Anti-Ramsey number of matchings in r-partite r-uniform hypergraphs*

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Abstract

An edge-colored hypergraph is rainbow if all of its edges have different colors. Given two hypergraphs $\mathcal{H}$ and $\mathcal{G}$, the anti-Ramsey number $ar(\mathcal{G}, \mathcal{H})$ of $\mathcal{H}$ in $\mathcal{G}$ is the maximum number of colors in a coloring of the edges of $\mathcal{G}$ so that there does not exist a rainbow copy of $\mathcal{H}$. Li et al. determined the anti-Ramsey number of k-matchings in complete bipartite graphs. Jin and Zang showed the uniqueness of the extremal coloring. In this paper, as a generalization of these results, we determine the anti-Ramsey number $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$ of k-matchings in complete r-partite r-uniform hypergraphs and show the uniqueness of the extremal coloring. Also, we show that $\mathcal{K}_{k-1,n_2,\ldots,n_r}$ is the unique extremal hypergraph for Turán number $ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$ and show that $ar_r(\mathcal{K}_{n_1,\ldots,n_r},\ M_k) = ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_{k-1}) + 1$, which gives a multi-partite version result of Özkahya and Young's conjecture.

Keywords: anti-Ramsey number; r-partite r-uniform hypergraph

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1 Introduction

An edge-colored graph G is called *rainbow* if every edge of G receives a different color. Given two graphs H and G, $ar(H, G)$ is defined to be the maximum number of colors in a coloring of the edges of H that has no rainbow copy of G. The number $ar(H, G)$ is called the *anti-Ramsey number* of G in H. When $H = K_n$, $ar(K_n, G)$ is the anti-Ramsey number of G. Let $ex(H, G)$ denote the maximum number of edges that a subgraph of H can have with no subgraph isomorphic to G.

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The study of anti-Ramsey number began by Erdős et al. [3] in 1970s. In the original work, they conjectured that $ar\left(K_n, C_k\right) = \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1)$, and proved the conjecture when $k = 3$. After that, Alon [1] proved the conjecture for $k = 4$. Jiang, Schiermeyer and West (unpublished manuscript) proved the conjecture for $k \leq 7$. Finally, Montellano-Ballesteros and Neumann-Lara [12] completely proved the conjecture in 2005.

For matchings, Schiermeyer [14] used a counting technique to determine $ar(K_n, kK_2)$ for all $k \geq 2$ and $n \geq 3k+3$. After that, Fujita et al. [5] solved this problem for $k \geq 2$ and $n \geq 2k+1$. In 2009, Chen et al. [2] extended Schiermeyer's result to all $k \geq 2$ and $n \geq 2k$ by using the Gallai-Edmonds structure theorem.

Taking complete bipartite graphs as the host graphs, Li et al [9] determined $ar(K_{n_1,n_2}, kK_2)$ for all $k \geq 1$. Denote by $B_{n,m}$ the set of all the m-regular bipartite graphs of order $2n$. Li and Xu [10] showed that $ar(B_{n,m}, kK_2) = m(k-2)+1$ for $k \geq 2$, $m \geq 3$ and $n > 3k-1$.

A *hypergraph* $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is a finite set $V(\mathcal{H})$ of elements, called *vertices*, together with a finite set $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$, called *hyperedges* or simply *edges*. The *union* of hypergraphs $\mathcal{G}$ and $\mathcal{H}$ is the graph $\mathcal{G} \cup \mathcal{H}$ with vertex set $V(\mathcal{G}) \cup V(\mathcal{H})$ and edge set $E(\mathcal{G}) \cup E(\mathcal{H})$. If each edge of $\mathcal{H}$ has exactly r vertices, $\mathcal{H}$ is called *r-uniform*. For a subset V' of $V(\mathcal{H})$, denoted by $\mathcal{H}[V']$ the subhypergraph of $\mathcal{H}$ induced by V'. For $v \in V(\mathcal{H})$, we use $\mathcal{H} - v$ to denote $\mathcal{H}[V(\mathcal{H}) \backslash \{v\}]$. For an edge e in $E(\mathcal{H})$, denoted by $\mathcal{H} - e$ the hypergraph obtained by deleting e from $\mathcal{H}$. For a vertex $v \in V(\mathcal{H})$, the *degree* $d_{\mathcal{H}}(v)$ is defined as the number of edges of $\mathcal{H}$ containing v. A vertex of degree zero is called an *isolated* vertex. For $u, v \in V(\mathcal{H})$, we define $d_{\mathcal{H}}(u, v)$ to be the number of edges of $\mathcal{H}$ containing $\{u, v\}$, and we call this number the *co-degree* of $\{u, v\}$. For a hypergraph $\mathcal{H}$, we denote the number of edges in $\mathcal{H}$ by $e(\mathcal{H})$. A *complete r-uniform hypergraph* is a hypergraph whose edge set consists of all r-subsets of the vertex set. A *matching* in a hypergraph is a set of edges in which no two edges have a common vertex. We call a matching with k edges a *k-matching*, denoted by M_k. An edge-colored hypergraph is called *rainbow hypergraph* if the all of its edges have different colors. The *representing hypergraph* of a hypergraph $\mathcal{H}$ with an edge coloring c is a spanning subhypergraph of $\mathcal{H}$ obtained by taking one edge of each color of c. For an edge set $E \subseteq E(\mathcal{H})$, let $c(E)$ denote the set of colors of edges in E. For simplicity, when $E = \{e\}$ and $E = E(\mathcal{H})$, we use $c(e)$ and $c(\mathcal{H})$ instead of $c(\{e\})$ and $c(E(\mathcal{H}))$, respectively.

Let $n_1, n_2, \ldots, n_r$ be integers and $V_1, V_2, \ldots, V_r$ be disjoint vertex sets with $|V_i| = n_i$ for each $i = 1, 2, \ldots, r$. A *complete r-partite r-uniform hypergraph* on vertex classes $V_1, V_2, \ldots, V_r$, denoted by $\mathcal{K}_{n_1,\ldots,n_r}$, is defined to be the r-uniform hypergraph whose edge set consists of all the r-element subsets S of $V_1 \cup \cdots \cup V_r$ such that $|S \cap V_i| = 1$ for all $i = 1, 2, \ldots, r$.

Given two hypergraphs $\mathcal{H}$ and $\mathcal{G}$, the *anti-Ramsey number* of $\mathcal{H}$ in $\mathcal{G}$, denoted by $ar(\mathcal{G}, \mathcal{H})$, is the maximum number of colors in a coloring of the edges of $\mathcal{G}$ with no rainbow copy of $\mathcal{H}$. When $\mathcal{G}$ is an r-uniform complete hypergraph on n vertices, $ar_r(\mathcal{G},\mathcal{H})$ is the anti-Ramsey number of $\mathcal{H}$. The *Turán number* $ex_r(\mathcal{G},\mathcal{H})$ is the maximum number of edges in an $\mathcal{H}$-free subhypergraph of $\mathcal{G}$, where $\mathcal{H}$-free hypergraph is one which contains no $\mathcal{H}$ as a subhypergraph.

Gu et al. [6] determined the anti-Ramsey numbers of linear paths/cycles and loose paths/cycles in hypergraphs for sufficiently large n and gave bounds on the anti-Ramsey numbers of Berge paths/cycles. For the anti-Ramsey number of matchings in hypergraphs, Özkahya and Young [13] stated a conjecture that $ar_r(\mathcal{K}_n, M_k) = ex_r(\mathcal{K}_n, M_{k-1}) + 1$ for all $n > sk$ and proved the conjecture when $k = 2, 3$ and n is sufficiently large. Recently, Frankl and Kupavskii [4] proved that the conjecture is true for $n \geq rk + (r-1)(k-1)$ and $k \geq 3$. Jin [7] determined the exact value of the anti-Ramsey number of a k-matching in a complete tripartite 3-uniform hypergraph.

Take a subhypergraph $\mathcal{K}_{k-2,n_2,\ldots,n_r}$ of $\mathcal{K}_{n_1,n_2,\ldots,n_r}$. Color the edges of $\mathcal{K}_{k-2,n_2,\ldots,n_r}$ by distinct colors and color the remaining edges of $\mathcal{K}_{n_1,\ldots,n_r}$ by a new color. Denote by ϕ_r the obtained $((k-2)n_2\cdots n_r+1)$-edge-coloring of $\mathcal{K}_{n_1,\ldots,n_r}$. Li, Tu and Jin [9] determined the following results in complete bipartite graphs.

Theorem 1 ([9])**.** *For $n_2 \geq n_1 \geq k \geq 1$,*

$$ex(K_{n_1,n_2}, kK_2) = (k-1)n_2.$$

Moreover, K_{k-1,n_2} is the unique such extremal graph.

Theorem 2 ([9])**.** *For $n_2 \geq n_1 \geq k \geq 3$,*

$$ar(K_{n_1,n_2}, kK_2) = (k-2)n_2 + 1.$$

In addition to the anti-Ramsey number, another interesting problem posed by Erdős is the uniqueness of the extremal coloring. In [8], Jin and Zang obtained the following result.

Theorem 3 ([8])**.** *For $n_2 \geq n_1 \geq k \geq 3$, every $((k-2)n_2+1)$-edge-coloring except for ϕ_2 of K_{n_1,n_2} contains a rainbow kK_2.*

The following proposition provides a lower and upper bound for $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$, and the proof of which is similar to that of [14].

Proposition 4. $ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_{k-1}) + 1 \leq ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) \leq ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$.

Proof. The upper bound is clear. For the lower bound, let $\mathcal{H}_0$ be an extremal hypergraph for $ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_{k-1})$ and color all edges of $\mathcal{H}_0$ differently and all the edges in $E\left(\mathcal{K}_{n_1,\ldots,n_r}\right) \backslash E\left(\mathcal{H}_0\right)$

with one extra color. The hypergraph $\mathcal{K}_{n_1,\ldots,n_r}$ with this coloring does not contain a rainbow k-matching. The result follows. $\square$

The proposition provides a lower bound for $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$. In this paper we will show that $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_{k-1}) + 1$, which gives a multi-partite version result of Özkahya and Young's conjecture.

In [11], Liu and Wang determined $ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k)$.

Theorem 5 ([11])**.** *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq k \geq 1$,*

$$ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = (k-1)n_2 \cdots n_r.$$

We will show that $\mathcal{K}_{k-1,n_2,\ldots,n_r}$ is the unique extremal hypergraph in Theorem 5.

The following result is very useful for us.

Theorem 6. *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq k \geq 1$, every subhypergraph of $\mathcal{K}_{n_1,\ldots,n_r}$ with $(k-1)n_2 \cdots n_r$ edges and without isolated vertices, except for $\mathcal{K}_{k-1,n_2,\ldots,n_r}$, contains a k-matching.*

Motivated by Theorem 2, one may naturally ask what is the maximum number of colors in a complete r-partite r-uniform hypergraph without a rainbow k-matching, where $r \geq 3$. This paper focus on the anti-Ramsey number of k-matchings in complete r-partite r-uniform hypergraphs. The following are our main results.

Theorem 7. *(i) For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 3$,*

$$ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) = 1.$$

(ii) For $n_1 = 2$, let t be the largest integer such that $n_t = n_1 = 2$. Then

$$ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) = 2^{t-1}.$$

Theorem 8. *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 2k-1$ and $k \geq 3$,*

$$ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = (k-2)n_2 \cdots n_r + 1.$$

Moreover, every $((k-2)n_2 \cdots n_r + 1)$-edge-coloring except for ϕ_r of $\mathcal{K}_{n_1,\ldots,n_r}$ contains a rainbow k-matching.

Combining Theorems 5, 7 and 8, we have the following corollary.

Corollary 9. *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 2k-1$ and $k \geq 2$,*

$$ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = ex_r(\mathcal{K}_{n_1,\ldots,n_r}, M_{k-1}) + 1.$$

2 Proofs of Theorems 6 and 7

Proof of Theorem 6. We use induction on r. The base case of $r = 2$ is true for all $n_2 \geq n_1 \geq k$ by Theorem 1. Suppose that the assertion holds for all $r' < r$. Assume that $\mathcal{G}$ is a subhypergraph of $\mathcal{K}_{n_1,\ldots,n_r}$ with $(k-1)n_2 \cdots n_r$ edges and without isolated vertices, and does not contain a k-matching. Let $V_s = \{v_{s1}, v_{s2}, \ldots, v_{sn_s}\}$ for $s = 1, 2, \ldots, r$. We consider two different cases.

Case 1. $n_1 = n_2$.

For $1 \leq i, j \leq n_1$, let $F_{i,j} = \big\{\{v_{1i}, v_{2j}, w_3, \ldots, w_r\} \in E(\mathcal{G})|\ w_s \in V_s \text{ for } 3 \leq s \leq r\big\}$ and $F_i = F_{i,1} \cup F_{i+1,2} \cup \cdots \cup F_{i+n_1-1,n_1}$, where $F_{i,j} = F_{i-n_1,j}$ if $i > n_1$.

For each F_i, $i = 1, 2, \ldots, n_1$, we construct an $(r-1)$-partite $(r-1)$-uniform hypergraph $\mathcal{G}_i$ on vertex classes $V_1, V_3, \ldots, V_r$, and $e = \{v_{1l}, w_3, \ldots, w_r\}$ is an edge of $\mathcal{G}_i$ if and only if $e' = \{v_{1l}, v_{2l'}, w_3, \ldots, w_r\}$ is an edge of F_i, where $l - l' \equiv i - 1 \pmod{n_1}$. Therefore, there is a bijection between F_i and $E(\mathcal{G}_i)$. Note that if two edges e_1 and e_2 in $\mathcal{G}_i$ are independent, then the corresponding edges e_1' and e_2' in F_i are also independent. Then we have the following fact.

Fact A. Any matching in $\mathcal{G}_i$ corresponds to a matching in $F_i \subseteq E(\mathcal{G})$.

First, we prove the following claims.

Claim 1. For $i \neq j$, $F_i \cap F_j = \emptyset$.

Proof. If there exists an edge $\{v_{1l}, v_{2l'}, w_3, \ldots, w_r\} \in F_i \cap F_j$, then $l - l' \equiv i - 1 \pmod{n_1}$ and $l - l' \equiv j - 1 \pmod{n_1}$ which implies $i = j$. $\square$

It follows from Claim 1 that $e(\mathcal{G}) = \sum\limits_{i=1}^{n_1} |F_i| = \sum\limits_{i=1}^{n_1} e(\mathcal{G}_i)$.

Claim 2. For any $1 \leq i \leq n_1$, $e(\mathcal{G}_i) = (k-1)n_3 \cdots n_r$.

Proof. First, we have $e(\mathcal{G}_i) \leq (k-1)n_3 \cdots n_r$. Otherwise, $\mathcal{G}_i$ contains a k-matching by Theorem 5, so does $\mathcal{G}$ by Fact A, a contradiction. Hence,

$$(k-1)n_1 n_3 \cdots n_r = e(\mathcal{G}) = \sum_{i=1}^{n_1} e(\mathcal{G}_i) \leq n_1(k-1)n_3 \cdots n_r,$$

which implies that $e(\mathcal{G}_i) = (k-1)n_3 \cdots n_r$ for each $1 \leq i \leq n_1$. $\square$

According to Fact A and Claim 2, $\mathcal{G}_i$ is a subhypergraph of $\mathcal{K}_{n_1,n_3,\ldots,n_r}$ with $(k-1)n_3 \cdots n_r$ edges and does not contain a k-matching. By the induction hypothesis, $\mathcal{G}_i \cong \mathcal{K}_{k-1,n_3,\ldots,n_r} \cup (n_1 - k + 1)\mathcal{K}_1$ for $i = 1, 2, \ldots, n_1$. Recall the construction of $\mathcal{G}_i$, we deduce that $d_{\mathcal{G}}(v_{1i}, v_{2j}) = 0$ or $d_{\mathcal{G}}(v_{1i}, v_{2j}) = n_3 \cdots n_r$ for $1 \leq i, j \leq n_1$. Construct an auxiliary bipartite graph G with bipartition (V_1, V_2), where $e_{ij} = v_{1i}v_{2j} \in E(G)$ if and only if $d_{\mathcal{G}}(v_{1i}, v_{2j}) = n_3 \cdots n_r$. Then $e(\mathcal{G}) = (k-1)n_2 \cdots n_r$ implies that $e(G) = (k-1)n_2 = (k-1)n_1$. We claim that there is no k-matching in G. If there exists a k-matching e_{i_1,j_1}, e_{i_2,j_2}, $\ldots$, e_{i_k,j_k} in G, we can find k

edges e'_{i_1,j_1}, e'_{i_2,j_2}, ..., e'_{i_k,j_k} to form a k-matching in $\mathcal{G}$, where $e'_{i_l,j_l} = \{v_{1,i_l}, v_{2,j_l}, v_{3l}, \ldots, v_{rl}\}$ for $l = 1, 2, \ldots, r$. This contradicts the choice of $\mathcal{G}$. It follows from Theorem 1 that $G \cong K_{k-1,n_1} \cup (n_1 - k + 1)K_1$. Without loss of generality, let $E(G) = \{v_{1i}v_{2j} | \ 1 \leq i \leq k-1, 1 \leq j \leq n_2\}$. By the construction of G, every edge in $E(\mathcal{K}_{n_1,\ldots,n_r})$ containing $\{v_{1i}, v_{2j}\}$ is an edge in $\mathcal{G}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n_2$. Hence, $\mathcal{G} \cong \mathcal{K}_{k-1,n_2,n_3,\ldots,n_r}$ when $n_1 = n_2$.

Case 2. $n_1 < n_2$.

Claim 3. For $u \in V_2$, $d_{\mathcal{G}}(u) = (k-1)n_3 \cdots n_r$.

Proof. If there exists a vertex $u \in V_2$ such that $d_{\mathcal{G}}(u) < (k-1)n_3 \cdots n_r$, then

$$e(\mathcal{G} - u) = e(\mathcal{G}) - d_{\mathcal{G}}(u) > (k-1)(n_2 - 1)n_3 \cdots n_r.$$

By Theorem 5, $\mathcal{G} - u$ contains a k-matching, so does $\mathcal{G}$, a contradiction. Hence, $d_{\mathcal{G}}(u) \geq (k-1)n_3 \cdots n_r$ for all $u \in V_2$. Note that

$$(k-1)n_2 \cdots n_r = e(\mathcal{G}) = \sum_{u \in V_2} d_{\mathcal{G}}(u) \geq (k-1)n_2 n_3 \cdots n_r.$$

We deduce that $d_{\mathcal{G}}(u) = (k-1)n_3 \cdots n_r$ for all $u \in V_2$. $\square$

Set $V'_2 \subseteq V_2$ such that $|V'_2| = n_1$. Let $\mathcal{G}' = \mathcal{G}[V_1, V'_2, V_3, \ldots, V_r]$. According to Claim 3, we have $e(\mathcal{G}') = \sum_{u \in V'_2} d_{\mathcal{G}}(u) = (k-1)n_1 n_3 \cdots n_r$. It follows from Case 1 that

$$\mathcal{G}' \cong \mathcal{K}_{k-1,n_1,n_3,\ldots,n_r} \cup (n_1 - k + 1)\mathcal{K}_1.$$

Combining this with the arbitrariness of V'_2, we have $d_{\mathcal{G}}(v_{1i}, v_{2j}) = 0$ or $d_{\mathcal{G}}(v_{1i}, v_{2j}) = n_3 \cdots n_r$ for $1 \leq i \leq n_1, 1 \leq j \leq n_2$. Construct an auxiliary bipartite graph G with bipartition (V_1, V_2), where $e_{ij} = v_{1i}v_{2j} \in E(G)$ if and only if $d_{\mathcal{G}}(v_{1i}, v_{2j}) = n_3 \cdots n_r$. Then $e(\mathcal{G}) = (k-1)n_2 n_3 \cdots n_r$ implies that $e(G) = (k-1)n_2$. We claim that there is no k-matching. Otherwise, if there is a k-matching $e_{i_1,j_1}, \ldots, e_{i_k,j_k}$ in G, we can find k edges in $F_{i_1,j_1}, \ldots, F_{i_k,j_k}$ to form a k-matching in $\mathcal{G}$, a contradiction. By Theorem 1, $G \cong K_{k-1,n_2} \cup (n_1 - k + 1)K_1$. Without loss of generality, let $E(G) = \{v_{1i}v_{2j} | \ 1 \leq i \leq k-1, 1 \leq j \leq n_2\}$. By the construction of G, every edge in $E(\mathcal{K}_{n_1,\ldots,n_r})$ containing $\{v_{1i}, v_{2j}\}$ is an edge in $\mathcal{G}$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n_2$. Hence, $\mathcal{G} \cong \mathcal{K}_{k-1,n_2,n_3,\ldots,n_r}$. $\square$

We now turn to the proofs of the main results of this paper. Theorem 7 gives the value of anti-Ramsey number of k-matching in complete r-partite r-uniform hypergraphs when $k = 2$.

Proof of Theorem 7.

(i) $n_1 \geq 3$.

Suppose to the contrary that $\mathcal{H}$ is a complete r-partite r-uniform hypergraph colored by more than one color, and containing no rainbow 2-matching. Set $e, f \in E(\mathcal{H})$ such that $c(e) \neq c(f)$. Clearly, $E(\mathcal{H}-e) \cap E(\mathcal{H}-f) \neq \emptyset$ as $n_1 \geq 3$. Choose an edge $g \in E(\mathcal{H}-e) \cap E(\mathcal{H}-f)$, then $c(g) = c(e)$ as $\mathcal{H}$ does not contain a rainbow 2-matching. Similarly, we have $c(g) = c(f)$, then $c(e) = c(f)$, contradicting the fact $c(e) \neq c(f)$. Therefore, $|c(\mathcal{H})| = 1$, i.e., $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) = 1$.

(ii) $n_1 = 2$.

By the choice of t, $n_1 = n_2 = \cdots = n_t = 2$. Let $\mathcal{H} \cong \mathcal{K}_{n_1,\ldots,n_r}$ be a complete r-partite r-uniform hypergraph. Let $R(\mathcal{H}, t) = \{\{v_1, v_2, \ldots, v_t\} |\ v_i \in V_i \text{ and } i \in [t]\}$. For any edge $e = \{v_1, v_2, \ldots, v_r\} \in E(\mathcal{H})$, let $R(e, t) = \{v_1, v_2, \ldots, v_t\}$. For $\alpha = \{v_1, v_2, \ldots, v_t\} \in R(\mathcal{H}, t)$, let $\overline{\alpha} = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_t\}$, where $\overline{v}_i$ is the remaining vertex in V_i except v_i. For $\alpha = \{v_1, v_2, \ldots, v_t\} \in R(\mathcal{H}, t)$, let

$$E_{\{\alpha,\overline{\alpha}\}} = \{e \in E(\mathcal{H}) \,|\, R(e,t) = \alpha \text{ or } R(e,t) = \overline{\alpha}\}.$$

Then we decompose the edge set of $\mathcal{H}$ into $E(\mathcal{H}) = \cup_{\{\alpha,\overline{\alpha}\}\in Q} E_{\{\alpha,\overline{\alpha}\}}$, where $Q = \{\{\alpha, \overline{\alpha}\} |\ \alpha \in R(\mathcal{H}, t)\}$. It is easily checked that $|Q| = 2^{t-1}$ and $E_{\{\alpha,\overline{\alpha}\}} \cap E_{\{\beta,\overline{\beta}\}} = \emptyset$ for any two different elements $\{\alpha, \overline{\alpha}\}, \{\beta, \overline{\beta}\}$ in Q.

We consider the following coloring of $\mathcal{H}$: For any two edges $e, f \in E(\mathcal{H})$, $c(e) = c(f)$ if and only if there exists an element $\{\alpha, \overline{\alpha}\} \in Q$ such that $e, f \in E_{\{\alpha,\overline{\alpha}\}}$. By the definition of the coloring of $\mathcal{H}$, any two independent edges have the same color and $|c(\mathcal{H})| = |Q| = 2^{t-1}$. So $\mathcal{H}$ does not contain a rainbow 2-matching. Therefore, $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) \geq 2^{t-1}$.

Let $\mathcal{H} = \mathcal{K}_{n_1,\ldots,n_r}$ be an edge-colored complete r-partite r-uniform hypergraph without rainbow 2-matchings. To obtain the upper bound, we consider the following cases.

Case 1. $t = r$.

In this case, $E_{\{\alpha,\overline{\alpha}\}} = \{\alpha, \overline{\alpha}\}$. Then $E(\mathcal{H})$ can be decomposed into 2^{r-1} pairs of independent edges, and each pair of edges must be colored by the same color. Hence $|c(\mathcal{H})| \leq 2^{r-1}$, then $ar_r(\mathcal{K}_{2,\ldots,2}, M_2) \leq 2^{r-1}$, which implies that $ar_r(\mathcal{K}_{2,\ldots,2}, M_2) = 2^{r-1}$.

Case 2. $t < r$.

Recall that $E(\mathcal{H}) = \cup_{\{\alpha,\overline{\alpha}\}\in Q} E_{\{\alpha,\overline{\alpha}\}}$. We have the following claims.

Claim 1. For any $\{\alpha, \overline{\alpha}\} \in Q$, if $e, f \in E_{\{\alpha,\overline{\alpha}\}}$ and $R(e,t) = R(f,t)$, then $c(e) = c(f)$.

Proof. For any $\{\alpha, \overline{\alpha}\} \in Q$, since $n_r \geq \cdots \geq n_{t+1} \geq 3$, there exists an edge $g = \{u_1, u_2, \ldots, u_r\} \in E_{\{\alpha,\overline{\alpha}\}}$, such that $R(g,t) = \overline{R(e,t)}$ and $u_j \in V_j - e \cup f$ for $j = t+1, \ldots, r$. Then $g \cap e = g \cap f = \emptyset$. Therefore, $c(e) = c(g) = c(f)$. Otherwise, $\mathcal{H}$ contains a rainbow 2-matching. $\square$

Claim 2. For any $\{\alpha, \overline{\alpha}\} \in Q$, if $e, f \in E_{\{\alpha,\overline{\alpha}\}}$ and $R(e,t) = \overline{R(f,t)}$, then $c(e) = c(f)$.

Proof. Since $n_r \geq \cdots \geq n_{t+1} \geq 3$, there exists an edge $g = \{u_1, u_2, \ldots, u_r\} \in E_{\{\alpha,\overline{\alpha}\}}$ with $R(g,t) = R(e,t)$ and $u_j \in V_j - f$ for $j = t+1, \ldots, r$. It follows from Claim 1 that $c(g) = c(e)$. Note that $g \cap f = \emptyset$, we have $c(f) = c(g)$. Then $c(e) = c(f)$. $\square$

Combining these two claims, we conclude that $c(E_{\{\alpha,\overline{\alpha}\}}) = 1$ for any $\{\alpha, \overline{\alpha}\} \in Q$. Therefore, $|c(\mathcal{H})| \leq 2^{t-1}$, which implies that $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) \leq 2^{t-1}$. Thus $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_2) = 2^{t-1}$. $\square$

3 Proof of Theorem 8

In this section we will determine the value of the anti-Ramsey number of k-matchings in complete r-partite r-uniform hypergraphs, and also give the uniqueness of extremal coloring.

We need the following lemma.

Lemma 10. *For $n_1 \geq 2k-1$ and $k \geq 3$,*

$$ar_r(\mathcal{K}_{n_1,\ldots,n_1}, M_k) = (k-2)n_1^{r-1} + 1,$$

and every $((k-2)n_1^{r-1}+1)$-edge-coloring except for ϕ_r of $\mathcal{K}_{n_1,\ldots,n_1}$ contains a rainbow k-matching.

Proof. We use induction on r. The base case of $r = 2$ is true for all $n_1 \geq 2k-1$ by Theorem 2 and Theorem 3. Suppose that the lemma holds for all $r' < r$. Assume, by way of contradiction, that $\mathcal{H} = \mathcal{K}_{n_1,\ldots,n_1}$ is a hypergraph with a $((k-2)n_1^{r-1}+2)$-edge-coloring and does not contain a rainbow k-matching.

Let $V_s = \{v_{s1}, v_{s2}, \ldots, v_{sn_1}\}$, $s = 1, 2, \ldots, r$. For $1 \leq i, j \leq n_1$, let

$$E_{i,j} = \big\{\{v_{1i}, v_{2j}, w_3, \ldots, w_r\} \in E(\mathcal{H}) \,|\, w_s \in V_s \text{ for } 3 \leq s \leq r\big\},$$

and $E_i = E_{i,1} \cup E_{i+1,2} \cup \cdots \cup E_{i+n_1-1,n_1}$, where $E_{i,j} = E_{i-n_1,j}$ if $i > n_1$.

For each E_i, $i = 1, 2, \ldots, n_1$, we construct a complete $(r-1)$-partite $(r-1)$-uniform hypergraph $\mathcal{H}_i$ on vertex classes $V_1, V_3, \ldots, V_r$ such that $e = \{v_{1l}, w_3, \ldots, w_r\}$ is an edge of $\mathcal{H}_i$ if and only if $e' = \{v_{1l}, v_{2l'}, w_3, \ldots, w_r\}$ is an edge of E_i, where $l - l' \equiv i-1 \pmod{n_1}$, and we color e by $c(e')$. Therefore, there is a bijection between E_i and $E(\mathcal{H}_i)$ and $c(E_i) = c(\mathcal{H}_i)$. Note that if two edges e_1 and e_2 in $\mathcal{H}_i$ are independent, then the corresponding edges e'_1 and e'_2 in E_i are also independent. Then we have the following fact.

Fact B. Any rainbow matching in $\mathcal{H}_i$ corresponds to a rainbow matching in $E_i \subseteq E(\mathcal{H})$.

Obviously, $E(\mathcal{H}) = \bigcup_{i=1}^{n_1} E_i$. Then

$$\sum_{i=1}^{n_1} |c(\mathcal{H}_i)| = \sum_{i=1}^{n_1} |c(E_i)| \geq |c(\mathcal{H})| = (k-2)n_1^{r-1} + 2.$$

Without loss of generality, we assume that $\mathcal{H}_1$ has the most colors in $\mathcal{H}_1, \ldots, \mathcal{H}_{n_1}$.

Claim 1. $|c(\mathcal{H}_1)| = (k-2)n_1^{r-2} + 1$.

Proof. First, we have $|c(\mathcal{H}_i)| \leq (k-2)n_1^{r-2} + 1$ for $1 \leq i \leq n_1$. Otherwise, by induction hypothesis, $\mathcal{H}_i$ contains a rainbow k-matching, so does $\mathcal{H}$ by Fact B, a contradiction. If $|c(\mathcal{H}_1)| \leq (k-2)n_1^{r-2}$, then

$$(k-2)n_1^{r-1} + 2 = |c(\mathcal{H})| \leq \sum_{i=1}^{n_1} |c(\mathcal{H}_i)| \leq n_1(k-2)n_1^{r-2},$$

a contradiction. Hence, $|c(\mathcal{H}_1)| = (k-2)n_1^{r-2} + 1$. $\square$

We next show that there exists an integer $2 \leq t \leq n_1$ such that $|c(\mathcal{H}_t)| = (k-2)n_1^{r-2} + 1$ and $c(\mathcal{H}_1) \cap c(\mathcal{H}_t) = \emptyset$. Otherwise,

$$|c(\mathcal{H})| \leq ((k-2)n_1^{r-2} + 1) + (k-2)n_1^{r-2} \cdot (n_1 - 1) < (k-2)n_1^{r-1} + 2,$$

contradicting the assumption of $\mathcal{H}$. Since $\mathcal{H}_1$ and $\mathcal{H}_t$ do not contain a rainbow k-matching and with $(k-2)n_1^{r-2} + 1$ colors, by the induction hypothesis, they are both colored by ϕ_{r-1}. Let H_t be the representing hypergraph of $\mathcal{H}_t$. Then

$$e(H_t) = |c(\mathcal{H}_t)| > (k-2)n_1^{r-2},$$

so H_t contains a $(k-1)$-matching by Theorem 5. Hence, there is a rainbow $(k-1)$-matching in $\mathcal{H}_t$, which corresponds to a rainbow $(k-1)$-matching, denoted by M_{k-1}, in E_t. The $(k-1)$-matching M_{k-1} meets $k-1$ vertices in V_1 and $k-1$ vertices in V_2, respectively. As $n_1 \geq 2k-1$, there exists an integer s, such that $v_{1s} \in V_1 - V(M_{k-1})$ and $v_{2s} \in V_2 - V(M_{k-1})$. Then we can find an edge $e \in E_{s,s} \subseteq E_1$ such that $e \cap V(M_{k-1}) = \emptyset$. Recall that $c(\mathcal{H}_1) \cap c(\mathcal{H}_t) = \emptyset$, $c(\mathcal{H}_1) = c(E_1)$ and $c(\mathcal{H}_t) = c(E_t)$, then $e \cup M_{k-1}$ is a rainbow k-matching in $\mathcal{H}$, a contradiction. Hence $ar_r(\mathcal{K}_{n_1,\ldots,n_1}, M_k) \leq (k-2)n_1^{r-1} + 1$. Combining this with Proposition 4 and Theorem 5, we have $ar_r(\mathcal{K}_{n_1,\ldots,n_1}, M_k) = (k-2)n_1^{r-1} + 1$.

Now we prove the uniqueness of the extremal coloring. Suppose that $|c(\mathcal{H})| = (k-2)n_1^{r-1} + 1$ and $\mathcal{H}$ does not contain a rainbow k-matching. Let H be the representing hypergraph of $\mathcal{H}$. Then $e(H) = (k-2)n_1^{r-1} + 1$. As the discussion above, we construct $E_i \subseteq E(\mathcal{H})$ and a complete $(r-1)$-partite $(r-1)$-uniform hypergraph $\mathcal{H}_i$ on vertex classes $V_1, V_3, \ldots, V_r$ for $i = 1, 2, \ldots, n_1$. Without loss of generality, assume that $\mathcal{H}_1$ has the most colors in $\mathcal{H}_1$, …, $\mathcal{H}_{n_1}$. Then $|c(\mathcal{H}_1)| = (k-2)n_1^{r-2} + 1$.

Notice that $\mathcal{H}_1$ does not contain a rainbow k-matching and $|c(\mathcal{H}_1)| = (k-2)n_1^{r-2}+1$. By the induction hypothesis, $\mathcal{H}_1$ is colored by configuration ϕ_{r-1}. Without loss of generality, we assume that in $\mathcal{H}_1$, $E^* = \big\{\{v_{1i}, w_3, \ldots, w_r\} \in E(\mathcal{H}_1) \mid k-1 \le i \le n_1 \text{ and } w_s \in V_s \text{ for } 3 \le s \le r\big\}$ are colored by one color and all the remaining edges of $\mathcal{H}_1$ colored by distinct colors. Then, in $\mathcal{H}$, $c(E_{k-1,k-1}) = \cdots = c(E_{n_1,n_1}) = c(E^*)$, and the colors of the edges in $E_{1,1} \cup \cdots \cup E_{k-2,k-2}$ are different from each other. Let $\mathcal{H}_0$ be the rainbow subhypergraph of $\mathcal{H}$ obtained by taking one edge of each color of $c(\mathcal{H})$ except $c(E^*)$ and such that $E_{1,1} \cup \cdots \cup E_{k-2,k-2}$ are contained in $\mathcal{H}_0$. Then $|E(\mathcal{H}_0)| = (k-2)n_1^{r-1}$. We have the following claim.

Claim 2. $\mathcal{H}_0 \cong \mathcal{K}_{k-2,n_1,\ldots,n_1}$.

Proof. If $\mathcal{H}_0 \ncong \mathcal{K}_{k-2,n_1,\ldots,n_1}$, then there exists a $(k-1)$-matching, denoted by M'_{k-1}, in $\mathcal{H}_0$ by Theorem 6. Notice that M'_{k-1} meets at most $2(k-1)$ vertices in $V_1 \cup V_2$ and $n_1 \ge 2k-1$, we can find an edge e'_k in E_1, such that $V(M'_{k-1}) \cap e'_k = \emptyset$. If $e'_k \in E_1 \backslash (E_{1,1} \cup \cdots \cup E_{k-2,k-2})$, then $c(M'_{k-1}) \cap c(e'_k) = \emptyset$. If $e'_k \in E_{1,1} \cup \cdots \cup E_{k-2,k-2}$, then $e'_k \in E(\mathcal{H}_0)$. We also have $c(M'_{k-1}) \cap c(e'_k) = \emptyset$ as $\mathcal{H}_0$ is a rainbow subhypergraph of $\mathcal{H}$. Then $e'_k \cup M'_{k-1}$ is a rainbow k-matching in $\mathcal{H}$, a contradiction. □

By Claim 2, there is a rainbow subhypergraph $\mathcal{H}_0 \cong \mathcal{K}_{k-2,n_2,\ldots,n_r}$ of $\mathcal{H}$. Combining the fact that all the edges in $E(\mathcal{H}) \backslash E(\mathcal{H}_0)$ are colored by $c(E^*)$, we deduce that $\mathcal{H}$ is colored by configuration ϕ_r. □

We now prove Theorem 8.

Proof of Theorem 8. We first prove that $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = (k-2)n_2 \cdots n_r + 1$ for $n_1 \ge 2k-1$ and $k \ge 3$. Since ϕ_r is an edge coloring of $\mathcal{K}_{n_1,\ldots,n_r}$ such that there is no rainbow k-matching in $\mathcal{K}_{n_1,\ldots,n_r}$, $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) \ge (k-2)n_2 \cdots n_r + 1$.

The upper bound is proven by induction on the number of vertices of $\mathcal{K}_{n_1,\ldots,n_r}$. The base case of $n_1 = n_r$ is true by Lemma 10, we now suppose that $n_1 < n_r$. Assume the assertion holds for all $n'_1, \ldots, n'_r$ such that $\sum_{i=1}^r n'_i < \sum_{i=1}^r n_i$. Suppose that $\mathcal{H} = \mathcal{K}_{n_1,\ldots,n_r}$ is a hypergraph with a $((k-2)n_2 \cdots n_r + 2)$-edge-coloring, and containing no rainbow k-matchings. Let H be the representing hypergraph of $\mathcal{H}$. Then $e(H) = (k-2)n_2 \cdots n_r + 2$. For any vertex $v \in V_r$, $H - v$ is a subhypergraph of some representing hypergraph of $\mathcal{H} - v$. Since $\mathcal{H} - v$ contains no rainbow k-matchings, by the induction hypothesis, we have

$$e(H - v) \le |c(\mathcal{H} - v)| \le (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 1.$$

Then $d_H(v) = e(H) - e(H-v) \geq (k-2)n_2 \cdots n_{r-1} + 1$. Thus,

$$\begin{aligned} e(H) &= \sum_{v \in V_r} d_H(v) \geq n_r \cdot \big((k-2)n_2 \cdots n_{r-1} + 1\big) \\ &= (k-2)n_2 \cdots n_r + n_r > (k-2)n_2 \cdots n_r + 2, \end{aligned}$$

which is a contradiction. Therefore, $ar_r(\mathcal{K}_{n_1,\ldots,n_r}, M_k) = (k-2)n_2 \cdots n_r + 1$.

Next we prove the uniqueness of the extremal coloring. Suppose that $|c(\mathcal{H})| = (k-2)n_2 \cdots n_r + 1$ and $\mathcal{H}$ does not contain a rainbow k-matching. Then $e(H) = (k-2)n_2 \cdots n_r + 1$. We have the following claim.

Claim 1. There exists a vertex $v^* \in V_r$ with $d_H(v^*) = (k-2)n_2 \cdots n_{r-1} + 1$, and $d_H(v) = (k-2)n_2 \cdots n_{r-1}$ for $v \in V_r \backslash \{v^*\}$.

Proof. For any vertex $v \in V_r$, $H - v$ is a subhypergraph of some representing hypergraph of $\mathcal{H} - v$. If $d_H(v) \leq (k-2)n_2 \cdots n_{r-1} - 1$, then

$$|c(\mathcal{H} - v)| \geq e(H - v) \geq (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 2 = ar_r(\mathcal{K}_{n_1,\ldots,n_{r-1},n_r-1}, M_k) + 1.$$

By the induction hypothesis, $\mathcal{H} - v$ contains a rainbow k-matching, a contradiction. Thus, $d_H(v) \geq (k-2)n_2 \cdots n_{r-1}$ for all $v \in V_r$. Note that

$$e(H) = \sum_{v \in V_r} d_H(v) = n_r \cdot (k-2)n_2 \cdots n_{r-1} + 1.$$

Therefore, there is exactly one vertex, say v^*, with degree $(k-2)n_2 \cdots n_{r-1} + 1$, and the remaining vertices in V_r with degree $(k-2)n_2 \cdots n_{r-1}$. $\square$

For $v \in V_r \backslash \{v^*\}$, we have

$$e(H - v) = e(H) - d_H(v) = (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 1.$$

Clearly, $H - v$ is a subhypergraph of some representing hypergraph of $\mathcal{H} - v$. Thus,

$$|c(\mathcal{H} - v)| \geq e(H - v) = (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 1.$$

By the induction hypothesis, $|c(\mathcal{H} - v)| \leq (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 1$. Hence, we deduce that $|c(\mathcal{H} - v)| = (k-2)n_2 \cdots n_{r-1}(n_r - 1) + 1$. By the induction hypothesis, $\mathcal{H} - v$ is colored by ϕ_r since $\mathcal{H} - v$ contains no rainbow k-matchings. Due to the arbitrariness of v, for any two vertices $v_1, v_2 \in V_r \backslash \{v^*\}$, $\mathcal{H} - v_1$ and $\mathcal{H} - v_2$ are colored by ϕ_r. Assume S_1 and S_2 are the $(k-2)$-subsets of V_1 such that the colors of the edges intersecting S_1 and S_2 are all different in $\mathcal{H} - v_1$ and $\mathcal{H} - v_2$, and edges not intersecting S_1 and S_2 are colored with a new color in $\mathcal{H} - v_1$

and $\mathcal{H} - v_2$, respectively. Then $S_1 = S_2$. So all edges intersecting S_1 are colored with different colors in $\mathcal{H}$ and edges not intersecting S_1 are colored with a new color in $\mathcal{H}$. So $\mathcal{H}$ is colored by ϕ_r. This completes the proof. $\square$

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