Weight hierarchies of a family of linear codes associated with degenerate quadratic forms

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Abstract We restrict a degenerate quadratic form f over a finite field of odd characteristic to subspaces. Thus, a quotient space related to f is introduced. Then we get a non-degenerate quadratic form induced by f over the quotient space. Some related results on the subspaces and quotient space are obtained. Based on this, we solve the weight hierarchies of a family of linear codes related to f.

Keywords Weight hierarchy \cdot Generalized Hamming weight \cdot Linear code \cdot Quadratic form \cdot Quotient space

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1 Introduction

Weight hierarchies of linear codes have been an interesting topic for their important value in theory and applications to cryptography. In 1991, Wei in

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the paper [25] presented his wonderful results about weight hierarchies. It has been shown that the weight hierarchy of a linear code completely characterizes the performance of the code on the type II wire-tap channel. Readers can refer to [22] for a detailed survey on the results up to 1995 about weight hierarchies. The interest towards the knowledge of the weight hierarchy of a linear code has been continually increasing. Many authors devoted themselves to weight hierarchies of particular classes of codes [1,2,3,7,10,13,14,26,27]. In general, it is hard to settle the weight hierarchy of a linear code.

Let p be an odd prime and \mathbb{F}_{p^m} be the finite field with p^m elements. Denote by $C \subset \mathbb{F}_p^n$ an [n, k, d] p-ary linear code with minimum Hamming distance d[12]. Let $[C, r]_p$ be the set of all r-dimensional subspaces of C. For $V \in [C, r]_p$, define

$$Supp(V) = \{i : x_i \neq 0 \text{ for some } x = (x_1, x_2, \dots, x_n) \in V\}.$$

Then we define the r-th $(1 \le r \le k)$ generalized Hamming weight $d_r(C)$ of linear code C by

$$d_r(C) = \min\{|Supp(V)| : V \in [C, r]_p\}.$$

In particular, $d_1(C) = d$. The weight hierarchy of C is defined as the set $\{d_i(C) : 1 \le i \le k\}$ (see [11,12,15]).

Denote by $\mathbb{F}_{p^m}^*$ the set of nonzero elements in the finite field \mathbb{F}_{p^m} . A generic construction of linear code was proposed by Ding et al.([4,6]). It is as follows. Let Tr denote the trace function from \mathbb{F}_{p^m} to \mathbb{F}_p and $D = \{d_1, d_2, \ldots, d_n\} \subset \mathbb{F}_{p^m}^*$. Define a linear code C_D with length n as follows:

$$C_D = \{(\operatorname{Tr}(xd_1), \operatorname{Tr}(xd_2), \dots, \operatorname{Tr}(xd_n)) : x \in \mathbb{F}_{p^m}\},\tag{1}$$

and D is called the defining set. Many classes of linear codes with a few weights were obtained by choosing properly defining sets [9,18,28,30,21].

In this paper, we discuss the generalized Hamming weights of a class of linear codes C_D , whose defining set is chosen to be

$$D = D_f^a = \{ x \in \mathbb{F}_{p^m} : f(x) = a \}, \ a \in \mathbb{F}_p^*.$$
(2)

Here f is a degenerate quadratic form over \mathbb{F}_{p^m} with values in \mathbb{F}_p .

In the paper, we settle the weight hierarchy of $C_{D_f^a}$, $a \in \mathbb{F}_p^*$. In our previous work [20], the weight hierarchy of $C_{D_f^a}$ ($a \in \mathbb{F}_p^*$) relating to non-degenerate quadratic forms was solved. In [23,24], Z. Wan and X. Wu calculated the weight hierarchies of the projective codes from quadrics by the theory of finite geometry. In the case a = 0, the weight hierarchy of $C_{D_f^a}$ can be deduced from Theorem 18 in [24].

The weight distributions of $C_{D_f^a}$ have been settled. In reference [5], K. Ding and C. Ding constructed the linear codes $C_{D_f^a}$ in the case a = 0 relating to the special quadratic form $\text{Tr}(X^2)$ and determined their weight distributions. In [8,30,29], the authors calculated the weight distributions of $C_{D_f^a}$ for general quadratic forms. In these articles, it was shown that the linear codes $C_{D_f^a}$ have a few weights and can be used to get association schemes, authentication codes, secret sharing schemes with interesting access structures.

Also, by these results, we know that $C_{D_f^{\alpha}}$ is an *m*-dimensional linear code. So we can employ a general formula for calculating the generalized Hamming weights of linear codes defined in (1). It is given as follows.

Lemma 1.(Theorem 1, [19]) For each r $(1 \le r \le m)$, if the dimension of C_D is m, then $d_r(C_D) = n - \max\{|D \bigcap H| : H \in [\mathbb{F}_{p^m}, m - r]_p\}.$

The rest of this paper is organized as follows: in Sect. 2, we present some basic definitions and results of quadratic forms restricted to subspaces and of induced quadratic forms over quotient spaces of finite fields; in Sect. 3, using the results in Sect. 2, we give all the generalized Hamming weights of linear codes defined in (2).

2 Quadratic Form, Dual Space and Quotient Space

2.1 Restricting Quadratic Forms to Subspaces

The finite field \mathbb{F}_{p^m} can be viewed as an *m*-dimensional vector space over \mathbb{F}_p . Fix a basis $v_1, v_2, \ldots, v_m \in \mathbb{F}_{p^m}$ and express each vector $X \in \mathbb{F}_{p^m}$ in the unique form $X = x_1v_1 + x_2v_2 + x_mv_m$, with $x_1, x_2, \ldots, x_m \in \mathbb{F}_p$. We can write $X = (x_1, x_2, \ldots, x_m)^T$, where *T* represents the transpose of a matrix.

Let $f: \mathbb{F}_{p^m} \to \mathbb{F}_p$ be a quadratic form over \mathbb{F}_{p^m} with values in \mathbb{F}_p [16]. Set

$$F(X,Y) = \frac{1}{2}[f(X+Y) - f(X) - f(Y)].$$

We can write $f(X) = X^T A X$, where A is the symmetric matrix $(a_{ij})_{1 \le i,j \le m}$ and $a_{ii} = f(v_i), a_{ij} = F(v_i, v_j)$.

The rank R_f of quadratic form f is defined to be the rank of matrix A. We say that f is non-degenerate if $R_f = m$ and degenerate, otherwise. We can find a invertible matrix M such that $M^T A M$ is a diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_{R_f}, 0, \ldots, 0)$. Let $\Delta_f = \lambda_1 \cdot \lambda_2 \cdots \lambda_{R_f}$. When $R_f = 0$, we define $\Delta_f = 1$. Let η be the quadratic character of \mathbb{F}_p , i.e., $\eta(a) = a^{\frac{p-1}{2}}$ for $a \in \mathbb{F}_p^*$. In the paper, $\eta(0)$ is defined to be zero. Under the congruent transformation of $A \to M^T A M$, $\eta(\Delta_f)$ is an invariant. We called $\eta(\Delta_f)$, denoted by ϵ_f , the sign of the quadratic form f.

For a subspace $H \subseteq \mathbb{F}_{p^m}$, define

$$H^{\perp} = \{ x \in \mathbb{F}_{p^m} : F(x, y) = 0 \text{ for each } y \in H \}$$

Then H^{\perp} is called the dual space of H. And R_f can also be defined as the codimension of $\mathbb{F}_{p^m}^{\perp}$. Namely, $R_f + \dim(\mathbb{F}_{p^m}^{\perp}) = m$.

Let H be a d-dimensional subspace of \mathbb{F}_{p^m} . Restricting the quadratic form f to H, we get a quadratic form over H in d variables. It is denoted by $f|_H$. Similarly, we define the dual space $H_{f|_H}^{\perp}$ of H under $f|_H$ in itself by

$$H_{f|_{H}}^{\perp} = \{x \in H : f(x+y) = f(x) + f(y) \text{ for each } y \in H\}.$$

Let R_H , ϵ_H be the rank and sign of $f|_H$ over H, respectively. Obviously, $H_{f|_H}^{\perp} = H \bigcap H^{\perp}$ and $R_H = d - \dim(H_{f|_H}^{\perp})$.

Example 1 Let $f(X) = x_1^2 - 2x_1x_2 + x_2^2$ with $X = (x_1, x_2)$. It is a degenerate quadratic form over \mathbb{F}_p^2 . After simple calculation, we have $F(X, Y) = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$, where $Y = (y_1, y_2)$. Let $H = \{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = x_2\}$. It is not hard to get $\mathbb{F}_{p^m}^\perp = H, R_f = 1, f|_H = 0, H_{f|_H}^\perp = H, R_H = 0$ and $\epsilon_H = 1$.

For $a \in \mathbb{F}_p$, the following lemma tells us the number of solutions in H of the equation f(x) = a.

Lemma 2.(Proposition 1, [20]) Let f be a quadratic form over \mathbb{F}_{p^m} , $a \in \mathbb{F}_p$ and H be a d-dimensional (d > 0) subspace of \mathbb{F}_{p^m} , then the number of solutions of f(X) = a in H is

$$|H \bigcap D_f^a| = \begin{cases} p^{d-1} + v(a)\eta((-1)^{\frac{R_H}{2}})\epsilon_H p^{d-\frac{R_H+2}{2}}, & \text{if } R_H \equiv 0 \pmod{2}, \\ p^{d-1} + \eta((-1)^{\frac{R_H-1}{2}}a)\epsilon_H p^{d-\frac{R_H+1}{2}}, & \text{if } R_H \equiv 1 \pmod{2}, \end{cases}$$

where v(a) = p - 1 if a = 0, otherwise v(a) = -1.

Remark In Proposition 1 of [20], f is set to be non-degenerate. In fact, we know from the proof that this condition is unnecessary. Namely, f can be any quadratic form.

For later use, we need two results in the case that f is a non-degenerate quadratic form. They are listed as follows.

Lemma 3.(Proposition 2, [20]) Let f be a non-degenerate quadratic form over \mathbb{F}_{p^m} . For each r with 0 < 2r < m, there exist an r-dimensional subspace $H \subseteq \mathbb{F}_{p^m}$ (m > 2) such that $H \subseteq H^{\perp}$.

For k elements $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{F}_{p^m}$, the matrix $M(\beta_1, \beta_2, \ldots, \beta_k)$ of them is defined as the $k \times k$ square matrix $(F(\beta_i, \beta_j))_{1 \le i,j \le k}$. The discriminant $\Delta(\beta_1, \beta_2, \ldots, \beta_k)$ of them is defined to be $\det(M(\beta_1, \beta_2, \ldots, \beta_k))$. We denote by $\langle \beta_1, \beta_2, \ldots, \beta_k \rangle$ the subspace spanned by $\beta_1, \beta_2, \ldots, \beta_k$.

Proposition 1. Let f be a non-degenerate quadratic form over \mathbb{F}_{p^m} and $H \subset \mathbb{F}_{p^m}$ a subspace with $\dim(H \cap H^{\perp}) = e$. Then, $\epsilon_H \epsilon_{H^{\perp}} = (-1)^{\frac{e(p-1)}{2}} \epsilon_f$.

Proof. Suppose $\dim(H) = r$. By hypothesis, we can set

$$H = \langle \alpha_1, \alpha_2, \dots, \alpha_{r-e}, \beta_1, \beta_2, \dots, \beta_e \rangle,$$
$$H^{\perp} = \langle \gamma_1, \gamma_2, \dots, \gamma_{m-r-e}, \beta_1, \beta_2, \dots, \beta_e \rangle,$$

 $\langle \alpha_1, \alpha_2, \dots, \alpha_{r-e}, \gamma_1, \gamma_2, \dots, \gamma_{m-r-e} \rangle^{\perp} = \langle \eta_1, \eta_2, \dots, \eta_e, \beta_1, \beta_2, \dots, \beta_e \rangle.$ We have $\epsilon_H = \eta(\Delta(\alpha_1, \alpha_2, \dots, \alpha_{r-e})), \ \epsilon_{H^{\perp}} = \eta(\Delta(\gamma_1, \gamma_2, \dots, \gamma_{m-r-e})).$ And

$$\epsilon_f = \eta(\Delta(\alpha_1, \alpha_2, \dots, \alpha_{r-e}, \gamma_1, \gamma_2, \dots, \gamma_{m-r-e}, \beta_1, \beta_2, \dots, \beta_e, \eta_1, \eta_2, \dots, \eta_e))$$

= $\eta(\Delta(\alpha_1, \dots, \alpha_{r-e}, \gamma_1, \dots, \gamma_{m-r-e}))\eta(\Delta(\beta_1, \dots, \beta_e, \eta_1, \dots, \eta_e))$
= $\eta(\Delta(\alpha_1, \dots, \alpha_{r-e}))\eta(\Delta(\gamma_1, \dots, \gamma_{m-r-e}))\eta((-1)^e)\eta(\det(M^2)))$
= $\epsilon_H \epsilon_{H^{\perp}} \eta((-1)^e)\eta(\det(M_e^2)).$

Here M_e is the square matrix $(F(\beta_i, \eta_j))_{1 \le i,j \le e}$. Then the desired result follows and we complete the proof.

2.2 Induced Quadratic Form over a Quotient Space

From now on, we suppose f is a degenerate quadratic form. Let $\overline{\mathbb{F}}_{p^m}$ be the quotient space $\mathbb{F}_{p^m}/\mathbb{F}_{p^m}^{\perp}$. For $\overline{\alpha} \in \overline{\mathbb{F}}_{p^m}$, define $\overline{f}(\overline{\alpha}) = f(\alpha)$. It is welldefined. We obtain a non-degenerate quadratic form \overline{f} , induced by f, over $\overline{\mathbb{F}}_{p^m}$. Without confusion, we still use f to denote \overline{f} . Let $R_{\overline{f}}$ and $\epsilon_{\overline{f}}$ denote the rank and sign of f over $\overline{\mathbb{F}}_{p^m}$, respectively. It is easy to see $R_{\overline{f}} = R_f$ and $\epsilon_{\overline{f}} = \epsilon_f$.

Since f is a non-degenerate quadratic form over $\overline{\mathbb{F}}_{p^m}$, the results in Lemmas 2, 3 and Proposition 1 can be applied to $\overline{\mathbb{F}}_{p^m}$.

For $a \in \mathbb{F}_p$, set

$$\overline{D}_f^a = \{ \overline{x} \in \overline{\mathbb{F}}_{p^m} : f(\overline{x}) = a \}$$

Obviously, $|\overline{\mathbb{F}}_{p^m} \cap \overline{D}_f^a| = |\overline{D}_f^a|$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^m}$, we have

$$|\overline{D}_{f}^{a}| = \begin{cases} p^{\overline{m}-1} + v(a)\eta((-1)^{\frac{R_{f}}{2}})\epsilon_{f}p^{\overline{m}-\frac{R_{f}+2}{2}}, \text{ if } R_{f} \equiv 0 \pmod{2}, \\ p^{\overline{m}-1} + \eta((-1)^{\frac{R_{f}-1}{2}}a)\epsilon_{f}p^{\overline{m}-\frac{R_{f}+1}{2}}, \text{ if } R_{f} \equiv 1 \pmod{2}, \end{cases}$$

where $\overline{m} = R_f = \dim(\overline{\mathbb{F}}_{p^m}) = m - \dim(\mathbb{F}_{p^m}^{\perp}).$

Example 2 Just like Example 1, let $f(X) = x_1^2 - 2x_1x_2 + x_2^2$ with $X = (x_1, x_2)$, a degenerate quadratic form over $\mathbb{F}_p^2 \cong \mathbb{F}_{p^2}$. Because $\mathbb{F}_{p^2}^{\perp} = \{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = x_2\}$, $\overline{\mathbb{F}}_{p^2}$ is isomorphic to $\{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = -x_2\}$. So, $\overline{f} = 4x_1^2$ is a non-degenerate quadratic form over $\overline{\mathbb{F}}_{p^2}$. It is not hard to get $R_{\overline{f}} = R_f = 1$, $\epsilon_{\overline{f}} = \epsilon_f = 1$, and

$$|\overline{D}_{f}^{a}| = \begin{cases} 1, \text{ if } a = 0, \\ 2, \text{ if } \eta(a) = 1, \\ 0, \text{ if } \eta(a) = -1 \end{cases}$$

Let φ be the canonical map from \mathbb{F}_{p^m} to $\overline{\mathbb{F}}_{p^m}$ [17]. For a subspace $H \subset \mathbb{F}_{p^m}$, denote by \overline{H} the image of H under φ , i.e., $\overline{H} = \varphi(H)$. In the absence of confusion, also we use \overline{H} to represent a subspace of $\overline{\mathbb{F}}_{p^m}$. Let $R_{\overline{H}}$ and $\epsilon_{\overline{H}}$ denote the rank and sign of f over \overline{H} , respectively. **Proposition 2.** Let H be a subspace of \mathbb{F}_{p^m} and $\overline{H} = \varphi(H) \subseteq \overline{\mathbb{F}}_{p^m}$, then $R_{\overline{H}} = R_H, \ \epsilon_{\overline{H}} = \epsilon_H.$

Proof. Suppose dim(H) = r, dim $(H \bigcap H^{\perp}) = t$. Then we set

$$H\bigcap H^{\perp} = \langle \beta_1, \beta_2, \dots, \beta_t \rangle, \quad H = \langle \alpha_1, \alpha_2, \dots, \alpha_{r-t}, \beta_1, \beta_2, \dots, \beta_t \rangle.$$

So we have $\overline{H} = \langle \overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{r-t}, \overline{\beta}_1, \overline{\beta}_2, \dots, \overline{\beta}_t \rangle.$

The matrix $M(\alpha_1, \alpha_2, \ldots, \alpha_{r-t}, \beta_1, \beta_2, \ldots, \beta_t)$ is the block matrix

$$\begin{pmatrix} M_1 & O \\ O & O \end{pmatrix},$$

where $M_1 = M(\alpha_1, \alpha_2, \dots, \alpha_{r-t})$. Then $R_H = Rank(M_1), \ \epsilon_H = \eta(\det(M_1))$.

And the matrix $M(\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{r-t}, \overline{\beta}_1, \overline{\beta}_2, \dots, \overline{\beta}_t)$ is the block matrix

$$\begin{pmatrix} \overline{M}_1 & O \\ O & O \end{pmatrix},$$

where $\overline{M}_1 = M(\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{r-t})$. Then $R_{\overline{H}} = Rank(\overline{M}_1)$, $\epsilon_{\overline{H}} = \eta(\det(\overline{M}_1))$. In fact, $\overline{M}_1 = M_1$, since $f(\overline{x}) = f(x)$ for each $x \in \mathbb{F}_{p^m}$. Hence the desired results follow directly and we complete the proof.

Define the dual space \overline{H}^{\perp} of \overline{H} by

$$\overline{H}^{\perp} = \{ \overline{x} \in \overline{\mathbb{F}}_{p^m} : f(\overline{x} + \overline{y}) = f(\overline{x}) + f(\overline{y}) \text{ for each } \overline{y} \in \overline{H} \}$$

For the dual spaces, we have an interesting conclusion as below.

Proposition 3. Let H be a subspace of \mathbb{F}_{p^m} , then $\overline{H}^{\perp} = \overline{H^{\perp}}$.

Proof. Let \overline{x} be an element of $\overline{H^{\perp}}$ with $x \in H^{\perp}$. We have f(x+y) = f(x) + f(y) for each $y \in H$. So $f(\overline{x+y}) = f(\overline{x}) + f(\overline{y})$. Since $\overline{x+y} = \overline{x} + \overline{y}$, $f(\overline{x}+\overline{y}) = f(\overline{x}) + f(\overline{y})$. By definition, $\overline{x} \in \overline{H}^{\perp}$, which means $\overline{H^{\perp}} \subset \overline{H}^{\perp}$. On the other hand, let \overline{x} be an element of \overline{H}^{\perp} . For each $\overline{y} \in \overline{H}$, we have $f(\overline{x}+\overline{y}) = f(\overline{x}) + f(\overline{y})$. So $f(\overline{x+y}) = f(\overline{x}) + f(\overline{y})$ and f(x+y) = f(x) + f(y). Thus $x \in H^{\perp}$ and $\overline{x} \in \overline{H^{\perp}}$. Therefore $\overline{H^{\perp}} \supset \overline{H}^{\perp}$. In a word, $\overline{H}^{\perp} = \overline{H^{\perp}}$. The proof is finished.

3 Weight Hierarchies of Linear Codes Defined in (2)

By our method, we have successfully settled the weight hierarchies of $C_{D_f^a}$. In this case a = 0, the weight hierarchies can be derived from Theorem 18 in [24]. In this section, we will just present the weight hierarchies of $C_{D_f^a}$ in the case $a \in \mathbb{F}_p^*$.

Theorem 1. Let f be a degenerate quadratic form over \mathbb{F}_{p^m} with rank $R_f = 2s$ and a a non-zero element in \mathbb{F}_p^* . Suppose m = 2s + l, $l = \dim(\mathbb{F}_{p^m}^{\perp})$, then for the linear codes defined in (2), we have

$$d_r(C_{D_f^a}) = \begin{cases} p^{m-1} - p^{m-r-1} - ((-1)^{\frac{s(p-1)}{2}} \epsilon_f + 1)p^{s+l-1}, & \text{if } 1 \le r \le s, \\ p^{m-1} - 2p^{m-r-1} - (-1)^{\frac{s(p-1)}{2}} \epsilon_f p^{s+l-1}, & \text{if } s \le r < m, \\ p^{m-1} - (-1)^{\frac{s(p-1)}{2}} \epsilon_f p^{s+l-1}, & \text{if } r = m. \end{cases}$$

Proof. We will use Lemma 1 to compute $d_r(C_{D_f^a})$. To do so, we need to know the value of $\max\{|D_f^a \cap H| : H \in [\mathbb{F}_{p^m}, m-r]_p\}$.

Case : $s \leq r < m$. If H_{m-r} is an (m-r)-dimensional subspace of \mathbb{F}_{p^m} , then, by Lemma 2, we have

$$|H_{m-r}\bigcap D_f^a| \le 2p^{m-r-1}$$

and $|H_{m-r} \bigcap D_f^a|$ may reach the upper bound $2p^{m-r-1}$ if $R_{H_{m-r}} = 1$ or 0. We assert that there exists an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ satisfying $R_{H_{m-r}} = 1$ and $\epsilon_{H_{m-r}}$ may take values -1 or 1. Applying Lemma 3 to $\overline{\mathbb{F}}_{p^m}$, there is an (s-1)-dimensional subspace $\overline{H}_{s-1} \subset \overline{\mathbb{F}}_{p^m}$ with $\overline{H}_{s-1} \subset \overline{H}_{s-1}^{\perp}$. So $\dim(\overline{H}_{s-1}^{\perp}) = s + 1$, $R_{\overline{H}_{s-1}^{\perp}} = 2$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^m}$, for each $b \in \mathbb{F}_p^*$, $|\overline{D}_f^b \bigcap \overline{H}_{s-1}^{\perp}| > p^{s-1}$. We choose an element $\alpha \in (\overline{D}_f^b \bigcap \overline{H}_{s-1}^{\perp}) \setminus \overline{H}_{s-1}$ and let $\overline{H}_s = \langle \alpha \rangle \bigoplus \overline{H}_{s-1}$. Then $\dim(\overline{H}_s) = s, R_{\overline{H}_s} = 1$ and the values of $\epsilon_{\overline{H}_s} = \eta(b)$ may take -1 or 1. Note that the hypothesis $l = \dim(\mathbb{F}_p^{\perp})$. Therefore, there exists an (s+l)-dimensional subspace $H_{s+l} \subset \mathbb{F}_{p^m}$ with $\overline{H}_{s+l} =$ $\varphi(H_{s+l}) = \overline{H}_s$. Thus the assertion is true since $1 \leq m-r \leq s+l$. By Lemma 2, for $s \leq r < m$, we have that $\max\{|D_f^a \bigcap H|: H \in [\mathbb{F}_{p^m}, m-r]_p\} = 2p^{m-r-1}$.

Case : $1 \leq r < s$. For an (m - r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$, we have

$$\dim(\overline{H}_{m-r}) = \dim(H_{m-r}/(H_{m-r}\bigcap \mathbb{F}_{p^m}^{\perp})) \ge m-r-l = 2s-r.$$

So, we have $\dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^{\perp}) \leq r$, since $\dim(\overline{H}_{m-r}) + \dim(\overline{H}_{m-r}^{\perp}) = 2s$. Noting that $R_{\overline{H}_{m-r}} = \dim(\overline{H}_{m-r}) - \dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^{\perp})$. By Proposition 2, we have $R_{H_{m-r}} = R_{\overline{H}_{m-r}} \geq 2s - 2r$. By Lemma 2, we have

$$|H_{m-r} \bigcap D_f^a| \le p^{m-r-1} + p^{s+l-1},$$

and $|H_{m-r} \bigcap D_f^a|$ may reach the upper bound $p^{m-r-1} + p^{s+l-1}$ if $R_{H_{m-r}} = 2s - 2r + 1$ or 2s - 2r. We assert that there is such an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ with $|H_{m-r} \bigcap D_f^a| = p^{m-r-1} + p^{s+l-1}$. By the construction of \overline{H}_s as above, we have an r-dimensional subspace $\overline{H}_r \subset \overline{\mathbb{F}}_{p^m}$ satisfying $R_{\overline{H}_r} = 1$ and $\epsilon_{\overline{H}_r}$ may take values -1 or 1. And $\dim(\overline{H}_r^{\perp}) = 2s - r, R_{\overline{H}_r^{\perp}} = 2s - 2r + 1$. By Proposition 1, the values of $\epsilon_{\overline{H}_r^{\perp}}$ may take -1 or 1, too. Note that $l = \dim(\mathbb{F}_{p^m}^{\perp})$ and m-r = 2s - r + l. Thus we can construct an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ satisfying $\overline{H}_{m-r} = \overline{H}_r^{\perp}$. Notice that $\epsilon_{H_{m-r}} = \epsilon_{\overline{H}_r^{\perp}}$. Therefore, $|D_f^a \bigcap H_{m-r}| = p^{m-r-1} \pm p^{s+l-1}$. By Lemma 2, we have that $\max\{|D_f^a \bigcap H| : H \in [\mathbb{F}_{p^m}, m-r]_p\} = p^{m-r-1} + p^{s+l-1}$.

By Lemma 2, we have $|D_f^a| = p^{m-1} - \epsilon_f(-1)^{\frac{s(p-1)}{2}} p^{s+l-1}$. Then the desired results follow directly from Lemma 1. And we complete the proof.

Example 3 Let (p,m) = (3,4) and $f(x) = \text{Tr}(x^{12}) = \text{Tr}(x^{3^2+3})$. Then s = 1, l = 2, $\epsilon_f = 1$ and the weight hierarchy of $C_{D_f^1}$ is $d_1 = 18, d_2 = 30, d_3 = 34, d_4 = 36$.

Theorem 2. Let f be a degenerate quadratic form over \mathbb{F}_{p^m} with rank $R_f = 2s + 1$ and a a non-zero element in \mathbb{F}_p^* . Suppose m = 2s + 1 + l, $l = \dim(\mathbb{F}_{p^m}^{\perp})$. If $\eta(a) = (-1)^{\frac{s(p-1)}{2}} \epsilon_f$, then for the linear codes defined in (2) we have

$$d_r(C_{D_f^a}) = \begin{cases} p^{m-1} - p^{m-r-1}, & \text{if } 1 \le r \le s, \\ p^{m-1} + p^{s+l} - 2p^{m-r-1}, & \text{if } s < r < m, \\ p^{m-1} + p^{s+l}, & \text{if } r = m. \end{cases}$$

Proof. Case : $1 \le r \le s$. For an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$, we have

$$\dim(\overline{H}_{m-r}) = \dim(H_{m-r}/(H_{m-r}\bigcap \mathbb{F}_{p^m}^{\perp})) \ge m-r-l = 2s+1-r.$$

We have $\dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^{\perp}) \leq r$, since $\dim(\overline{H}_{m-r}) + \dim(\overline{H}_{m-r}^{\perp}) = 2s + 1$. Noting that $R_{\overline{H}_{m-r}} = \dim(\overline{H}_{m-r}) - \dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^{\perp})$. By Proposition 2, we have $R_{H_{m-r}} = R_{\overline{H}_{m-r}} \geq 2s + 1 - 2r$. By Lemma 2, we have

$$|H_{m-r}\bigcap D_f^a| \le p^{m-r-1} + p^{s+l},$$

and $|H_{m-r} \cap D_f^a|$ may reach the upper bound $p^{m-r-1} + p^{s+l}$ if $R_{H_{m-r}} = 2s + 1 - 2r$. We assert that there is such an (m - r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ with $|H_{m-r} \cap D_f^a| = p^{m-r-1} + p^{s+l}$, which is constructed as follows. Applying Lemma 3 to $\overline{\mathbb{F}}_{p^m}$, we know there is an *r*-dimensional subspace $\overline{H}_r \subset \overline{\mathbb{F}}_{p^m}$ with $\overline{H}_r \subset \overline{H}_r^{\perp}$. So $\dim(\overline{H}_r^{\perp}) = 2s + 1 - r$, $R_{\overline{H}_r^{\perp}} = 2s - 2r + 1$. Note that $l = \dim(\mathbb{F}_{p^m}^{\perp})$ and m - r = 2s - r + 1 + l. Thus we have an (m - r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ satisfying $\overline{H}_{m-r} = \overline{H}_r^{\perp}$. By Proposition 1, we have $\epsilon_{H_{m-r}} = \eta(-1)^r \epsilon_f$, since $\epsilon_{\overline{H}_r} = 1$, $\epsilon_f = \epsilon_{\overline{f}}$ and $\epsilon_{H_{m-r}} = \epsilon_{\overline{H}_{m-r}} = \epsilon_{\overline{H}_r^{\perp}}$. Therefore, by hypothesis and Lemma 2, $|D_f^a \cap H_{m-r}| = p^{m-r-1} + p^{s+l}$. By Lemma 2, we have that $\max\{|D_f^a \cap H| : H \in [\mathbb{F}_{p^m}, m-r]_p\} = p^{m-r-1} + p^{s+l}$.

Case : s < r < m. The proof is similar to that of Theorem 1.

By Lemma 2, we have $|D_f^a| = p^{m-1} + p^{s+l}$. Then the desired conclusions follow from Lemma 1. And the proof is completed.

Example 4 Let (p,m) = (3,4) and $f(x) = \text{Tr}(x^2 + x^{3+1})$. Then $s = 1, l = 1, \epsilon_f = -1$ and the weight hierarchy of $C_{D_f^1}$ is $d_1 = 18, d_2 = 30, d_3 = 34, d_4 = 36$.

Theorem 3. Let f be a degenerate quadratic form over \mathbb{F}_{p^m} with rank $R_f = 2s + 1$ and a a non-zero element in \mathbb{F}_p^* . Suppose m = 2s + 1 + l, $l = \dim(\mathbb{F}_{p^m}^{\perp})$. If $\eta(a) = -(-1)^{\frac{s(p-1)}{2}} \epsilon_f$, then for the linear codes defined in (2) we have

$$d_r(C_{D_f^a}) = \begin{cases} p^{m-1} - p^{m-r-1} - p^{s+l} - p^{s+l-1}, & \text{if } 1 \le r \le s, \\ p^{m-1} - p^{s+l} - 2p^{m-r-1}, & \text{if } s < r < m, \\ p^{m-1} - p^{s+l}, & \text{if } r = m. \end{cases}$$

Proof. Case : $1 \le r \le s$. for an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$, we have $R_{H_{m-r}} \ge 2s - 2r + 1$. By the corresponding proof of Theorem 2, we know that $\epsilon_{H_{m-r}} = \eta(-1)^r \epsilon_f$ if $R_{H_{m-r}} = 2s - 2r + 1$. By hypothesis and Lemma 2, we have $|D_f^a \bigcap H_{m-r}| = p^{m-r-1} - p^{s+l}$.

Next we will construct an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ with $R_{H_{m-r}} = 2s - 2r + 2$ and discuss the value of $|D_f^a \cap H_{m-r}|$. Applying Lemma 3 to $\overline{\mathbb{F}}_{p^m}$, there is an (r-1)-dimensional subspace $\overline{H}_{r-1} \subset \overline{\mathbb{F}}_{p^m}$ with $\overline{H}_{r-1} \subset \overline{H}_{r-1}^{\perp}$. So $\dim(\overline{H}_{r-1}^{\perp}) = 2s - r + 2$, $R_{\overline{H}_{r-1}^{\perp}} = 2s - 2r + 3$. Applying Lemma 2 to $\overline{\mathbb{F}}_{p^m}$, we have, for each $b \in \mathbb{F}_p^*$, $|\overline{D}_f^b \cap \overline{H}_{r-1}^{\perp}| > 1$. We choose an element $\alpha \in (\overline{D}_f^b \cap \overline{H}_{r-1}^{\perp})$ and let $\overline{H}_r = \langle \alpha \rangle \bigoplus \overline{H}_{r-1}$. Then $\dim(\overline{H}_r) = r, R_{\overline{H}_r} = 1$ and the values of $\epsilon_{\overline{H}_r} = \eta(b)$ may take -1 or 1. So $\dim(\overline{H}_r^{\perp}) = 2s + 1 - r, R_{\overline{H}_r^{\perp}} = 2s + 2 - 2r$ and $\epsilon_{\overline{H}_r^{\perp}}$ may take values -1 or 1, too. Therefore, there exists an (m-r)-dimensional subspace $H_{m-r} \subset \mathbb{F}_{p^m}$ with $\overline{H}_{m-r} = \overline{H}_r^{\perp}$. By Lemma 2, we have that $|D_f^a \cap H| = p^{m-r-1} \pm p^{s+l-1}$. Therefore, also by Lemma 2, we have $\max\{|D_f^a \cap H| : H \in [\mathbb{F}_{p^m}, m-r]_p\} = p^{m-r-1} + p^{s+l-1}$.

Case : s < r < m. The proof is similar to that of Theorem 1. We omit the details.

By Lemma 2, we have $|D_f^a| = p^{m-1} - p^{s+l}$. Then the desired conclusions follow from Lemma 1. And the proof is completed.

Example 5 Let (p,m) = (3,4) and $f(x) = \text{Tr}(x^2 - x^{3+1})$. Then $s = 1, l = 1, \epsilon_f = 1$ and the weight hierarchy of $C_{D_t^1}$ is $d_1 = 6, d_2 = 12, d_3 = 16, d_4 = 18$.

Examples 3-5 have been verified by Magma.

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