \epsilon_{H^{\perp}}=(-1)^{\frac{e(p-1)}{2}} \epsilon_{f}$.

Proof. Suppose $\operatorname{dim}(H)=r$. By hypothesis, we can set

$$
\begin{gathered}
H=\left\langle\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-e}, \beta_{1}, \beta_{2}, \ldots, \beta_{e}\right\rangle \\
H^{\perp}=\left\langle\gamma_{1}, \gamma_{2}, \ldots, \gamma_{m-r-e}, \beta_{1}, \beta_{2}, \ldots, \beta_{e}\right\rangle \\
\left\langle\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-e}, \gamma_{1}, \gamma_{2}, \ldots, \gamma_{m-r-e}\right\rangle^{\perp}=\left\langle\eta_{1}, \eta_{2}, \ldots, \eta_{e}, \beta_{1}, \beta_{2}, \ldots, \beta_{e}\right\rangle .
\end{gathered}
$$

We have $\epsilon_{H}=\eta\left(\Delta\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-e}\right)\right)$, $\epsilon_{H^{\perp}}=\eta\left(\Delta\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{m-r-e}\right)\right)$. And

$$
\begin{aligned}
\epsilon_{f} & =\eta\left(\Delta\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-e}, \gamma_{1}, \gamma_{2}, \ldots, \gamma_{m-r-e}, \beta_{1}, \beta_{2}, \ldots, \beta_{e}, \eta_{1}, \eta_{2}, \ldots, \eta_{e}\right)\right) \\
& =\eta\left(\Delta\left(\alpha_{1}, \ldots, \alpha_{r-e}, \gamma_{1}, \ldots, \gamma_{m-r-e}\right)\right) \eta\left(\Delta\left(\beta_{1}, \ldots, \beta_{e}, \eta_{1}, \ldots, \eta_{e}\right)\right) \\
& =\eta\left(\Delta\left(\alpha_{1}, \ldots, \alpha_{r-e}\right)\right) \eta\left(\Delta\left(\gamma_{1}, \ldots, \gamma_{m-r-e}\right)\right) \eta\left((-1)^{e}\right) \eta\left(\operatorname{det}\left(M^{2}\right)\right) \\
& =\epsilon_{H} \epsilon_{H^{\perp}} \eta\left((-1)^{e}\right) \eta\left(\operatorname{det}\left(M_{e}^{2}\right)\right) .
\end{aligned}
$$

Here $M_{e}$ is the square matrix $\left(F\left(\beta_{i}, \eta_{j}\right)\right)_{1 \leq i, j \leq e}$. Then the desired result follows and we complete the proof.
2.2 Induced Quadratic Form over a Quotient Space

From now on, we suppose $f$ is a degenerate quadratic form. Let $\overline{\mathbb{F}}_{p^{m}}$ be the quotient space $\mathbb{F}_{p^{m}} / \mathbb{F}_{p^{m}}^{\perp}$. For $\bar{\alpha} \in \overline{\mathbb{F}}_{p^{m}}$, define $\bar{f}(\bar{\alpha})=f(\alpha)$. It is welldefined. We obtain a non-degenerate quadratic form $\bar{f}$, induced by $f$, over $\overline{\mathbb{F}}_{p^{m}}$. Without confusion, we still use $f$ to denote $\bar{f}$. Let $R_{\bar{f}}$ and $\epsilon_{\bar{f}}$ denote the rank and sign of $f$ over $\overline{\mathbb{F}}_{p^{m}}$, respectively. It is easy to see $R_{\bar{f}}=R_{f}$ and $\epsilon_{\bar{f}}=\epsilon_{f}$.

Since $f$ is a non-degenerate quadratic form over $\overline{\mathbb{F}}_{p^{m}}$, the results in Lemmas 2, 3 and Proposition 1 can be applied to $\overline{\mathbb{F}}_{p^{m}}$.

For $a \in \mathbb{F}_{p}$, set

$$
\bar{D}_{f}^{a}=\left\{\bar{x} \in \overline{\mathbb{F}}_{p^{m}}: f(\bar{x})=a\right\} .
$$

Obviously, $\left|\overline{\mathbb{F}}_{p^{m}} \bigcap \bar{D}_{f}^{a}\right|=\left|\bar{D}_{f}^{a}\right|$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^{m}}$, we have

$$
\left|\bar{D}_{f}^{a}\right|=\left\{\begin{array}{l}
p^{\bar{m}-1}+v(a) \eta\left((-1)^{\frac{R_{f}}{2}}\right) \epsilon_{f} p^{\bar{m}-\frac{R_{f}+2}{2}}, \text { if } \quad R_{f} \equiv 0(\bmod 2) \\
p^{\bar{m}-1}+\eta\left((-1)^{\frac{R_{f}-1}{2}} a\right) \epsilon_{f} p^{\bar{m}-\frac{R_{f}+1}{2}}, \quad \text { if } \quad R_{f} \equiv 1(\bmod 2)
\end{array}\right.
$$

where $\bar{m}=R_{f}=\operatorname{dim}\left(\overline{\mathbb{F}}_{p^{m}}\right)=m-\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$.
Example 2 Just like Example 1, let $f(X)=x_{1}^{2}-2 x_{1} x_{2}+x_{2}^{2}$ with $X=\left(x_{1}, x_{2}\right)$, a degenerate quadratic form over $\mathbb{F}_{p}^{2} \cong \mathbb{F}_{p^{2}}$. Because $\mathbb{F}_{p^{2}}^{\perp}=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{F}_{p}^{2}\right.$ : $\left.x_{1}=x_{2}\right\}, \overline{\mathbb{F}}_{p^{2}}$ is isomorphic to $\left\{\left(x_{1}, x_{2}\right) \in \mathbb{F}_{p}^{2}: x_{1}=-x_{2}\right\}$. So, $\bar{f}=4 x_{1}^{2}$ is a non-degenerate quadratic form over $\overline{\mathbb{F}}_{p^{2}}$. It is not hard to get $R_{\bar{f}}=R_{f}=$ $1, \epsilon_{\bar{f}}=\epsilon_{f}=1$, and

$$
\left|\bar{D}_{f}^{a}\right|=\left\{\begin{array}{l}
1, \text { if } \quad a=0, \\
2, \text { if } \eta(a)=1, \\
0, \text { if } \eta(a)=-1 .
\end{array}\right.
$$

Let $\varphi$ be the canonical map from $\mathbb{F}_{p^{m}}$ to $\overline{\mathbb{F}}_{p^{m}}$ 17. For a subspace $H \subset \mathbb{F}_{p^{m}}$, denote by $\bar{H}$ the image of $H$ under $\varphi$, i.e., $\bar{H}=\varphi(H)$. In the absence of confusion, also we use $\bar{H}$ to represent a subspace of $\overline{\mathbb{F}}_{p^{m}}$. Let $R_{\bar{H}}$ and $\epsilon_{\bar{H}}$ denote the rank and sign of $f$ over $\bar{H}$, respectively.

Proposition 2. Let $H$ be a subspace of $\mathbb{F}_{p^{m}}$ and $\bar{H}=\varphi(H) \subseteq \overline{\mathbb{F}}_{p^{m}}$, then $R_{\bar{H}}=R_{H}, \epsilon_{\bar{H}}=\epsilon_{H}$.

Proof. Suppose $\operatorname{dim}(H)=r, \operatorname{dim}\left(H \bigcap H^{\perp}\right)=t$. Then we set

$$
H \bigcap H^{\perp}=\left\langle\beta_{1}, \beta_{2}, \ldots, \beta_{t}\right\rangle, \quad H=\left\langle\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-t}, \beta_{1}, \beta_{2}, \ldots, \beta_{t}\right\rangle .
$$

So we have $\bar{H}=\left\langle\bar{\alpha}_{1}, \bar{\alpha}_{2}, \ldots, \bar{\alpha}_{r-t}, \bar{\beta}_{1}, \bar{\beta}_{2}, \ldots, \bar{\beta}_{t}\right\rangle$.
The matrix $M\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-t}, \beta_{1}, \beta_{2}, \ldots, \beta_{t}\right)$ is the block matrix

$$
\left(\begin{array}{cc}
M_{1} & O \\
O & O
\end{array}\right)
$$

where $M_{1}=M\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r-t}\right)$. Then $R_{H}=\operatorname{Rank}\left(M_{1}\right), \epsilon_{H}=\eta\left(\operatorname{det}\left(M_{1}\right)\right)$.
And the matrix $M\left(\bar{\alpha}_{1}, \bar{\alpha}_{2}, \ldots, \bar{\alpha}_{r-t}, \bar{\beta}_{1}, \bar{\beta}_{2}, \ldots, \bar{\beta}_{t}\right)$ is the block matrix

$$
\left(\begin{array}{cc}
\bar{M}_{1} & O \\
O & O
\end{array}\right)
$$

where $\bar{M}_{1}=M\left(\bar{\alpha}_{1}, \bar{\alpha}_{2}, \ldots, \bar{\alpha}_{r-t}\right)$. Then $R_{\bar{H}}=\operatorname{Rank}\left(\bar{M}_{1}\right), \epsilon_{\bar{H}}=\eta\left(\operatorname{det}\left(\bar{M}_{1}\right)\right)$. In fact, $\bar{M}_{1}=M_{1}$, since $f(\bar{x})=f(x)$ for each $x \in \mathbb{F}_{p^{m}}$. Hence the desired results follow directly and we complete the proof.

Define the dual space $\bar{H}^{\perp}$ of $\bar{H}$ by

$$
\bar{H}^{\perp}=\left\{\bar{x} \in \overline{\mathbb{F}}_{p^{m}}: f(\bar{x}+\bar{y})=f(\bar{x})+f(\bar{y}) \text { for each } \bar{y} \in \bar{H}\right\} .
$$

For the dual spaces, we have an interesting conclusion as below.
Proposition 3. Let $H$ be a subspace of $\mathbb{F}_{p^{m}}$, then $\bar{H}^{\perp}=\overline{H^{\perp}}$.
Proof. Let $\bar{x}$ be an element of $\overline{H^{\perp}}$ with $x \in H^{\perp}$. We have $f(x+y)=$ $f(x)+f(y)$ for each $y \in H$. So $f(\overline{x+y})=f(\bar{x})+f(\bar{y})$. Since $\overline{x+y}=\bar{x}+$ $\bar{y}, f(\bar{x}+\bar{y})=f(\bar{x})+f(\bar{y})$. By definition, $\bar{x} \in \bar{H}^{\perp}$, which means $\overline{H^{\perp}} \subset \bar{H}^{\perp}$. On the other hand, let $\bar{x}$ be an element of $\bar{H}^{\perp}$. For each $\bar{y} \in \bar{H}$, we have $f(\bar{x}+\bar{y})=f(\bar{x})+f(\bar{y})$. So $f(\overline{x+y})=f(\bar{x})+f(\bar{y})$ and $f(x+y)=f(x)+f(y)$. Thus $x \in H^{\perp}$ and $\bar{x} \in \overline{H^{\perp}}$. Therefore $\overline{H^{\perp}} \supset \bar{H}^{\perp}$. In a word, $\bar{H}^{\perp}=\overline{H^{\perp}}$. The proof is finished.

## 3 Weight Hierarchies of Linear Codes Defined in (2)

By our method, we have successfully settled the weight hierarchies of $C_{D_{f}^{a}}$. In this case $a=0$, the weight hierarchies can be derived from Theorem 18 in [24]. In this section, we will just present the weight hierarchies of $C_{D_{f}^{a}}$ in the case $a \in \mathbb{F}_{p}^{*}$.

Theorem 1. Let $f$ be a degenerate quadratic form over $\mathbb{F}_{p^{m}}$ with rank $R_{f}=2 s$ and $a$ a non-zero element in $\mathbb{F}_{p}^{*}$. Suppose $m=2 s+l, l=\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$, then for the linear codes defined in (2), we have

$$
d_{r}\left(C_{D_{f}^{a}}\right)= \begin{cases}p^{m-1}-p^{m-r-1}-\left((-1)^{\frac{s(p-1)}{2}} \epsilon_{f}+1\right) p^{s+l-1}, & \text { if } 1 \leq r \leq s \\ p^{m-1}-2 p^{m-r-1}-(-1)^{\frac{s(p-1)}{2}} \epsilon_{f} p^{s+l-1}, & \text { if } s \leq r<m \\ p^{m-1}-(-1)^{\frac{s(p-1)}{2}} \epsilon_{f} p^{s+l-1}, & \text { if } r=m\end{cases}
$$

Proof. We will use Lemma 1 to compute $d_{r}\left(C_{D_{f}^{a}}\right)$. To do so, we need to know the value of $\max \left\{\left|D_{f}^{a} \bigcap H\right|: H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}$.

Case : $s \leq r<m$. If $H_{m-r}$ is an $(m-r)$-dimensional subspace of $\mathbb{F}_{p^{m}}$, then, by Lemma 2, we have

$$
\left|H_{m-r} \bigcap D_{f}^{a}\right| \leq 2 p^{m-r-1},
$$

and $\left|H_{m-r} \bigcap D_{f}^{a}\right|$ may reach the upper bound $2 p^{m-r-1}$ if $R_{H_{m-r}}=1$ or 0 . We assert that there exists an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ satisfying $R_{H_{m-r}}=1$ and $\epsilon_{H_{m-r}}$ may take values -1 or 1 . Applying Lemma 3 to $\overline{\mathbb{F}}_{p^{m}}$, there is an $(s-1)$-dimensional subspace $\bar{H}_{s-1} \subset \overline{\mathbb{F}}_{p^{m}}$ with $\bar{H}_{s-1} \subset \bar{H}_{s-1}^{\perp}$. So $\operatorname{dim}\left(\bar{H}_{s-1}^{\perp}\right)=s+1, R_{\bar{H}_{s-1}^{\perp}}=2$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^{m}}$, for each $b \in \mathbb{F}_{p}^{*},\left|\bar{D}_{f}^{b} \bigcap \bar{H}_{s-1}^{\perp}\right|>p^{s-1}$. We choose an element $\alpha \in\left(\bar{D}_{f}^{b} \bigcap \bar{H}_{s-1}^{\perp}\right) \backslash \bar{H}_{s-1}$ and let $\bar{H}_{s}=\langle\alpha\rangle \bigoplus \bar{H}_{s-1}$. Then $\operatorname{dim}\left(\bar{H}_{s}\right)=s, R_{\bar{H}_{s}}=1$ and the values of $\epsilon_{\bar{H}_{s}}=\eta(b)$ may take -1 or 1 . Note that the hypothesis $l=\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$. Therefore, there exists an $(s+l)$-dimensional subspace $H_{s+l} \subset \mathbb{F}_{p^{m}}$ with $\bar{H}_{s+l}=$ $\varphi\left(H_{s+l}\right)=\bar{H}_{s}$. Thus the assertion is true since $1 \leq m-r \leq s+l$. By Lemma 2, for $s \leq r<m$, we have that $\max \left\{\left|D_{f}^{a} \bigcap H\right|: H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}=2 p^{m-r-1}$.

Case : $1 \leq r<s$. For an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$, we have

$$
\operatorname{dim}\left(\bar{H}_{m-r}\right)=\operatorname{dim}\left(H_{m-r} /\left(H_{m-r} \bigcap \mathbb{F}_{p^{m}}^{\perp}\right)\right) \geq m-r-l=2 s-r
$$

So, we have $\operatorname{dim}\left(\bar{H}_{m-r} \bigcap \bar{H}_{m-r}^{\perp}\right) \leq r$, since $\operatorname{dim}\left(\bar{H}_{m-r}\right)+\operatorname{dim}\left(\bar{H}_{m-r}^{\perp}\right)=2 s$. Noting that $R_{\bar{H}_{m-r}}=\operatorname{dim}\left(\bar{H}_{m-r}\right)-\operatorname{dim}\left(\bar{H}_{m-r} \bigcap \bar{H}_{m-r}^{\perp}\right)$. By Proposition 2, we have $R_{H_{m-r}}=R_{\bar{H}_{m-r}} \geq 2 s-2 r$. By Lemma 2, we have

$$
\left|H_{m-r} \bigcap D_{f}^{a}\right| \leq p^{m-r-1}+p^{s+l-1}
$$

and $\left|H_{m-r} \bigcap D_{f}^{a}\right|$ may reach the upper bound $p^{m-r-1}+p^{s+l-1}$ if $R_{H_{m-r}}=2 s-$ $2 r+1$ or $2 s-2 r$. We assert that there is such an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ with $\left|H_{m-r} \bigcap D_{f}^{a}\right|=p^{m-r-1}+p^{s+l-1}$. By the construction of $\bar{H}_{s}$ as above, we have an $r$-dimensional subspace $\bar{H}_{r} \subset \overline{\mathbb{F}}_{p^{m}}$ satisfying $R_{\bar{H}_{r}}=1$ and $\epsilon_{\bar{H}_{r}}$ may take values -1 or 1 . And $\operatorname{dim}\left(\bar{H}_{r}^{\perp}\right)=2 s-r, R_{\bar{H}_{r}^{\perp}}=2 s-2 r+1$. By Proposition 1 , the values of $\epsilon_{\bar{H}_{r}^{\perp}}$ may take -1 or 1 , too. Note that $l=\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$ and $m-r=2 s-r+l$. Thus we can construct an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ satisfying $\bar{H}_{m-r}=\bar{H}_{r}^{\perp}$. Notice that $\epsilon_{H_{m-r}}=\epsilon_{\bar{H}_{r}^{\perp}}$. Therefore, $\left|D_{f}^{a} \bigcap H_{m-r}\right|=p^{m-r-1} \pm p^{s+l-1}$. By Lemma 2, we have that $\max \left\{\left|D_{f}^{a} \bigcap H\right|\right.$ : $\left.H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}=p^{m-r-1}+p^{s+l-1}$.

By Lemma 2, we have $\left|D_{f}^{a}\right|=p^{m-1}-\epsilon_{f}(-1)^{\frac{s(p-1)}{2}} p^{s+l-1}$. Then the desired results follow directly from Lemma 1. And we complete the proof.

Example 3 Let $(p, m)=(3,4)$ and $f(x)=\operatorname{Tr}\left(x^{12}\right)=\operatorname{Tr}\left(x^{3^{2}+3}\right)$. Then $s=$ $1, l=2, \epsilon_{f}=1$ and the weight hierarchy of $C_{D_{f}^{1}}$ is $d_{1}=18, d_{2}=30, d_{3}=$ $34, d_{4}=36$.

Theorem 2. Let $f$ be a degenerate quadratic form over $\mathbb{F}_{p^{m}}$ with rank $R_{f}=2 s+1$ and $a$ a non-zero element in $\mathbb{F}_{p}^{*}$. Suppose $m=2 s+1+l, l=$ $\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$. If $\eta(a)=(-1)^{\frac{s(p-1)}{2}} \epsilon_{f}$, then for the linear codes defined in (2) we have

$$
d_{r}\left(C_{D_{f}^{a}}\right)= \begin{cases}p^{m-1}-p^{m-r-1}, & \text { if } 1 \leq r \leq s, \\ p^{m-1}+p^{s+l}-2 p^{m-r-1}, & \text { if } s<r<m, \\ p^{m-1}+p^{s+l}, & \text { if } r=m .\end{cases}
$$

Proof. Case : $1 \leq r \leq s$. For an $(m-r)$-dimensional subspace $H_{m-r} \subset$ $\mathbb{F}_{p^{m}}$, we have

$$
\operatorname{dim}\left(\bar{H}_{m-r}\right)=\operatorname{dim}\left(H_{m-r} /\left(H_{m-r} \bigcap \mathbb{F}_{p^{m}}^{\perp}\right)\right) \geq m-r-l=2 s+1-r
$$

We have $\operatorname{dim}\left(\bar{H}_{m-r} \bigcap \bar{H}_{m-r}^{\perp}\right) \leq r$, since $\operatorname{dim}\left(\bar{H}_{m-r}\right)+\operatorname{dim}\left(\bar{H}_{m-r}^{\perp}\right)=2 s+1$. Noting that $R_{\bar{H}_{m-r}}=\operatorname{dim}\left(\bar{H}_{m-r}\right)-\operatorname{dim}\left(\bar{H}_{m-r} \bigcap \bar{H}_{m-r}^{\perp}\right)$. By Proposition 2, we have $R_{H_{m-r}}=R_{\bar{H}_{m-r}} \geq 2 s+1-2 r$. By Lemma 2, we have

$$
\left|H_{m-r} \bigcap D_{f}^{a}\right| \leq p^{m-r-1}+p^{s+l}
$$

and $\left|H_{m-r} \bigcap D_{f}^{a}\right|$ may reach the upper bound $p^{m-r-1}+p^{s+l}$ if $R_{H_{m-r}}=$ $2 s+1-2 r$. We assert that there is such an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ with $\left|H_{m-r} \bigcap D_{f}^{a}\right|=p^{m-r-1}+p^{s+l}$, which is constructed as follows. Applying Lemma 3 to $\overline{\mathbb{F}}_{p^{m}}$, we know there is an $r$-dimensional subspace $\bar{H}_{r} \subset \overline{\mathbb{F}}_{p^{m}}$ with $\bar{H}_{r} \subset \bar{H}_{r}^{\perp}$. So $\operatorname{dim}\left(\bar{H}_{r}^{\perp}\right)=2 s+1-r, R_{\bar{H}_{r}^{\perp}}=2 s-2 r+1$. Note that $l=\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$ and $m-r=2 s-r+1+l$. Thus we have an $(m-r)-$ dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ satisfying $\bar{H}_{m-r}=\bar{H}_{r}^{\perp}$. By Proposition 1, we have $\epsilon_{H_{m-r}}=\eta(-1)^{r} \epsilon_{f}$, since $\epsilon_{\bar{H}_{r}}=1, \epsilon_{f}=\epsilon_{\bar{f}}$ and $\epsilon_{H_{m-r}}=\epsilon_{\bar{H}_{m-r}}=\epsilon_{\bar{H}_{r}^{\perp}}$. Therefore, by hypothesis and Lemma 2, $\left|D_{f}^{a} \bigcap H_{m-r}\right|=p^{m-r-1}+p^{s+l}$. By Lemma 2, we have that $\max \left\{\left|D_{f}^{a} \bigcap H\right|: H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}=p^{m-r-1}+p^{s+l}$.

Case : $s<r<m$. The proof is similar to that of Theorem 1 .
By Lemma 2, we have $\left|D_{f}^{a}\right|=p^{m-1}+p^{s+l}$. Then the desired conclusions follow from Lemma 1. And the proof is completed.

Example 4 Let $(p, m)=(3,4)$ and $f(x)=\operatorname{Tr}\left(x^{2}+x^{3+1}\right)$. Then $s=1, l=$ $1, \epsilon_{f}=-1$ and the weight hierarchy of $C_{D_{f}^{1}}$ is $d_{1}=18, d_{2}=30, d_{3}=34, d_{4}=$ 36.

Theorem 3. Let $f$ be a degenerate quadratic form over $\mathbb{F}_{p^{m}}$ with rank $R_{f}=2 s+1$ and $a$ a non-zero element in $\mathbb{F}_{p}^{*}$. Suppose $m=2 s+1+l, l=$ $\operatorname{dim}\left(\mathbb{F}_{p^{m}}^{\perp}\right)$. If $\eta(a)=-(-1)^{\frac{s(p-1)}{2}} \epsilon_{f}$, then for the linear codes defined in (2) we have

$$
d_{r}\left(C_{D_{f}^{a}}\right)= \begin{cases}p^{m-1}-p^{m-r-1}-p^{s+l}-p^{s+l-1}, & \text { if } 1 \leq r \leq s \\ p^{m-1}-p^{s+l}-2 p^{m-r-1}, & \text { if } s<r<m \\ p^{m-1}-p^{s+l}, & \text { if } r=m\end{cases}
$$

Proof. Case : $1 \leq r \leq s$. for an $(m-r)$-dimensional subspace $H_{m-r} \subset$ $\mathbb{F}_{p^{m}}$, we have $R_{H_{m-r}} \geq 2 s-2 r+1$. By the corresponding proof of Theorem 2, we know that $\epsilon_{H_{m-r}}=\eta(-1)^{r} \epsilon_{f}$ if $R_{H_{m-r}}=2 s-2 r+1$. By hypothesis and Lemma 2, we have $\left|D_{f}^{a} \bigcap H_{m-r}\right|=p^{m-r-1}-p^{s+l}$.

Next we will construct an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ with $R_{H_{m-r}}=2 s-2 r+2$ and discuss the value of $\left|D_{f}^{a} \bigcap H_{m-r}\right|$. Applying Lemma 3 to $\overline{\mathbb{F}}_{p^{m}}$, there is an $(r-1)$-dimensional subspace $\bar{H}_{r-1} \subset \overline{\mathbb{F}}_{p^{m}}$ with $\bar{H}_{r-1} \subset$ $\bar{H}_{r-1}^{\perp}$. So $\operatorname{dim}\left(\bar{H}_{r-1}^{\perp}\right)=2 s-r+2, R_{\bar{H}_{r-1}}^{\perp}=2 s-2 r+3$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^{m}}$, we have, for each $b \in \mathbb{F}_{p}^{*},\left|\bar{D}_{f}^{b} \bigcap \bar{H}_{r-1}^{\perp}\right|>1$. We choose an element $\alpha \in\left(\bar{D}_{f}^{b} \bigcap \bar{H}_{r-1}^{\perp}\right)$ and let $\bar{H}_{r}=\langle\alpha\rangle \bigoplus \bar{H}_{r-1}$. Then $\operatorname{dim}\left(\bar{H}_{r}\right)=r, R_{\bar{H}_{r}}=1$ and the values of $\epsilon_{\bar{H}_{r}}=\eta(b)$ may take -1 or 1 . So $\operatorname{dim}\left(\bar{H}_{r}^{\perp}\right)=2 s+1-r, R_{\bar{H}_{r}^{\perp}}=$ $2 s+2-2 r$ and $\epsilon_{\bar{H}_{r}^{\perp}}^{\perp}$ may take values -1 or 1 , too. Therefore, there exists an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^{m}}$ with $\bar{H}_{m-r}=\bar{H}_{r}^{\perp}$. By Lemma 2, we have that $\left|D_{f}^{a} \bigcap H\right|=p^{m-r-1} \pm p^{s+l-1}$. Therefore, also by Lemma 2, we have $\max \left\{\left|D_{f}^{a} \bigcap H\right|: H \in\left[\mathbb{F}_{p^{m}}, m-r\right]_{p}\right\}=p^{m-r-1}+p^{s+l-1}$.

Case : $s<r<m$. The proof is similar to that of Theorem 1. We omit the details.

By Lemma 2, we have $\left|D_{f}^{a}\right|=p^{m-1}-p^{s+l}$. Then the desired conclusions follow from Lemma 1. And the proof is completed.

Example 5 Let $(p, m)=(3,4)$ and $f(x)=\operatorname{Tr}\left(x^{2}-x^{3+1}\right)$. Then $s=1, l=$ $1, \epsilon_{f}=1$ and the weight hierarchy of $C_{D_{f}^{1}}$ is $d_{1}=6, d_{2}=12, d_{3}=16, d_{4}=18$.

Examples 3-5 have been verified by Magma.

## References

1. M. Bras-Amorós, K. Lee, and A. Vico-Oton, New lower bounds on the generalized Hamming weights of AG codes, IEEE Trans. Inf. Theory, 60(10), 5930-5937(2014).
2. A. I. Barbero and C. Munuera, The weight hierarchy of Hermitian codes, SIAM J. Discrete Math., 13(1), 79-104(2000).
3. J. Cheng and C.-C. Chao, On generalized Hamming weights of binary primitive BCH codes with minimum distance one less than a power of two, IEEE Trans. Inf. Theory, 43(1), 294-298(1997).
4. C. Ding, Linear codes from some 2-designs, IEEE Trans. Inf. Theory, 61(6), 32653275(2015).
5. K. Ding, C. Ding, A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Trans. Inf. Theory, 61(11), 5835-5842(2015).
6. K. Ding, C. Ding, Bianry linear codes with three weights, IEEE Communication Letters, 18(11), 1879-1882(2014).
7. M. Delgado, J. I. Farrán, P. A. García-Sánchez, and D. Llena, On the weight hierarchy of codes coming from semigroups with two generators, IEEE Trans. Inf. Theory, 60(1), 282-295(2014).
8. X. Du and Y. Wan, Linear codes from quadratic forms, Applicable Algebra in Engineering, Communication and Computing, 28(6), 535-547(2017).
9. C. Ding, C. Li, N.Li, Z. Zhou, Three-weight cyclic codes and their weight distributions. Discret. Math., 339(2), 415-427 (2016).
10. P. Heijnen and R. Pellikaan, Generalized Hamming weights of q-ary Reed-Muller codes, IEEE Trans. Inf. Theory, 44(1), 181-196(1998).
11. Helleseth, T. , T. Kl $\phi$ ve and O. Ytrehus, Generalized Hamming Weights of Linear Codes, IEEE Trans. Inf. Theory, 38(3),1133-1140(1992).
12. W. C. Huffman and V. Pless, Fundamentals of error-correcting codes, Cambridge University Press, Cambridge(2003).
13. G. Jian, R. Feng and H. Wu, Generalized Hamming weights of three classes of linear codes, Finite Fields and Their Applications, 45, 341-354(2017).
14. H. Janwa and A. K. Lal, On the generalized Hamming weights of cyclic codes, IEEE Trans. Inf. Theory, 43(1), 299-308(1997).
15. T. Kl $\phi \mathrm{ve}$, The weight distribution of linear codes over $G F\left(q^{l}\right)$ having generator matrix over $G F(q)^{*}$, Discrete Math., 23(2), 159-168(1978).
16. R. Lidl, H. Niederreiter, Finite fields, Cambridge University Press, New York(1997).
17. S. Lang, Algebra. 3rd edition, Springer-verlag, New York(2002).
18. C. Li, Q. Yue, and F. Li, Hamming weights of the duals of cyclic codes with two zeros, IEEE Trans. Inf. Theory, 60(7), 3895-3902(2014).
19. F. Li, A class of cyclotomic linear codes and their generalized Hamming weights, Applicable Algebra in Engineering, Communication and Computing, 29(2018) 501-511.
20. F. Li, Weight hierarchies of a class of linear codes related to non-degenerate quadratic forms, IEEE Trans. Inf. Theory, doi:10.1109/TIT.2020.3021730(2020).
21. C. Tang, C. Xiang and K. Feng, Linear codes with few weights from inhomogeneous quadratic functions, Des. Codes Cryptogr., 83(3), 691-714(2017).
22. M. A. Tsfasman, S. G. Vlădut, Geometric approach to higher weights, IEEE Trans. Inf. Theory, 41(6), 1564-1588(1995).
23. Z. Wan, The weight hierarchies of the projective codes from nondegenerate quadrics, Des. Codes Cryptogr., 4(4), 283-300(1994).
24. Z. Wan, X. Wu, The weight hierarchies and generalized weight spectra of the projective codes from degenerate quadrics, Discrete Mathematics, 177,223-243(1997).
25. V.K.Wei, Generalized Hamming weights for linear codes, IEEE Trans. Inf. Theory, 37(5), 1412-1418(1991).
26. M. Xiong, S. Li, and G. Ge. The weight hierarchy of some reducible cyclic codes. IEEE Trans. Inf. Theory, 62(7), 4071-4080(2016).
27. M. Yang, J. Li, K. Feng and D. Lin, Generalized Hamming weights of irreducible cyclic codes, IEEE Trans. Inf. Theory, 61(9), 4905-4913(2015).
28. S. Yang, Z. Yao, C. Zhao, The weight distributions of two classes of p-ary cyclic codes with few weights, Finite Fields and Their Applications 44, 76-91(2017).
29. D. Zhang, C. Fan, D, Peng, X, Tang, Complete weight enumerators of some linear codes from quadratic forms. Cryptogr. Commun., 9(1), 151-163(2017).
30. Z. Zhou, N. Li, C. Fan and T. Helleseth, Linear codes with two or three weights from quadratic bent functions, Des. Codes Cryptogr., 81(2), 283-295(2016).

[^0]:    This research is supported in part by Anhui Provincial Natural Science Foundation (No. 1908085MA02) and National Natural Science Foundation of China (No. 11701001).

    Fei Li
    E-mail: cczxlf@163.com
    Faculty of School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu, 233041, Anhui, P.R.China
    Xiumei Li
    E-mail: lxiumei2013@mail.qfnu.edu.cn; Faculty of School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, Shandong, P.R.China

